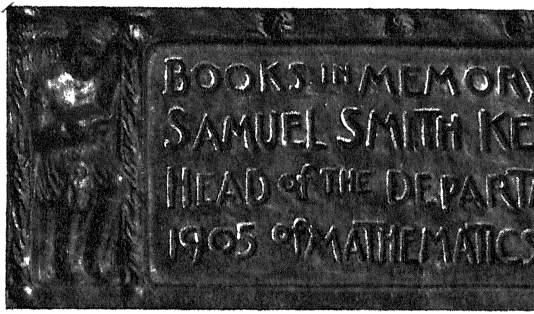
BOOKS IN MEMORY
SAMUEL SMITH KE
HEAD of THE DEPART
1905 of MATHEMATICS

LECTURES

ON

THE THEORY OF FUNCTIO[NS OF]

REAL VARIABLES

VOLUME II

BY

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PREFACE

THE present volume has been writ animated the first. The author has treatise or a manual; he has aimed versity lectures with necessary mod freedom in the choice of subjects and tion allowable in a lecture room may p to a larger audience.

A distinctive feature of these Lectu the theory of functions with referen definition. The first functions to combinations of the elementary functi paper of 1854, "Ueber die Darstellba eine trigonometrische Reihe," was th functions whose singularities ceased searches of later mathematicians have of such functions, whose existence so l tionized the older notion of a functio creation of finer tools of research. was paid to the singular character of none was accorded to the domain ove defined. After the epoch-making dis by G. Cantor in the theory of point sary to consider a function of one va terval, a function of two variables as

prevails in the theory of determinants. One ma
only two and three rowed determinants, but h
ground of complaint if another prefers to state l
demonstrations for general n. On the other ha
case may present unexpected and serious problems
Jordan has introduced the notion of functions of
having limited variation. How is this notion to
two or more variables? An answer is far from si
given by the author in Volume I; its serviceal
been shown by B. Camp. Another has been essay
The reader must be warned, however, against e
the development always extended to the gener
in the first place, would be quite impracticable
increasing the size of the present work. Secondl
be quite beyond the author's ability.

Another feature of the present work to which t
call attention is the novel theory of integratio
Chapter XVI of Volume I and Chapters I and I
It rests on the notion of a cell and the division of
any set, into unmixed partial sets. The definit
multiple integrals leads to results more general i
than yet obtained with Riemann integrals.

Still another feature is a new presentation o
measure. The demonstrations which the author
much to be desired in the way of completeness, r
In attempting to find a general and rigorous tr
at last led to adopt the form given in Chapter XI

The author also claims as original the theo
integrals developed in Chapter XII. Lebesgue h
functions such that the points e at which $a \leq f(x$

be divided into a finite number of metric sets
then

$$\underline{\int}_{\mathfrak{A}} f = \text{Max } \Sigma m_\iota \delta_\iota \quad , \quad \overline{\int}_{\mathfrak{A}} f = \text{Min}$$

where m_ι, M_ι are the minimum and maximum
then is more natural than to ask what will
δ_1, $\delta_2 \cdots$ are infinite instead of finite in nu
apparently trivial question results a theory o
contains the Lebesgue integrals as a specia
furthermore, has the great advantage that not
of the new integrals to the ordinary or Ri
perfectly obvious, but also the form of reas
Riemann's theory may be taken over to dev
of the new integrals.

Finally the author would call attention t
the area of a curved surface given at the e
Though the above are the main features of
that the experienced reader will discover som
lacking in originality, but not of sufficient
phasize here.

It is now the author's pleasant duty to a
valuable assistance derived from his colleague
Dr. W. A. Wilson. He has read the enti
proof with great care, corrected many errors
the demonstrations, besides contributing the s
373, 401–406, 414–424.

Unstinted praise is also due to the house
pany, who have met the author's wishes with u
and have given the utmost care to the press w

J

NEW HAVEN, December, 1911

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CHAPTER I

POINT SETS AND PROPER INTEGRALS

1. In this short chapter we wish to complete our
proper multiple integrals and give a few theorems
which we shall either need now or in the next chap
take up the important subject of improper multiple i

In Volume I, 702, we have said that a limited poi
upper and lower contents are the same is measurabl
best to reserve this term for another notion which h
great prominence of late. We shall therefore call th
sets whose upper and lower contents are equal, *metric*
a set $\mathfrak{A}$ is metric, either symbol

$$\overline{\mathfrak{A}} \quad \text{or} \quad \underline{\mathfrak{A}}$$

expresses its content. In the following it will b

given requires each cell to be metr
is not necessary; it suffices that the
sion of the given set $\mathfrak{A}$. Such divisi
visions of norm δ. [I, 711.] Under
now theorems analogous to I, 714, 7

2. *Let* $\mathfrak{B}$ *contain the limited point*
mixed division of $\mathfrak{B}$ *of norm* δ. *Let*
taining points of $\mathfrak{A}$. *Then*

$$\lim_{\delta=0} \mathfrak{A}_\delta = \mathfrak{}$$

The proof is entirely analogous to

3. *Let* $\mathfrak{B}$ *contain the limited point*
limited in $\mathfrak{A}$. *Let* Δ *be an unmixed*
cells $\delta_1, \delta_2, \cdots$. *Let* M_ι, m_ι *be respect*
mum of f in δ_ι. *Then*

$$\lim_{\delta=0} S_\Delta = \lim_{\delta=0} \Sigma M_\iota \delta$$

$$\lim_{\delta=0} S_\Delta = \lim_{\delta=0} \Sigma m_\iota \delta_\iota$$

Let us prove 1); the relation 2) may
manner. In the first place we show
gous to I, 722, that

$$S_\Delta < \int_{\mathfrak{A}} f d\mathfrak{A} + \epsilon,$$

The cells of E containing points of $\mathfrak{A}$ fall int
1° the cells $e_{\iota\kappa}$ containing points of the cell δ_ι but o
of Δ; 2° the cells e'_ι containing points of two or m
Thus we have

$$S_E = \Sigma M_{\iota\kappa} e_{\iota\kappa} + \Sigma M'_\iota e'_\iota,$$

where $M_{\iota\kappa}$, M'_ι are the maxima of f in $e_{\iota\kappa}$, e'_ι. Th
have

$$S_E < \Sigma \mathfrak{M}_\iota e_{\iota\kappa} + \frac{\epsilon}{4}, \qquad e < e_0,$$

if e_0 is taken sufficiently small.

On the other hand, we have

$$\left| S_\Delta - \Sigma \mathfrak{M}_\iota e_{\iota\kappa} \right| < F \left| \sum_\iota \delta_\iota - \sum_\iota e_{\iota\kappa} \right|.$$

Now we may suppose δ_0, e_0 are taken so small th

$$\Sigma \delta_\iota, \quad \Sigma e_{\iota\kappa}$$

differ from $\mathfrak{A}$ by as little as we choose. We hav
properly chosen δ_0, e_0,

$$S_\Delta > \Sigma \mathfrak{M}_\iota e_{\iota\kappa} - \frac{\epsilon}{4}.$$

This with 6) gives

$$S_\Delta > S_E - \frac{\epsilon}{2},$$

which with 5) proves 4).

4. *Let $f(x_1 \cdots x_m)$ be limited in the limited fiel
an unmixed division of $\mathfrak{A}$ of norm δ, into cells δ_1, δ_2*

has both signs in $\mathfrak{A}$. For exa
whose center call C. Let us eff
equal squares and let $\mathfrak{A}$ be forme
and of C. Let us define f as foll

$$f = 1 \text{ withi}$$
$$= -100 \text{ a}$$

For the division E,

$$S_E = -1 + $$

Hence, $\quad \text{Min } S_D <$

On the other hand, $\quad \lim_{d=0} S_D =$

The theorems I, 723, and its an
for unmixed divisions of space.
be unmixed divisions of the field
so, is shown by the example just

6. In certain cases the field $\mathfrak{A}$
In such a case we define $\quad \int_{\mathfrak{A}} f =$

7. From 4 we have at once:

Let Δ be an unmixed division of

$$\mathfrak{A} = \text{Mi}$$

with respect to the class of all divisi

with respect to the class of rectangular division of
But the class E is a subclass of the class D.

Thus

$$\underset{\Delta}{\text{Min}}\, \Sigma\delta_\iota \leq \underset{D}{\text{Min}}\, \Sigma d_\iota \leq \underset{E}{\text{Min}}\, \Sigma e_\iota .$$

Here the two end terms have the value $\mathfrak{A}$.

3. Let $f(x_1 \cdots x_m)$, $g(x_1 \cdots x_m)$ be limited in the
We have then the following theorems:

1. *Let $f = g$ in $\mathfrak{A}$ except possibly at the points of a*
Then,

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{A}} g .$$

For let $|f|, |g| < M$. Let D be a cubical divis
Let M_ι, N_ι denote the maximum of f, g in the cell
note the cells containing points of $\mathfrak{D}$, while A n
other cells of $\mathfrak{A}_D$.

Then,

$$\sum_{\mathfrak{A}} M_\iota d_\iota = \sum_{A} M_\iota d_\iota + \sum_{\Delta} M_\iota d_\iota ;$$

$$\sum_{\mathfrak{A}} N_\iota d_\iota = \sum_{A} M_\iota d_\iota + \sum_{\Delta} N_\iota d_\iota .$$

Hence,

$$\left| \sum_{\mathfrak{A}} M_\iota d_\iota - \sum_{\mathfrak{A}} N_\iota d_\iota \right| \leq \sum_{\Delta} |M_\iota - N_\iota| d_\iota \leq 2M$$

and the term on the right $\doteq 0$ as $d \doteq 0$.

2. *Let $f = g$ in $\mathfrak{A}$ except possibly at the points of a*
Then

3. *If* $c > 0$,
$$\overline{\int_{\mathfrak{A}}} cf = c \overline{\int_{\mathfrak{A}}} f;$$

but if $c < 0$,
$$\overline{\int_{\mathfrak{A}}} cf = c \underline{\int_{\mathfrak{A}}} f; \quad \underline{\int_{\mathfrak{A}}} cf = c$$

For in any cell d_ι
$$\mathrm{Max} \cdot cf = c \,\mathrm{Max}\, f; \qquad \mathrm{Min} \cdot c$$

when $c > 0$; while
$$\mathrm{Max} \cdot cf = c \,\mathrm{Min}\, f; \qquad \mathrm{Min} \cdot c$$
when $c < 0$.

4. *If* g *is integrable in* $\mathfrak{A}$,
$$\int_{\mathfrak{A}} (f + g) = \int_{\mathfrak{A}} f + \int_{\mathfrak{A}}$$

For from
$$\mathrm{Max}\, f + \mathrm{Min}\, g \leq \mathrm{Max}\, (f + g) \leq \mathrm{M}$$

we have
$$\int_{\mathfrak{A}} f + \int_{\mathfrak{A}} g \leq \overline{\int_{\mathfrak{A}}} (f + g) \leq \int_{\mathfrak{A}}$$

But g being integrable,
$$\underline{\int_{\mathfrak{A}}} g = \overline{\int_{\mathfrak{A}}} g.$$

Hence 2) gives

6. *Let* $f = g + h$, $|h| \le H$ *a constant, in* $\mathfrak{A}$. *Then*

$$\left| \overline{\int_{\mathfrak{A}}} f - \overline{\int_{\mathfrak{A}}} g \right| \le H\mathfrak{A}.$$

For
$$-H + g \le f \le g + H.$$

Then by 2 and 4

$$-\overline{\int_{\mathfrak{A}}} H + \overline{\int_{\mathfrak{A}}} g \le \overline{\int_{\mathfrak{A}}} f \le \overline{\int_{\mathfrak{A}}} g + \overline{\int_{\mathfrak{A}}} H,$$

or
$$-H\mathfrak{A} + \overline{\int_{\mathfrak{A}}} g \le \overline{\int_{\mathfrak{A}}} f \le \overline{\int_{\mathfrak{A}}} g + H\mathfrak{A}.$$

4. *Let* $f(x_1 \cdots x_m)$ *be limited in limited* $\mathfrak{A}$. *Then,*

$$\left| \overline{\int_{\mathfrak{A}}} f \right| \le \overline{\int_{\mathfrak{A}}} |f|,$$

$$\underline{\int_{\mathfrak{A}}} f \le \overline{\int_{\mathfrak{A}}} f,$$

$$\overline{\int_{\mathfrak{A}}} f \le \overline{\int_{\mathfrak{A}}} f + \overline{\int_{\mathfrak{A}}} f,$$

$$\underline{\int_{\mathfrak{A}}} f \ge \underline{\int_{\mathfrak{A}}} f + \underline{\int_{\mathfrak{A}}} f.$$

If $|f| \le M$, *we have also,*

$$\overline{\int_{\mathfrak{A}}} f \le M\mathfrak{A}.$$

To prove 3), we use the rel

$$-|f|$$

Hence $$\int_{\mathfrak{A}} -|f| \cdot$$

from which 3) follows on usin

The demonstration of 4) is s

To prove 5), we observe tha

$$\Sigma M_i d$$

5. 1. *Let $f \geq 0$ be limited i*
the aggregate formed of the poin

$$\int_{\mathfrak{A}} f$$

This is obvious since the sum

$$\sum_{\mathfrak{B}} M_i d_i$$

may have terms in common.
twice on the right of 1) and o
to the limit.

Remark. The relation 1) m

Example. Let $\mathfrak{A} = (0, 1)$,
tional points in $\mathfrak{A}$. Let $f = 1$ i

For $$\overline{\int}_{\mathfrak{B}} g = \overline{\int}_{\mathfrak{A}} g + \overline{\int}_{\mathfrak{C}} g, \qquad \text{by I}$$

But $$\overline{\int}_{\mathfrak{A}} g = \overline{\int}_{\mathfrak{A}} f, \qquad \text{by 3, 4,}$$

and obviously $$\overline{\int}_{\mathfrak{C}} g = 0.$$

3. The reader should note that the above theo
true if $\mathfrak{A}$ is not an unmixed part of $\mathfrak{B}$.

Example. Let $\mathfrak{A}$ denote the rational points in
$\mathfrak{B}$.

Let $$f = g = 1 \quad \text{in } \mathfrak{A}.$$

Then $$\int_{\mathfrak{A}} f = 1, \qquad \overline{\int}_{\mathfrak{B}} g = 1$$

4. *Let $\mathfrak{A}$ be a part of the limited field $\mathfrak{B}$. Let f*
$\mathfrak{A}$. *Let $g = f$ in $\mathfrak{A}$ and $= 0$ in $\mathfrak{C} = \mathfrak{B} - \mathfrak{A}$. Then*

$$\overline{\int}_{\mathfrak{A}} f = \overline{\int}_{\mathfrak{B}} g;$$

$$\underline{\int}_{\mathfrak{A}} f = \underline{\int}_{\mathfrak{B}} g.$$

For let M_ι, N_ι be the maxima of f, g in the cell

$$\Sigma N_\iota d_\iota = \Sigma N_\iota d_\iota + \Sigma N_\iota d_\iota.$$

For let $|f| \le M$ in $\mathfrak{A}$. Let $\mathfrak{C}$

$$\int_{\mathfrak{A}} f =$$

But

$$\left| \int_{\mathfrak{C}_u} f \right.$$

Hence passing to the limit u

2. We note that 1 may be in
For let $\mathfrak{A}$ be the unit square.
concentric square whose side is
points of $\mathfrak{A}$ and $= 2$ for the othe

$$\int_{\mathfrak{A}} f = 2,$$

7. In I, 716 we have given
when each $\mathfrak{B}_u < \mathfrak{A}$. A similar t
viz.:

Let $\mathfrak{B}_u \le \mathfrak{B}_{u'}$ if $u < u'$. Let $\mathfrak{A}$
$\mathfrak{A}$ as $u \doteq 0$. Then for each $\epsilon > 0$,

$$\mathfrak{B}_{u,D} - \mathfrak{A}_D, \quad \mathfrak{B}_{u,D} \quad \mathfrak{A}$$

For $$\mathfrak{B}_{u_0} < \mathfrak{A} + \frac{\epsilon}{2}, \qquad u_0$$

Also for any division D

belonging to a given point b of $\mathfrak{B}$, we denote b
by $\mathfrak{C}$. We write

$$\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C},$$

and call $\mathfrak{B}$, $\mathfrak{C}$ *components* of $\mathfrak{A}$.

We note that the fundamental relations of I

$$\underline{\int}_{\mathfrak{A}} f \leq \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} f \leq \overline{\int}_{\mathfrak{A}} f$$

hold not only for the components $\mathfrak{x}$, $\mathfrak{P}$, etc.,
also for the general components $\mathfrak{A}$, $\mathfrak{B}$.

In what follows we shall often give a proof
for the sake of clearness, but in such cases th
admit an easy generalization. In such cases
the x-projection or component of $\mathfrak{A}$.

2. *If* $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$ *is limited and* $\mathfrak{B}$ *is discrete,*

For let $\mathfrak{A}$ lie within a cube of edge $\frac{1}{2}C >$
space. Then for any $d <$ some d_0,

$$\overline{\mathfrak{B}}_D < \frac{\epsilon}{C^s}.$$

Then $$\overline{\mathfrak{A}}_D < C^s \overline{\mathfrak{B}}_D < \epsilon.$$

3. That the converse of 2 is not necessar
the two following examples, which we shall u

Example 1. Let $\mathfrak{A}$ denote the points x

For
$$x = \frac{m}{n}, \qquad m, n \text{ relative}$$
let
$$0 \le y \le \frac{1}{n}.$$

Then, $\mathfrak{B}$ denoting the projection of $\mathfrak{A}$
$$\overline{\mathfrak{A}} = 0, \qquad \mathfrak{B} = 1.$$

9. 1. *Let* $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$ *be a limited point*
$$\mathfrak{A} \le \int_{\mathfrak{B}} \mathfrak{C} \le \mathfrak{A}.$$

For let $f = 1$ in $\mathfrak{A}$. Let $g = 1$ at eac
other points of a cube $A = B \cdot C$ containi
$$\mathfrak{A} = \int_A g, \qquad \mathfrak{A} = \int$$

By I, 733,
$$\int_A g = \int_B \int_C g = \int$$

But by 5, 4,
$$\int_B \int_C g = \int_{\mathfrak{B}} \int_{\mathfrak{C}} f.$$
$$\int_B \int_C g = \int_{\mathfrak{B}} \int_{\mathfrak{C}} f.$$

Thus
$$\mathfrak{A} = \int_A g \le \int \int f \le \int \int f \le \int$$

3. In this connection we should note, however, that
of 2 is not always true, *i.e.* if $\mathfrak{C}$ is integrable, then $\mathfrak{A}$
and 2, 2) holds. This is shown by the following:

Example. In the unit square we define the points x,

For rational x, $$0 \le y \le \tfrac{1}{2}.$$

For irrational x, $$\tfrac{1}{2} < y \le 1.$$

Then $\mathfrak{C} = \frac{1}{2}$ for every x in $\mathfrak{B}$. Hence

$$\int_{\mathfrak{B}} \mathfrak{C} = \tfrac{1}{2}.$$

But $$\mathfrak{A} = 0, \qquad \mathfrak{A} = 1.$$

10. 1. *Let $f(x_1 \cdots x_m)$ be limited in the limited field*

If $f \ge 0$, $$\int_{\mathfrak{B}} \int_{\mathfrak{C}} f \le \int_{\mathfrak{A}} f.$$

If $f \le 0$, $$\int_{\mathfrak{A}} f \le \int_{\mathfrak{B}} \int_{\mathfrak{C}} f.$$

Let us first prove 1). Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ lie in the spac
$r + s = m$. Then any cubical division D divides the
cubical cells d, d', d'' of volumes d, d', d'' respe
viously $d = d'd''$. D also divides $\mathfrak{B}$ and each $\mathfrak{C}$ into

2. To illustrate the necessity
$\mathfrak{A}$ to be the Pringsheim set of I
Then

$$\int_{\mathfrak{A}}$$

On the other hand $$\int_{\mathfrak{C}}$$

Hence $$\int_{\mathfrak{B}}\int_{\mathfrak{C}}$$

and the relation 1) does not hol

Iterabl

11. 1. There is a large class o
have content and yet

$$\mathfrak{A}$$

Any limited point set satisfyin
or more specifically *iterable with r*

Example 1. Let $\mathfrak{A}$ consist of
square. Obviously

$$\mathfrak{A} \quad \int_{\mathfrak{A}} \mathfrak{C}$$

so that $\mathfrak{A}$ is iterable both with res

Example 3. Let $\mathfrak{A}$ consist of the points in the un
fined thus:

For rational x let $0 \leq y \leq \frac{1}{2}$.
For irrational x let $\frac{1}{2} \leq y \leq 1$.

Here $\mathfrak{A} = 1$, while

$$\int_{\mathfrak{A}}\mathfrak{C} = \frac{1}{2}; \qquad \int_{\mathfrak{C}}\mathfrak{B} = 1.$$

Hence $\mathfrak{A}$ is iterable with respect to $\mathfrak{C}$ but not with

Example 4. Let $\mathfrak{A}$ consist of the sides of the un
the rational points within the square.

Here $\mathfrak{A} = 1$, while

$$\int_{\mathfrak{A}}\mathfrak{C} = 0, \qquad \int_{\mathfrak{A}}\mathfrak{C} = 1,$$

and similar relations for $\mathfrak{C}$. Thus $\mathfrak{A}$ is not iterable
to either $\mathfrak{B}$ or $\mathfrak{C}$.

Example 5. Let $\mathfrak{A}$ be the Pringsheim set of I, 71

Here $\mathfrak{A} = 1$, while

$$\int_{\mathfrak{A}}\mathfrak{C} = 0, \qquad \int_{\mathfrak{C}}\mathfrak{B} = 0.$$

Hence $\mathfrak{A}$ is not iterable with respect to either $\mathfrak{B}$ o

2. *Every limited metric point set is iterable with re
its projections.*

This follows at once from the definition and 3, 5

13. 1. Let $\mathfrak{A}$ be a limited p
space $\Re_m$. Let $\mathfrak{B}$, $\mathfrak{C}$ be com
A cubical division D of norm
d and $\Re_r$ and $\Re_s$ into cells of
b be any point of $\mathfrak{B}$, lying in a
of all the cells d_s, containing
Let $\sum\limits_{dr} d_s$ denote the sum of all
whose projection falls in d_r, not

We have now the following t

If $\mathfrak{A}$ *is iterable with respect to*

$$\lim_{\delta=0} \sum_{\mathfrak{B}} d_r \{ \sum_{d_r}$$

For

$$\mathfrak{C}_b \cdot \sum$$

Hence

$$\sum_{\mathfrak{B}} d_r \mathfrak{C}_b \cdot \sum_{\mathfrak{B}}$$

Let now $\delta \doteq 0$. The first an
since $\mathfrak{A}$ is iterable. Thus, the
the same limit, and this gives 1

2. *If* $\mathfrak{A}$ *is iterable with respect*

$$\lim_{\delta=0} \sum_{\mathfrak{B}}$$

This follows at once from 1).

Let S denote, as in 13, 1, the cells of $\sum_{d_r} d_s$ which c

and F the cells containing points of both $\mathfrak{C}$, $\mathfrak{C}'$

fall in d_r. Then from

$$C'_{b_1} + C''_{b_1} \le C'_{b_1} + S \le C'_{b_2} + S \le \sum_{d_r} d_s$$

we have

$$\mathfrak{C}_{b_1} \le C'_{b_1} + C''_{b_1} \le \mathrm{Min}\, C'_b + \beta' + S \le \mathrm{Max}\, C'_b + \beta$$

Multiplying by d_r and summing over $\mathfrak{B}$ we h

$$\sum d_r \mathfrak{C}_{b_1} \le \sum d_r \mathrm{Min}\, C'_b + \sum \beta' d_r + \mathfrak{C}'_D \le \sum d_r \mathrm{Max}$$

$$\le \mathfrak{A}_D + \sum d_r$$

Passing to the limit, we have

$$\mathfrak{A} \le \int_{\mathfrak{B}} C' + \eta' + \mathfrak{C}' \le \int_{\mathfrak{B}} C' + \eta''$$

the limit of the last term vanishing since $\mathfrak{C}$, $\mathfrak{C}'$

of $\mathfrak{A}$. Here η', η'' are as small as we please on t

small. From 2) we now have

$$\int_{\mathfrak{B}} C' = \mathfrak{A} - \mathfrak{C}' - \mathfrak{C}.$$

4. *Let $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$ be iterable with respect to $\mathfrak{B}$ of $\mathfrak{B}$ and A all those points of $\mathfrak{A}$ whose projection A is iterable with respect to B.*

For let D be a cubical division of space of m

As each of the braces is ≥ 0 w

A

14. We can now generalize t
733 as follows:

Let $f(x_1 \cdots x_m)$ *be limited in t*
with respect to $\mathfrak{B}$. *Then*

$$\int_{\mathfrak{A}} f \cdot \int_{\mathfrak{B}}$$

For let us choose the positive c

$$f + A > 0, \quad f$$

Let us effect a cubical division
cells d. As in 13, this divides
as well as their contents, by d_e.
As usual let m, M denote the m
cell d containing a point of $\mathfrak{A}$.
extremes of f when we consider
projection is b. Let $|f| < F$ in
Then for any b, we have by 1

$$\Sigma(m' - B)d_e \cdot \int_{\mathfrak{C}}$$

or

$$- B(\Sigma d, \quad \mathfrak{C}$$

since $m \geq m'$.

In a similar manner

where $|\beta_1|, |\beta_2| < \beta$ and $\mathfrak{C}_1$, and $\mathfrak{C}_2$ stand for $\mathfrak{C}_{b_1}$, …
where

$$j = \mathrm{Min}\int_{\mathfrak{C}} f, \qquad J = \mathrm{Max}\int_{\mathfrak{C}} f$$

for all points b in d_r.

Let $c = \mathrm{Min}\,\mathfrak{C}$ in d_r, then 4) gives

$$-B(\sum_1 d_s - c) + \sum_1 m d_s < j + \beta_1 < J + \beta_2 < \sum_2 M d_s + \ldots$$

where the indices 1, 2 indicate that in $\sum_k$ we have …
b_1, b_2.

Multiplying by d_r and summing over all the cell…
points of $\mathfrak{B}$, the last relation gives

$$-B\sum_{\mathfrak{B}} d_r(\sum_1 d_s - c) + \sum_{\mathfrak{B}} d_r \sum_1 m d_s < \sum_{\mathfrak{B}} j d_r + \ldots$$

$$< \sum_{\mathfrak{B}} J d_r + \sum_{\mathfrak{B}} \beta_2 d_r < \sum_{\mathfrak{B}} d_r \sum_2 M d_s + A\sum_{\mathfrak{B}} d_r(\sum_2 d_s \ldots$$

Now as $\delta \doteq 0$, $\quad \sum_{\mathfrak{B}} d_r \sum_1 d_s \doteq \mathfrak{A}, \qquad \sum_{\mathfrak{B}} d_r \sum_2 d_s \doteq \mathfrak{A},$

$$\sum_{\mathfrak{B}} d_r c \doteq \int \mathfrak{C} = \mathfrak{A}, \qquad \text{since } \mathfrak{A} \ldots$$

Thus the first and last sums in 5) are evanescent…
the other hand

2. If $\mathfrak{A}$ is not iterable wit
ing the discrete set $\mathfrak{D}$. Let
ponents B, C. Then 1 give

$$\underline{\int}_{\mathfrak{A}} f <$$

since

3. The reader should gu
only $\mathfrak{A}$ is iterable on remo
the following:

Example. Let the points
Let $\mathfrak{A}_1$ consist of all the p
Let $\mathfrak{D}$ lie on the rational or

$$x = \frac{m}{n},$$

$$0 < y$$

Let us define f over $\mathfrak{A}$ th

$$f =$$

$$f =$$

The relation 1) is false in

while

$$\int_{\mathfrak{B}}$$

15. 1. Let $f(x_1 \cdots x_m)$
Let D denote the rectangul
of $\mathfrak{A}$, except possibly thos

To show that 1) exist we need onl
there exists a d_0 such that for any re
norms $< d_0$

$$\Delta = \left| \int_{\mathfrak{A}_{D'}} - \int_{\mathfrak{A}}\right.$$

To this end, we denote by E the di
ing D'' on D'. Then E is a rectangu
Let

$$\mathfrak{A}_E - \mathfrak{A}_{D'} = A', \qquad \mathfrak{A}_{\ldots}$$

If d_0 is sufficiently small, A', A''

an arbitrarily small positive number.

$$\Delta = \left| \left(\int_{\mathfrak{A}_D} - \int_{\mathfrak{A}_E} \right) - \left(\int_{\mathfrak{A}_{D''}} - \right.\right.$$

if η is taken small enough.

2. The integrals

$$\int_{\mathfrak{A}} f, \qquad$$

heretofore considered may be called
tegrals, in contradistinction.

3. *Let f be limited in the limited*
and outer lower (upper) integrals are

For $\mathfrak{A}_D$ is an unmixed part of $\mathfrak{A}$ su

$$\text{Cont } \mathfrak{A}_D \doteq \mathfrak{A},$$

Then by 6, 1,

$$\lim_{d=0} \int_{\mathfrak{A}_D} f =$$

For each $\underline{\mathfrak{A}}_D = 0$, and

16. *Let* $\mathfrak{A} = \mathfrak{B} + \mathfrak{C}$ *b*

For let D be a cubic
of $\overline{\mathfrak{A}}_D$ fall into three cla
these form $\underline{\mathfrak{B}}_D$. 2°, c
3°, cells containing fro
or 2°. Call these $\mathfrak{f}_D$.

Let now $d \doteq 0$. A
Front $\mathfrak{A}$ and this is dis

17. 1. Let

be point sets, limited
The aggregate formed
sets 1) is called their

or more shortly by

If $\mathfrak{A}$ is a general sy
may also be denoted b

or even more briefly b

or by

$$Dv\{\mathfrak{A}\},$$

if $\mathfrak{A}$ is a general symbol as before.

2. *Examples.*

Let $\mathfrak{A}$ be the interval $(0, 2)$; $\mathfrak{B}$ the interval $(1, \infty)$

$$U(\mathfrak{A}, \mathfrak{B}) = (0, \infty), \qquad Dv(\mathfrak{A}, \mathfrak{B}) = (1, 2).$$

Let

$$\mathfrak{A}_1 = (0, 1), \qquad \mathfrak{A}_2 = (1, 2) \cdots$$

Then

$$U(\mathfrak{A}_1, \mathfrak{A}_2 \cdots) = (0, \infty),$$

$$Dv(\mathfrak{A}_1, \mathfrak{A}_2 \cdots) = 0.$$

Let

$$\mathfrak{A}_1 = (\tfrac{1}{2}, 1), \qquad \mathfrak{A}_2 = (\tfrac{1}{3}, \tfrac{1}{2}), \qquad \mathfrak{A}_3 = (\tfrac{1}{4}, \tfrac{1}{3}$$

Then

$$U(\mathfrak{A}_1, \mathfrak{A}_2 \cdots) = (0^*, 1),$$

$$Dv(\mathfrak{A}_1, \mathfrak{A}_2 \cdots) = 0.$$

Let

$$\mathfrak{A}_1 = (\tfrac{1}{2}, 1\tfrac{1}{2}), \qquad \mathfrak{A}_2 = (\tfrac{1}{4}, 1\tfrac{3}{4}), \qquad \mathfrak{A}_3 = (\tfrac{1}{8}, 1$$

Then

$$U(\mathfrak{A}_1, \mathfrak{A}_2 \cdots) = (0^*, 2^*),$$

$$Dv(\mathfrak{A}_1, \mathfrak{A}_2 \cdots) = \mathfrak{A}_1.$$

3. Let

$$\mathfrak{A} \geq \mathfrak{A}_1 \geq \mathfrak{A}_2 \geq \mathfrak{A}_3 \geq \cdots$$

Let

$$\mathfrak{D} = Dv(\mathfrak{A}, \mathfrak{A}_1, \mathfrak{A}_2, \cdots).$$

Let

$$\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{C}_1, \ \mathfrak{A}_1 = \mathfrak{A}_2 + \mathfrak{C}_2, \cdots$$

Then

$$\mathfrak{A} = \mathfrak{D} + \mathfrak{C}_1 + \mathfrak{C}_2 + \cdots$$

Let us first exclude the = sign in 1). Then every

Example. Let $\mathfrak{A} =$

$\mathfrak{B} = ($

$\mathfrak{C} = ($

Then U

and $(U$

are different.

Thus if we write $+$ for

$\mathfrak{A} + (\mathfrak{B} -$

18. 1. *Let* $\mathfrak{A}_1 \geq \mathfrak{A}_2 \geq \mathfrak{A}_3$
aggregates. Then

$\mathfrak{B} = Dv$

Moreover $\mathfrak{B}$ *is complete.*

Let a_n be a point of $\mathfrak{A}_n$, n
Any limiting point α of
ing point of

$a_m,$

But all these points lie in $\mathfrak{A}$
$\mathfrak{A}_m$, and therefore in every
$\mathfrak{B} > 0$.

$\mathfrak{B}$ *is complete.* For let β

b_1, b

As each b_m is in each $\mathfrak{A}_n$, and
is in $\mathfrak{B}$.

For each point b of $\mathfrak{B}$ lies in some $\mathfrak{A}_n$, or it lies in
hence in every A_n. In the first case b lies in $\mathfrak{A}$, in th
A. Moreover it cannot lie in both A and $\mathfrak{A}$.

20. 1. Let

$$\mathfrak{A}_1 < \mathfrak{A}_2 < \mathfrak{A}_3 \cdots$$

be an infinite sequence of point sets whose union ca
fact may be more briefly indicated by the notation

$$\mathfrak{A} = U(\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3 \cdots).$$

Obviously when $\mathfrak{A}$ is limited,

$$\overline{\mathfrak{A}} \geq \lim \overline{\mathfrak{A}}_n.$$

That the inequality may hold as well as the equal
shown by the following examples.

Example 1. Let $\mathfrak{A}_n =$ the segment $\left(\frac{1}{n}, 1\right)$.

Then

$$\mathfrak{A} = U\{\mathfrak{A}_n\} = (0^*, 1).$$

$$\overline{\mathfrak{A}} = 1, \qquad \overline{\mathfrak{A}}_n = \frac{n-1}{n} \doteq 1.$$

Example 2. Let $\mathfrak{a}_n$ denote the points in the unit int
abscissæ are given by

$$x = \frac{m}{n}, \; m < n = 1, 2, 3, \cdots m, n \text{ relatively prim}$$

Let

$$\mathfrak{A}_n = \mathfrak{a}_1 + \cdots + \mathfrak{a}_n.$$

Here

$$\mathfrak{A} = U\{\mathfrak{A}_n\}$$

is the totality of rational numbers in $(0^*, 1^*)$.

As

Example 1. Let $\mathfrak{B}_n$ = the segment $\left(0, \frac{1}{n}\right)$.

Then $$\mathfrak{B} = Dv\{\mathfrak{B}_n\} = (0), \text{ the origin.}$$

Here $$\overline{\mathfrak{B}} = 0. \qquad \lim \overline{\mathfrak{B}}_n = \lim \frac{1}{n} = 0,$$

and $$\overline{\mathfrak{B}} = \lim \overline{\mathfrak{B}}_n.$$

Example 2. Let $\mathfrak{A}_n$ be as in 1, Example 2. Let

Let $$\mathfrak{B}_n = (1, 2) + \mathfrak{b}_n.$$

Here $$\mathfrak{B} = \text{the segment } (1, 2) \text{ and } \overline{\mathfrak{B}}_n = 2.$$

Hence $$\overline{\mathfrak{B}} < \lim \overline{\mathfrak{B}}_n.$$

3. *Let* $\mathfrak{B}_1 < \mathfrak{B}_2 < \cdots$ *be unmixed parts of* $\mathfrak{A}$. *Let* $\mathfrak{B} = U\{\mathfrak{B}_n\}$. *Then* $\mathfrak{C} = \mathfrak{A} - \mathfrak{B}$ *is discrete.*

For let $\mathfrak{A} = \mathfrak{B}_n + \mathfrak{C}_n$; then $\mathfrak{C}_n$ is an unmixed part of

$$\overline{\mathfrak{A}} = \overline{\mathfrak{B}}_n + \overline{\mathfrak{C}}_n.$$

Passing to the limit $n = \infty$, this gives

$$\lim \overline{\mathfrak{C}}_n = 0.$$

Hence $\mathfrak{C}$ is discrete by 2.

4. We may obviously apply the terms monotone increasing, monotone decreasing sequences, etc. [Cf. I, 108, 211] of the type 1), 3).

21. *Let* $\mathfrak{C} = \mathfrak{A} + \mathfrak{B}$. *If* $\mathfrak{A}, \mathfrak{B}$ *are complete,*

2. 1. *If $\mathfrak{A}$, $\mathfrak{B}$ are complete, so are also*

$$\mathfrak{C} = (\mathfrak{A}, \mathfrak{B}), \qquad \mathfrak{D} = Dv(\mathfrak{A}, \mathfrak{B}).$$

et us first show that $\mathfrak{C}$ is complete. Let c be a limiting p
$\mathfrak{C}$. Let $c_1, c_2, \cdots$ be points of $\mathfrak{C}$ which $\doteq c$. Let us sepa
c_n into two classes, according as they belong to $\mathfrak{A}$, or do
of these classes must embrace an infinite number of po
ch $\doteq c$. As both $\mathfrak{A}$ and $\mathfrak{B}$ are complete, c lies in either
Hence it lies in $\mathfrak{C}$.

o show that $\mathfrak{D}$ is complete. Let $d_1, d_2, \cdots$ be points of $\mathfrak{D}$ w
. As each d_n is in both $\mathfrak{A}$ and $\mathfrak{B}$, their limiting point d i
nd $\mathfrak{B}$, since these are complete. Hence d is in $\mathfrak{D}$.

. *If $\mathfrak{A}$, $\mathfrak{B}$ are metric so are*

$$\mathfrak{C} = (\mathfrak{A}, \mathfrak{B}) \qquad \mathfrak{D} = Dv(\mathfrak{A}, \mathfrak{B}).$$

or the points of Front $\mathfrak{C}$ lie either in Front $\mathfrak{A}$ or in Fron
le the points of Front $\mathfrak{D} \leq$ Front $\mathfrak{A}$ and also $\leq$ Front $\mathfrak{B}$.
nt $\mathfrak{A}$ and Front $\mathfrak{B}$ are discrete since $\mathfrak{A}$, $\mathfrak{B}$ are metric.

3. *Let the complete set $\mathfrak{A}$ have a complete part $\mathfrak{B}$. Then
small $\epsilon > 0$ is taken, there exists a complete set $\mathfrak{C}$ in $\mathfrak{A}$, havin
t in common with $\mathfrak{B}$ such that*

$$\mathfrak{C} \geq \mathfrak{A} - \mathfrak{B}_\epsilon.$$

*reover there exists no complete set $\mathfrak{C}$, having no point in com
$\mathfrak{B}$ such that*

$$\mathfrak{C} = \mathfrak{A} - \mathfrak{B}.$$

But the cells of $\overline{\mathfrak{C}}_D$ may be subdivided, forming a new which does not change the cells of $\mathfrak{B}_D$, so that $\mathfrak{B}_D = \mathfrak{B}_\Delta$

$$\overline{\mathfrak{C}}_\Delta = \overline{\mathfrak{C}} + \epsilon''', \qquad 0 < \epsilon''' < \epsilon$$

Thus 2), 3), 4) give

$$\mathfrak{A} + \epsilon' = \mathfrak{B} + \epsilon'' + \mathfrak{C} + \epsilon'''.$$

or
$$\mathfrak{C} = \mathfrak{A} - \mathfrak{B} - (\epsilon'' + \epsilon''' - \epsilon'),$$
$$> \mathfrak{A} - \mathfrak{B} - \epsilon.$$

24. *Let* $\mathfrak{A}$, $\mathfrak{B}$ *be complete. Let*

$$\mathfrak{U} = (\mathfrak{A}, \mathfrak{B}), \qquad \mathfrak{D} = Dv(\mathfrak{A}, \mathfrak{B}).$$

Then
$$\mathfrak{A} + \mathfrak{B} = \mathfrak{U} + \mathfrak{D}.$$

For let
$$\mathfrak{U} = \mathfrak{A} + A.$$

Then A contains complete sets C, such that

$$C > \mathfrak{U} - \mathfrak{A} - \epsilon,$$

but no complete set such that

$$C > \mathfrak{U} - \mathfrak{A},$$

by 23. On the other hand,

$$\mathfrak{B} = A + \mathfrak{D}.$$

Hence A contains complete sets C, such that

$$C > \mathfrak{B} - \mathfrak{D} - \epsilon,$$

but no complete set such that

For suppose $l - k = \zeta > 0.$

Let $l = \epsilon + \eta, \quad \epsilon, \eta > 0.$

Then by 23 there exists in $\mathfrak{A}_1$ a complete set $\mathfrak{C}_1$, having no common with $\mathfrak{D}$ such that

$$\mathfrak{C}_1 > \mathfrak{A}_1 - \zeta - \epsilon,$$

as $\mathfrak{A}_1 = k$, such that $\mathfrak{C}_1 > \eta.$

Let $\mathfrak{C}_2 = Dv(\mathfrak{A}_2, \mathfrak{C}_1), \quad \mathfrak{U} = (\mathfrak{A}_2, \mathfrak{C}_1).$

Then by 24, $\mathfrak{A}_2 + \mathfrak{C}_1 = \mathfrak{U} + \mathfrak{C}_2.$

Thus
$$\begin{aligned}
\mathfrak{C}_2 &= \mathfrak{A}_2 + \mathfrak{C}_1 - \mathfrak{U}\\
&\geq \mathfrak{A}_2 + \mathfrak{C}_1 - \mathfrak{A}_1\\
&> \mathfrak{A}_2 + (\mathfrak{A}_1 - \zeta - \epsilon) - \mathfrak{A}_1 = \mathfrak{A}_2 - \zeta - \epsilon\\
&= k - \zeta - \epsilon\\
&= \eta.
\end{aligned}$$

Thus $\mathfrak{A}_2$ contains the non vanishing complete set $\mathfrak{C}_2$ having no point in common with $\mathfrak{D}$. In this way we may continue, $\mathfrak{A}_3, \ldots$ contain a non vanishing complete component not which is absurd.

Corollary. *Let* $\mathfrak{A} = (\mathfrak{A}_1, \mathfrak{A}_2, \ldots)$ *be complete. Then* $\mathfrak{A}_n \ldots$

This follows easily from 23, 25.

CHAPTER II

IMPROPER MULTIPLE INTEGRALS

26. Up to the present we have considered only p
integrals. We take up now the case when the integ
is not limited. Such integrals are called improper.
we get the integrals treated in Vol. I, Chapter 14.
application of the theory we are now to develop i
of the order of integration in iterated improper i
treatment of this question given in Vol. I may be
generalized by making use of the properties of imp
integrals.

27. Let $\mathfrak{A}$ be a limited point set in m-way space
point of $\mathfrak{A}$ let $f(x_1 \cdots x_m)$ have a definite value
The points of infinite discontinuity of f which lie
denote by $\mathfrak{J}$. In general $\mathfrak{J}$ is discrete, and this ca
most important. But it is not necessary. We c
singular points.

Example. Let $\mathfrak{A}$ be the unit square. At the
$y = \frac{r}{s}$, these fractions being irreducible, let $f = ns$.
points of $\mathfrak{A}$ let $f = 1$. Here every point of $\mathfrak{A}$ is a p
discontinuity and hence $\mathfrak{J} = \mathfrak{A}$.

Several types of definition of improper integ
proposed. We shall mention only three.

28. *Type I.*

or all possible complete divisions Δ of norm δ, are call
lower and *upper integrals* of f in $\mathfrak{A}$, and are denoted by

$$\underline{\int}_{\mathfrak{A}} f d\mathfrak{A}, \qquad \overline{\int}_{\mathfrak{A}} f d\mathfrak{A},$$

r more shortly by

$$\underline{\int}_{\mathfrak{A}} f, \qquad \overline{\int}_{\mathfrak{A}} f.$$

When the limits 1) are finite, the corresponding integr
are *convergent*. We also say *f admits a lower or an upper in*
tegral in $\mathfrak{A}$. When the two integrals 2) are equal, we s
f is integrable in $\mathfrak{A}$ and denote their common value by

$$\int_{\mathfrak{A}} f d\mathfrak{A} \qquad \text{or by} \qquad \int_{\mathfrak{A}} f$$

We call 3) the *improper integral* of f in $\mathfrak{A}$; we also s
f *admits an improper integral in* $\mathfrak{A}$ and that the integra
convergent.

The definition of an improper integral just given is an e
of that given in Vol. I, Chapter 14. It is the natural d
ment of the idea of an improper integral which goes back
beginnings of the calculus.

It is convenient to speak of the symbols 2) as upper and
integrals, even when the limits 1) are not finite. A similar
applies to the symbol 3).

Let us replace f by $-f$ in one of the symbols 2), 3)
resulting symbol is called the *adjoint* of the latter symbo
We write

$$\int_{\mathfrak{A}} f = \mathrm{Adj} \int_{\mathfrak{A}} f, \text{ etc.}$$

We define now the lower integral as

$$\underline{\int_{\mathfrak{A}}} f = \lim_{\lambda, \mu \to \infty} \underline{\int_{\mathfrak{A}}} f_{\lambda\mu}.$$

A similar definition holds for the upper integral. terms introduced in 28 apply here without change.

This definition of an improper integral is due to *de la Poussin.* It has been employed by him and *E. G. D.* with great success.

30. *Type III.* Let $\alpha, \beta > 0$. Let $\mathfrak{A}_{\alpha\beta}$ denote the p at which

$$-\alpha \le f(x_1 \cdots x_m) \le \beta.$$

We define now

$$\underline{\int_{\mathfrak{A}}} f = \lim_{\alpha, \beta \to \infty} \underline{\int_{\mathfrak{A}_{\alpha\beta}}} f \quad ; \quad \overline{\int_{\mathfrak{A}}} f = \lim_{\alpha, \beta \to \infty} \overline{\int_{\mathfrak{A}_{\alpha\beta}}} f.$$

The other terms introduced in 28 apply here witho This type of definition originated with the author an developed in his lectures.

31. When the points of infinite discontinuity $\mathfrak{J}$ a and the upper integrals are absolutely convergent, all t tions lead to the same result, as we shall show.

When this condition is not satisfied, the results ma different.

Example. Let $\mathfrak{A}$ be the unit square. Let $\mathfrak{A}_1, \mathfrak{A}_2$ den tively the upper and lower halves. At the rational*

$x = \dfrac{m}{n}$, $y = \dfrac{r}{s}$, in $\mathfrak{A}_1$, let $f = ns$. At the other points $\mathfrak{C}$

$f = -2$. In $\mathfrak{A}_2$ let $f = 0$.

1° *Definition.* Here

3° *Definition.* Here $\mathfrak{A}_\beta$ embraces all the points of
finite number of points of $\mathfrak{B}$ for $\alpha = 2$, β arbitrarily la

$$\overline{\int}_{\mathfrak{A}} f = 1, \qquad \underline{\int}_{\mathfrak{A}} f = 1,$$

and thus

$$\int_{\mathfrak{A}} f = 1.$$

32. In the following we shall adopt the third type
as it seems to lead to more general results when trea
portant subject of inversion of the order of integratio
integrals.

We note that if f is limited in $\mathfrak{A}$,

$$\lim_{\alpha,\beta=\infty} \int_{\mathfrak{A}} f = \text{the proper integral} \int_{\mathfrak{A}} f.$$

For α, β being sufficiently large, $\mathfrak{A}_{\alpha\beta} = \mathfrak{A}$.

Also, if $\mathfrak{A}$ is discrete,

$$\overline{\int}_{\mathfrak{A}} f = \underline{\int}_{\mathfrak{A}} f = 0.$$

For $\mathfrak{A}_{\alpha\beta}$ is discrete, and hence

$$\int_{\mathfrak{A}_{\alpha\beta}} f = 0.$$

Hence the limit of these integrals is 0.

33. *Let* $m = \mathrm{Min}\, f$, $M = \mathrm{Max}\, f$ in $\mathfrak{A}$.
Then

$$\lim_{\alpha,\beta=\infty} \int_{\mathfrak{A}_{\alpha\beta}} f = \lim_{\beta=\infty} \int_{\mathfrak{A}_{m,\beta}} f, \quad \textit{if } m \textit{ is finite}.$$

Thus in these cases we may simplify our notation by r

$$\mathfrak{A}_{\alpha, M} \quad , \quad \mathfrak{A}_{m\beta}$$

by

$$\mathfrak{A}_{\alpha} \quad , \quad \mathfrak{A}_{\beta} \quad ,$$

respectively.

2. Thus we have:

$$\int_{\mathfrak{A}} f = \lim_{\beta} \int_{\mathfrak{A}_\beta} f, \qquad \textit{when } \mathrm{Min} f \textit{ is finite.}$$

$$\int_{\mathfrak{A}} f = \lim_{\alpha} \int_{\mathfrak{A}_\alpha} f, \qquad \textit{when } \mathrm{Max} f \textit{ is finite.}$$

3. Sometimes we have to deal with several function
In this case the notation $\mathfrak{A}_{\alpha\beta}$ is ambiguous. To make it
let $\mathfrak{A}_{f, \alpha, \beta}$ denote the points of $\mathfrak{A}$ where

$$-\alpha < f < \beta.$$

Similarly, $\mathfrak{A}_{g, \alpha, \beta}$ denotes the points where

$$-\alpha < g < \beta, \text{ etc.}$$

34. $\int_{\mathfrak{A}_{\alpha\beta}} f$ *is a monotone decreasing function of* α *for each*

$\int_{\mathfrak{A}_{\alpha\beta}} f$ *is a monotone increasing function of* β *for each*

If Max f *is finite*

$\int_{\mathfrak{A}_\alpha} f$ *are monotone decreasing functions of* α.

If Min f *is finite*

$\int_{\mathfrak{A}_\beta} f$ *are monotone increasing functions of* β.

But each cell d_ι of $\mathfrak{A}_{\alpha\beta}$ lies among the cells
can break up the sum 2), getting

$$\sum_{\mathfrak{A}_{\alpha'\beta}} m_\iota' d_\iota' = \sum_{\mathfrak{A}_{\alpha\beta}} m_\iota' d_\iota' + \sum m_\iota'' d_\iota''.$$

Here the second term on the right is su
not containing points of $\mathfrak{A}_{\alpha\beta}$. It is thus $<$
on the right $m_\iota' < m_\iota$. It is thus less than t

$$\sum_{\mathfrak{A}_{\alpha'\beta}} m_\iota' d_\iota' < \sum_{\mathfrak{A}_{\alpha\beta}} m_\iota d_\iota.$$

Thus

$$\underline{\int}_{\mathfrak{A}_{\alpha'\beta}} < \underline{\int}_{\mathfrak{A}_{\alpha\beta}}, \qquad \alpha' > \alpha.$$

In a similar manner we may prove the s
us turn to the third.

We need only to show that

$$\overline{\int}_{\mathfrak{A}_{-\alpha}} f \text{ is monotone decreas}$$

Let $\alpha' > \alpha$. Then

$$\overline{\int}_{\mathfrak{A}_{-\alpha}} = \lim_{d=0} \sum_{\mathfrak{A}_{-\alpha}} M_\iota d_\iota.$$

$$\overline{\int}_{\mathfrak{A}_{-\alpha'}} = \lim_{d=0} \sum_{\mathfrak{A}_{-\alpha'}} M_\iota' d_\iota'.$$

As before

$$\sum_{\mathfrak{A}_{-\alpha'}} M_\iota' d_\iota' = \sum_{\mathfrak{A}_{-\alpha}} M_\iota' d_\iota' + \sum M_\iota''$$

But in the cells d_ι, $M_\iota' = M_\iota$. Hence th
the same as Σ in 3). The second term of 5
follows now as before.

For by 34

$$\overline{\int}_{\mathfrak{A}_n} f \quad , \quad \underline{\int}_{\mathfrak{A}_n} f$$

are limited monotone functions. Their limits exist

36. *If $M = \text{Max} f$ is finite, and $\int_{\mathfrak{A}} f$ is convergent, t*

ing upper integral is convergent and

$$\overline{\int}_{\mathfrak{A}} f \leq \int_{\mathfrak{A}} f + M \lim_{n=\infty} \mathfrak{A}_n,$$

where $f < -n$ in $\mathfrak{A}_n$.

Similarly, if $m = \text{Min} f$ is finite and $\int_{\mathfrak{A}} f$ is converg

sponding lower integral is convergent and

$$m \lim_{n=\infty} \mathfrak{A}_n \leq \underline{\int}_{\mathfrak{A}} f - \int_{\mathfrak{A}} f \quad , \quad f > n \text{ in } \mathfrak{A}_n$$

Let us prove the first half of the theorem.

We have

$$\int_{\mathfrak{A}} f = \lim_{n=\infty} \int_{\mathfrak{A}_n}$$

Now

$$\overline{\int}_{\mathfrak{A}} f \leq \int_{\mathfrak{A}_n} + \int_{\mathfrak{A}_n} \leq \quad M \mathfrak{A}_n.$$

We have now only to pass to the limit.

37. *If $\int_{\mathfrak{A}} f$ is convergent, and $\mathfrak{B} \leq \mathfrak{A}$,*

$$\int_{\mathfrak{B}} f$$

Let $$f = 1 \qquad \text{when } x \text{ is irr}$$

$$= \frac{1}{y} \qquad \text{when } x \text{ is ra}$$

Then $$\int_{\mathfrak{A}_\beta} f = 1 \quad ; \quad \text{hence } \int_{\mathfrak{A}} f$$

On the other hand,

$$\int_{\mathfrak{B}_\beta} = \int_0^1 dx \int_{\frac{1}{\beta}}^1 \frac{dy}{y} = \log \beta$$

Hence $$\int_{\mathfrak{B}} = \lim_{\beta = \infty} \int_{\mathfrak{B}_\beta} = \lim \log \beta$$

is divergent.

38. 1. In the future it will be convenie
points of $\mathfrak{A}$ where $f \geq 0$, and $\mathfrak{N}$ the points w
call them the *positive* and *negative componen*

2. *If* $\int_{\mathfrak{A}} f$ *converges, so do* $\int_{\mathfrak{P}} f$.

If $\int_{\mathfrak{A}} f$ *converges, so do* $\int_{\mathfrak{N}} f$.

For let us effect a cubical division of s
$\beta' > \beta$. Let e denote those cells containi
those cells containing a point of $\mathfrak{P}_{\beta'}$ but n
cells containing a point of $\mathfrak{A}_{\alpha\beta}$ but none of $\mathfrak{P}$

Then

We find similarly

$$\overline{\int}_{\mathfrak{P}_{\beta'}} - \overline{\int}_{\mathfrak{P}_{\beta}} = \lim_{d=0}\{\Sigma(M'_e - M_e)e + \Sigma M'_{e'}e'\}.$$

Now

$$\left|\overline{\int}_{\mathfrak{A}_{\alpha\beta'}} - \overline{\int}_{\mathfrak{A}_{\alpha\beta}}\right| < \epsilon$$

for a sufficiently large α, and for any $\beta, \beta' > \beta_0$.

Hence the same is true of the left side of 1).

As corollaries we have:

3. *If the upper integral of f is convergent in $\mathfrak{A}$, then*

$$\overline{\int}_{P} f \leq \overline{\int}_{\mathfrak{P}} f \qquad P < \mathfrak{P}.$$

If the lower integral of f is convergent in $\mathfrak{A}$,

$$\underline{\int}_{N} f \geq \underline{\int}_{\mathfrak{N}} f \qquad N < \mathfrak{N}.$$

For

$$\overline{\int}_{P_\beta} \leq \overline{\int}_{\mathfrak{P}_\beta} \leq \overline{\int}_{\mathfrak{P}} \qquad \text{etc.}$$

4. *If $f \geq 0$ and $\overline{\int}_{\mathfrak{A}} f$ is convergent, so is*

$$\overline{\int}_{\mathfrak{B}} f \quad , \quad \mathfrak{B} < \mathfrak{A}.$$

Moreover the second integral is $\leq$ the first.

This follows at once from 3, as $\mathfrak{A} = \mathfrak{P}$.

Let D be a cubical division of space of n
denote cells containing at least one point o
$f > 0$. Let $\mathfrak{n}_{\alpha'}$, $\mathfrak{n}_{\alpha''}$ denote cells containing
$\mathfrak{A}_{\alpha''\beta''}$ at which $f < 0$. We have

$$\sum_{\mathfrak{A}_{\alpha'\beta'}} M_\iota d_\iota = \sum_{\mathfrak{P}_{\beta'}} + \sum_{\mathfrak{n}_{\alpha'}} \quad ; \quad \sum_{\mathfrak{A}_{\alpha''\beta''}} M_\iota d_\iota =$$

Subtracting,

$$\left| \sum_{\mathfrak{A}_{\alpha'\beta'}} M_\iota d_\iota - \sum_{\mathfrak{A}_{\alpha''\beta''}} M_\iota d_\iota \right| \leq \left| \sum_{\mathfrak{P}_{\beta'}} M_\iota d_\iota - \sum_{\mathfrak{P}_{\beta''}} M_\iota d_\iota \right| +$$

Let $M'_\iota = \text{Max } f$ for points of $\mathfrak{R}$ in d_ι.
sign in $\mathfrak{R}$,

$$\left| \sum_{\mathfrak{n}_{\alpha'}} M_\iota d_\iota - \sum_{\mathfrak{n}_{\alpha''}} M_\iota d_\iota \right| < \left| \sum_{\mathfrak{R}_{\alpha'}} M'_\iota d_\iota - \right.$$

Letting $d \doteq 0$, 2) and 3) give

$$\left| \int_{\mathfrak{A}_{\alpha'\beta'}} - \int_{\mathfrak{A}_{\alpha''\beta''}} \right| < \left| \int_{\mathfrak{P}_{\beta'}} - \int_{\mathfrak{P}_{\beta''}} \right| + \Big|$$

Now if β is taken sufficiently large, the fir
$< \epsilon/2$. On the other hand, since $\int_{\mathfrak{R}} f$ is co
36. Hence for α sufficiently large, the last
$< \epsilon/2$. Thus 4) gives 1).

40. *If f is integrable in $\mathfrak{A}$, it is in any $\mathfrak{B}$*

Let us first show it is integrable in any $\mathfrak{A}$

Let $$A_{\alpha} = \int - \int .$$

Now any cell d_ι of $\mathfrak{A}_{\alpha\beta}$ is a cell of $\mathfrak{A}_{\alpha'\beta'}$, an
Hence $A_{\alpha'\beta'} > A_{\alpha\beta}$. Thus $A_{\alpha\beta}$ is a monotone inc
of α, β. On the other hand

$$\lim A_{\alpha\beta} = 0,$$

by hypothesis. Hence $A_{\alpha\beta} = 0$ and thus f is inte
Next let f be limited in $\mathfrak{B}$, then $|f| <$ some
$\mathfrak{B} < \mathfrak{A}_{\gamma,\gamma}$. But f being integrable in $\mathfrak{A}_{\gamma,\gamma}$, it is in
Let us now consider the general case. Since f

$$\int_{\mathfrak{B}} f, \quad \int_{\mathfrak{B}} f,$$

both converge by 38. Let now P, N be the poin
in $\mathfrak{B}$. Then

$$\int_P f = \int_{\mathfrak{B}} f, \quad \left|\int_N f\right| = \left|\int_{\mathfrak{B}} f\right|.$$

Thus

$$\int_P f, \quad \int_N f$$

both converge. Hence by 39,

$$\int_{\mathfrak{B}} f$$

both converge. But if $\mathfrak{B}_{a,b}$ denote the points
$-a < f < b$,

$$\int_{\mathfrak{B}} f = \lim_{a,b=\infty} \int_{\mathfrak{B}_{ab}} f,$$

by definition.

But as just seen,

$$\int_{\mathfrak{B}_{ab}} \quad \int_{\mathfrak{B}_{ab}}.$$

Hence

For let

$$\overline{\int_{\mathfrak{A}}} f = \lim_{\alpha,\beta} \overline{\int_{\mathfrak{A}_{\alpha\beta}}} f$$

exist. Since

$$\overline{\int_{\mathfrak{A}_{\alpha\beta}}} = \underline{\int_{\mathfrak{A}_{\alpha\beta}}},$$

necessarily

$$\underline{\int_{\mathfrak{A}}} f = \lim \underline{\int_{\mathfrak{A}_{\alpha\beta}}} f$$

exists and 1), 2) are equal.

42. 1. In studying the function f it is so
introduce two auxiliary functions defined as

$$g = f \qquad \text{where } f >$$
$$= 0 \qquad \text{where } f \leq$$
$$h = -f \qquad \text{where } f <$$
$$= 0 \qquad \text{where } f \geq$$

Thus g, h are both ≥ 0 and

$$f = g - h,$$
$$|f| = g + h.$$

We call them the *associated non-negative f*

2. *As usual let* $\mathfrak{A}_{\alpha\beta}$ *denote the points of* $\mathfrak{A}$
Let $\mathfrak{A}_\beta$ *denote the points where* $g \leq \beta$, *and* $\mathfrak{A}_\alpha$
Then

$$\int_{\mathfrak{A}} g = \lim_{\alpha,\beta} \int_{\mathfrak{A}_{\alpha\beta}} g,$$

$$\int_{\mathfrak{A}} h = \lim_{\alpha,\beta} \int_{\mathfrak{A}_{\alpha\beta}} h.$$

For

Let $\qquad f = 1$ at the irrational points in $\mathfrak{A} = (0, 1)$,

$$= -n, \text{ for } x = \frac{m}{n} \text{ in } \mathfrak{A}.$$

Then

$$\int_{\mathfrak{A}_\eta} g = 0 \quad , \quad \int_{\mathfrak{A}_{a\eta}} g = 1.$$

Again let $f = -1$ for the irrational points in $\mathfrak{A}$,

$$= n \text{ for the rational points } x = \frac{m}{n}.$$

Then

$$\int_{\mathfrak{A}_\eta} h = 0. \qquad \int_{\mathfrak{A}_{a\eta}} h = 1.$$

43. 1.

1) $\int_{\mathfrak{A}} g = \int_{\mathfrak{P}} f$; $\qquad \int_{\mathfrak{A}} g \quad \int_{\mathfrak{P}} f,$

3) $\int_{\mathfrak{A}} h = -\int_{\mathfrak{N}} f$; $\qquad \int_{\mathfrak{A}} h \quad \int_{\mathfrak{N}} f,$

provided the integral on either side of the equations con
provided the integrals on the right side of the inequalities

Let us prove 1); the others are similarly established.
a cubical division of space of norm d, we have for a fixe

$$\int_{\mathfrak{A}_\eta} g = \lim_{d=0} \{ \sum_{\mathfrak{P}_\eta} M_i d_i + \sum 0 \cdot d_i \}$$

$$= \lim_{d=0} \sum_{\mathfrak{P}_\eta} M_i d_i = \int_{\mathfrak{P}_\eta} f.$$

3. If $\overline{\int}_{\mathfrak{A}} f$ is convergent, we cannot say t

vergent. A similar remark holds for the lo

For let $\qquad f = 1$ at the rational points of

$\qquad = -\frac{1}{x}$ at the irrational poin

Then

$$\overline{\int}_{\mathfrak{A}} f = 1 \quad , \quad \underline{\int}_{\mathfrak{A}} f = -\infty$$

4. That the inequality sign in 2) or 4)

shown thus:

Let

$$f = \frac{1}{\sqrt{x}} \text{ for rational } x \text{ in } \mathfrak{A} =$$

$$= -\frac{1}{\sqrt{x}} \text{ for irrational } x.$$

Then

$$\overline{\int}_{\mathfrak{A}} g = 0 \quad , \quad \overline{\int}_{\mathfrak{B}} f = 2.$$

44. 1.

$$\overline{\int}_{\mathfrak{A}} f = \overline{\int}_{\mathfrak{A}} g - \lim_{\alpha,\beta\to\infty} \int_{\mathfrak{A}_{\alpha\beta}} h$$

$$\underline{\int}_{\mathfrak{A}} f = \lim_{\alpha,\beta\to\infty} \underline{\int}_{\mathfrak{A}_{\alpha\beta}} g - \overline{\int}_{\mathfrak{A}} h$$

provided, 1° *the integral on the left exists, or*

limit on the right exist.

For let us effect a cubical division of no

taining points of $\mathfrak{A}$ fall into two classes:

Let now $\alpha, \beta \doteq \infty$. If the integral on the left of 1
gent, the integral on the right of 1) is convergent by 43
the limit on the right of 1) exists. Using now 42, 2,

Let us now look at the 2° hypothesis. By 42, 2,

$$\lim_{\alpha, \beta \to \infty} \int_{\mathfrak{A}_{\alpha\beta}} g = \int_{\mathfrak{A}} g.$$

Thus passing to the limit in 4), we get 1).

2. A relation of the type

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{A}} g + \int_{\mathfrak{A}} h$$

does not always hold as the following shows.

Example. Let $f = n$ at the points $x = \frac{m}{2^{2n}}$

$$= -n \text{ for } x = \frac{m}{2^{2n+1}}$$

$$= -1 \text{ at the other points of } \mathfrak{A}$$

Then $\int_{\mathfrak{A}} f = -1$ $\int_{\mathfrak{A}} g = 0$ $\int_{\mathfrak{A}} h = 0.$

45. *If $\int_{\mathfrak{A}} f$ is convergent, it is in any unmixed part*

Let us consider the upper integral first. By 43, 2,

$$\int_{\mathfrak{A}} g$$

As the limit of the right side exists, that
From this fact, and because 1) exists,

$$\overline{\int}_{\mathfrak{B}} f$$

exists by 44, 1.

A similar demonstration holds for the low

46. *If* $\mathfrak{A}_1, \mathfrak{A}_2 \cdots \mathfrak{A}_m$ *form an unmixed divis*

$$\overline{\int}_{\mathfrak{A}} f = \overline{\int}_{\mathfrak{A}_1} f + \cdots + \overline{\int}_{\mathfrak{A}_m}$$

provided the integral on the left exists or a right exist.

For if $\mathfrak{A}_{m,\alpha\beta}$ denote the points of $\mathfrak{A}_{\alpha\beta}$ in $\mathfrak{A}_m$

$$\overline{\int}_{\mathfrak{A}_{\alpha\beta}} = \overline{\int}_{\mathfrak{A}_{1\alpha\beta}} + \cdots + \overline{\int}_{\mathfrak{A}_{m\alpha\beta}}$$

Now if the integral on the left of 1) is con on the right of 1) all converge by 45. Pas gives 1). On the other hypothesis, the int 1) existing, a passage to the limit in 2) sl this case also.

47. *If* $\overline{\int}_{\mathfrak{P}} f$ *and* $\overline{\int}_{\mathfrak{N}} f$ *converge, so does* $\overline{\int}_{\mathfrak{A}}$

$$\overline{\int}_{\mathfrak{A}} f = \overline{\int}_{\mathfrak{P}} f - \overline{\int}_{\mathfrak{N}} f$$

$$= \overline{\int}_{\mathfrak{A}} g + \overline{\int}_{\mathfrak{A}} h.$$

For let A_β denote the points of $\mathfrak{A}$ where

$$0 \leq f \leq \beta.$$

48. 1. *If* $\overline{\underline{\int}}_{\mathfrak{A}} |f|$ *converges, both* $\overline{\underline{\int}}_{\mathfrak{A}} f$ *converge.*

For as usual let $\mathfrak{P}$ denote the points of $\mathfrak{A}$ where $f \geq 0$. Then

$$\overline{\underline{\int}}_{\mathfrak{P}} f = \overline{\underline{\int}}_{\mathfrak{P}} |f|$$

is convergent by 38, 3, since $\overline{\underline{\int}}_{\mathfrak{A}} |f|$ is convergent.

Similarly,

$$\overline{\underline{\int}}_{\mathfrak{N}} (-f) = -\overline{\underline{\int}}_{\mathfrak{N}} f$$

is convergent. The theorem follows now by 39.

2. *If* $\overline{\underline{\int}}_{\mathfrak{A}} |f|$ *converges, so do*

$$\overline{\underline{\int}}_{\mathfrak{A}} g \quad , \quad \overline{\underline{\int}}_{\mathfrak{A}} h.$$

For by 1,

$$\overline{\underline{\int}}_{\mathfrak{A}} f$$

both converge. The theorem now follows by 43, 2.

3. *For*

$$\overline{\underline{\int}}_{\mathfrak{A}} f$$

both to converge it is necessary and sufficient that

$$\overline{\underline{\int}}_{\mathfrak{A}} |f|$$

is convergent.

For if 3) converges, the integrals 2) both converge. On the other hand if both the integrals 2) converge

But by 41, 1, f is integrable in A_β. Hence
A_β by I, 720. Thus

$$\overline{\int}_{\mathfrak{A}} |f| = \lim_\beta \overline{\int}_{A_\beta} |f|.$$

49. From the above it follows that if both

$$\overline{\int}_{\mathfrak{A}} f$$

converge, they converge absolutely. Thus, i

$$\int_{\mathfrak{A}} f$$

converges, it is absolutely convergent.

We must, however, guard the reader agai
posing that *only absolutely* convergent upper
exist.

Example. At the rational points of $\mathfrak{A} = ($

$$f(x) = \frac{1}{2\sqrt{x}}.$$

At the irrational points let

$$f(x) = \frac{1}{x}.$$

Here

$$\overline{\int}_{\mathfrak{A}} f = 1, \qquad \underline{\int}_{\mathfrak{A}} f = \infty$$

Thus, f admits an upper, but not a low
other hand the upper integral of f does not

For obviously
$$\overline{\int}_{\mathfrak{A}} f = +\infty.$$

Let us consider, for example,

$$J = \int_0^1 \frac{\sin \frac{1}{x}}{x} dx = \int_{\mathfrak{A}} f dx.$$

If we set $x = \frac{1}{u}$, we get

$$J = \int_1^\infty \frac{\sin u}{u} du,$$

which converges by I, 667, but is not absolutely c
I, 646.

This apparent discrepancy at once disappears wh
that according to the definition laid down in Vol. I,

$$J = R \lim_{\alpha = 0} \int_\alpha^1 f dx,$$

while in the present chapter

$$J = \lim_{\alpha, \beta = \infty} \int_{\mathfrak{A}_{\alpha\beta}} f dx.$$

Now it is easy to see that, taking α large at pleasur

$$\int_{\mathfrak{A}_{\alpha\beta}} f dx \geq \eta \qquad \text{as } \beta \doteq \epsilon,$$

so that J does not converge according to our presen

In the theory of integration as ordinarily devel
on the calculus a similar phenomenon occurs, viz.
convergent integrals exist when $m > 1$.

51. 1. *If* $\int |f|$ *is convergent,*

2. *If* $\int_{\mathfrak{A}} |f|$ *is convergent,* $\int_{\mathfrak{A}} f$ *are convergent for an*

For $\int_{\mathfrak{A}} |f|$ is convergent by 38, 4.

Hence

$$\int_{\mathfrak{A}} f$$

converge by 48, 3.

3. *If,* 1°, $\int_{\mathfrak{A}} f$ *is convergent and* Min f *is finite, o*
convergent and Max f *is finite, then*

$$\int_{\mathfrak{A}} |f|$$

is convergent.

This follows by 36 and 48, 3.

52. *Let* $f \geq 0$ *in* $\mathfrak{A}$. *Let the integral*

$$\int_{\mathfrak{A}} f$$

converge. If

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{A}_\lambda} f + \alpha,$$

then for any unmixed part $\mathfrak{B} < \mathfrak{A}$

$$\int_{\mathfrak{B}} f = \int_{\mathfrak{B}_\lambda} f + \alpha',$$

where

$$0 \leq \alpha' \leq \alpha.$$

For let $\mathfrak{A} = \mathfrak{B} + \mathfrak{C}$. Then $\mathfrak{A}_\lambda = \mathfrak{B}_\lambda + \mathfrak{C}_\lambda$ is an unm
Also

Hence
$$\int_{\mathfrak{A}} + \int_{\mathfrak{C}} = \int_{\mathfrak{A}_1} + \int_{\mathfrak{C}_1} + \alpha.$$

From 2)
$$\alpha' = \int_{\mathfrak{A}} - \int_{\mathfrak{A}_1}$$
$$= \alpha - \left\{ \int_{\mathfrak{C}} - \int_{\mathfrak{C}_1} \right\}$$
$$= \alpha,$$

which establishes 3).

53. *If the integral*
$$\int_{\mathfrak{A}} |f|$$
converges, then
$$\epsilon > 0, \qquad \sigma > 0, \qquad \int_{\mathfrak{B}} |f| < \epsilon$$
for any $\mathfrak{B} < \mathfrak{A}$ *such that*
$$\overline{\mathfrak{B}} < \sigma.$$

Let us suppose first that $f \geq 0$. If the theor
there exists, however small $\sigma > 0$ is taken, a $\mathfrak{B}$ sat
that
$$\int_{\mathfrak{B}} f > \epsilon.$$

Then there exists a cubical division of space
points of $\mathfrak{A}$, call them $\mathfrak{C}$, which lie in cells conta
$\mathfrak{B}$, are such that $\overline{\mathfrak{C}} < \sigma$ also. Moreover $\mathfrak{C}$ is an un
Then from 4) follows, as $f \geq 0$, that
$$\int_{\mathfrak{C}} f > \epsilon$$
also.

Let us now take β so that

Let now $\beta\sigma > c$, then

$$\int_{\mathfrak{C}_\beta} > c,$$

which contradicts 5).

Let us now make no restrictions on the sign

$$\left|\int_{\mathfrak{A}'} f\right| \le \int_{\mathfrak{A}} |f| .$$

But since 1) converges, the present case is r

ceding.

54. 1. *Let* $\int_{\mathfrak{A}} f$ *converge.*

Let as usual $\mathfrak{A}_{\alpha\beta}$ *denote the points of* $\mathfrak{A}$ *at which*

$A_{\alpha\beta}$ *be such that each* $\mathfrak{A}_{\alpha\beta}$ *lies in some* $A_{\alpha\beta}$ *in whi*

Let $\mathfrak{D}_{\alpha\beta} = A_{\alpha\beta} - \mathfrak{A}_{\alpha\beta}$ *and let* $a, b \doteq \infty$ *with* α, β.

$$\lim_{\alpha,\beta=\infty} \overline{\mathfrak{D}}_{\alpha\beta} = 0.$$

For if not, let

$$\overline{\lim_{\alpha,\beta=\infty}} \overline{\mathfrak{D}}_{\alpha\beta} = l, \qquad l > 0.$$

Then for any $0 < \lambda < l$, there exists a monoto

such that

$$\overline{\mathfrak{D}}_{\alpha_n\beta_n} > \lambda \qquad \text{for } n > \text{some } m$$

Let $\mu_n = \text{Min}(\alpha_n, \beta_n)$, then $|f| > \mu_n$ in

Hence

$$\int_{\mathfrak{D}_{\alpha_n\beta_n}} |f| > \mu_n\lambda \doteq \infty .$$

But $\mathfrak{D}_{\alpha_n\beta_n}$ being a part of $\mathfrak{A}$

If, 1°,

$$\lim_{\beta=\infty} \int_{\mathfrak{A}_\beta} |f|,$$

or if, 2°,

$$\lim_{b=\infty} \int_{A_b} |f|$$

exist, then,

$$\lim_{a,b=\infty} \int_{A_{a,b}} f = \int_{\mathfrak{A}} f.$$

For, if 2° holds, 1° holds also, since

$$\int_{\mathfrak{A}_\beta} |f| \le \int_{A_b} |f|, \qquad \text{as } \mathfrak{A}_\beta \le A_b.$$

Thus case 2° is reduced to 1°. Let then the 1°
We have

$$\int_{\mathfrak{A}_{\alpha\beta}} f = \int_{\mathfrak{A}_{\alpha\beta}} g - \int_{\mathfrak{A}_{\alpha\beta}} h,$$

as 4) in 44, 1 shows. Let now

$$\mathfrak{D}_{\alpha\beta} = A_{ab} - \mathfrak{A}_{\alpha\beta}.$$

Then,

$$\int_{\mathfrak{A}_{\alpha\beta}} g \le \int_{A_{ab}} g \le \int_{\mathfrak{A}_{\alpha\beta}} g + \int_{\mathfrak{D}_{\alpha\beta}} g.$$

But $\overline{\mathfrak{D}}_{\alpha\beta} \doteq 0$, as $\alpha, \beta \doteq \infty$, by 54. Let us now pass
$\alpha, \beta = \infty$ in 3). Since the limit of the last term is 0 I
get

$$\lim_{\alpha,\beta=\infty} \int_{\mathfrak{A}_{\alpha\beta}} g = \lim_{a,b=\infty} \int_{A_{a,b}} g.$$

Similarly,

$$\lim \int h = \lim \int h.$$

2. The following example is instructive as show
the conditions imposed in 1 are not fulfilled, the re
not hold.

Example. Since

$$\int_0^1 \frac{dx}{x} = +\infty,$$

there exists, for any $b_n > 0$, a $0 < b_{n+1} < b_n$, such tha

$$\int_{b_{n+1}}^{b_n} \frac{dx}{x} > G_{n+1},$$

then

$$G_1 < G_2 < \cdots \doteq +\infty,$$

as $b_n \doteq 0$. Let now

$$f = 1 \text{ for the rational points in } \mathfrak{A}$$
$$= \frac{1}{x} \text{ for the irrational.}$$

Then

$$\int_{\mathfrak{A}_\beta} f = 1 - \frac{1}{\beta}.$$

Let

$$\int_{b_1}^1 \frac{dx}{x} = G_1, \qquad 0 < b_1 < 1.$$

Let A_n denote the points of $\mathfrak{A}$ in $(b_n, 1)$ and the ir
in (b_{n+1}, b_n).

Then

$$\int_{A_n} > G_{n+1} \doteq +\infty.$$

But obviously the set A_n is conjugate to $\mathfrak{A}_\beta$. On t

$$\int_{\mathfrak{A}} f = 1,$$

while

$$\lim_{n=\infty} \int_{A_n} f = +\infty.$$

56. *If the integral*

t us establish the theorem for the upper integral: sim
soning may be used for the lower. Since 1) is converg

$$\int_{\mathfrak{A}} g$$

1

$$\lambda = \lim_{\alpha, \beta \to \infty} \int_{\mathfrak{A}_{\alpha\beta}} h$$

st by 44, 1. Since 3) exists, we have by 53,

$$0 < \int_{\mathfrak{B}} g < \frac{\epsilon}{4},$$

any $\mathfrak{B} < \mathfrak{A}$ such that $\mathfrak{B} <$ some α'.

Since 4) exists, there exists a pair of values a, b such that

$$\lambda = \int_{\mathfrak{A}_{ab}} h + \eta \quad , \quad 0 < \eta < \frac{\epsilon}{4},$$

ce the integral on the right side of 4) is a monotone increas
ction of a, b.

Since $\mathfrak{A} = \mathfrak{B} + \mathfrak{C}$ is an unmixed division of $\mathfrak{A}$,

$$\int_{\mathfrak{A}_{\alpha\beta}} h = \int_{\mathfrak{B}_{\alpha\beta}} h + \int_{\mathfrak{C}_{\alpha\beta}} h.$$

Since $h \geq 0$, and the limit 4) exists, the above shows that

$$\mu = \lim_{\alpha, \beta \to \infty} \int_{\mathfrak{B}_{\alpha\beta}} h \quad , \quad \nu = \lim_{\alpha, \beta \to \infty} \int_{\mathfrak{C}_{\alpha\beta}} h$$

st and that

$$\lambda = \mu + \nu.$$

Then a, b being the same as in 6),

Thus,
$$\mu < \frac{\epsilon}{4}.$$

But
$$\int_{\mathfrak{A}} f - \int_{\mathfrak{A}} g = \mu$$

by 44, 1. Thus 2) follows on using 5), 11) an

57. *If the integral* $\int_{\mathfrak{A}} f$ *converges and* $\mathfrak{B}_u$ *is*

$\mathfrak{A}$ *such that* $\mathfrak{B}_u \doteq \mathfrak{A}$ *as* $u \doteq 0$, *then*

$$\lim_{u=0} \int_{\mathfrak{B}_u} f = \int_{\mathfrak{A}} f.$$

For if we set $\mathfrak{A} = \mathfrak{B}_u + \mathfrak{C}_u$, the last set is an
and $\mathfrak{C}_u \doteq 0$. Now

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{B}_u} + \int_{\mathfrak{C}_u}.$$

Passing to the limit, we get 1) on using 56.

58. 1. *Let* $\qquad \mathfrak{T}_{\alpha\beta} = Dv(\mathfrak{A}_{\alpha\beta}, \mathfrak{A}_{\alpha\beta}$

If, 1°, *the upper contents of*

$$\mathfrak{f}_{\alpha\beta} = \mathfrak{A}_{f\alpha\beta} \cdot \mathfrak{T}_{\alpha\beta}, \quad \mathfrak{g}_{\alpha\beta} = \mathfrak{A}_{g\alpha\beta} \cdot \mathfrak{T}_{\alpha\beta}, \quad \mathfrak{h}$$

$$\doteq 0 \text{ as } \alpha, \beta \doteq \infty,$$

and if, 2°, *the upper integrals of* $f, g, f+g$ *are*

$$\int_{\mathfrak{A}} (f+g) = \int_{\mathfrak{A}} f + \int_{\mathfrak{A}} g.$$

Then $D_{\alpha\beta}$ may be chosen so that $\mathfrak{F}_{\alpha\beta} \doteq 0$.

Now
$$\int_{\mathfrak{A}_{f,\alpha\beta}} f = \int_{\mathfrak{E}_{\alpha\beta}} + \int_{\mathfrak{F}_{\alpha\beta}}$$

since the fields are unmixed. By 56, the second integr
right $\doteq 0$ as $\alpha, \beta \doteq \infty$. Hence

$$\lim_{\alpha,\beta=\infty} \int_{\mathfrak{A}_{f,\alpha\beta}} f = \lim_{\alpha,\beta=\infty} \int_{\mathfrak{E}_{\alpha\beta}} f.$$

Similar reasoning applies to g and $f+g$.

Again,
$$\int_{\mathfrak{E}_{\alpha\beta}} (f+g) = \int_{\mathfrak{E}_{\alpha\beta}} f + \int_{\mathfrak{E}_{\alpha\beta}} g.$$

Thus, letting $\alpha, \beta \doteq \infty$ we get 2).

2. *When the singular points of f, g are discrete, the co*
holds.

3. *If g is integrable and the conditions* 1°, 2°, 3° *are sat*

$$\int_{\mathfrak{A}} (f+g) = \int_{\mathfrak{A}} f + \int_{\mathfrak{A}} g.$$

4. *If f, g are integrable and condition* 1° *is satisfied, f*
tegrable and
$$\int_{\mathfrak{A}} (f+g) = \int_{\mathfrak{A}} f + \int_{\mathfrak{A}} g.$$

5.
$$\int_{\mathfrak{A}} (f+C) = \int_{\mathfrak{A}} f + C \lim_{\alpha,\beta=\infty} \mathfrak{A}_{\alpha\beta}$$

Example. Let $\mathfrak{A}$ consist of the rational poi
Let
$$f = 1 + n \quad , \quad g = 1 - n$$
at the point $x = \frac{m}{n}$. Then
$$f + g = 2 \qquad \text{in } \mathfrak{A}.$$
Now
$$\mathfrak{A}_{f,\,\alpha\beta} \quad , \quad \mathfrak{A}_{g,\,\alpha\beta}$$
embrace only a finite number of points for a g
other hand,
$$\mathfrak{A}_{f+g,\,\alpha\beta} = \mathfrak{A} \qquad \text{for } \beta > 2.$$

Thus the upper content of the last set in
$\alpha, \beta \doteq \infty$ and condition 1° is not fulfilled. A
not hold in this case. For
$$\int_{\mathfrak{A}}^{*} (f + g) = 2 \quad , \quad \int_{\mathfrak{A}}^{*} f = 0 \quad , \quad \int_{\mathfrak{A}}^{*}$$

59. $$\textit{If } c > 0, \textit{ then } \int_{\mathfrak{A}}^{*} cf = c\int_{\mathfrak{A}}^{*} f,$$
$$\textit{if } c < 0, \textit{ then } \int_{\mathfrak{A}}^{*} cf = c\int_{\mathfrak{A}}^{*} f$$
provided the integral on either side is convergen

For
$$\int_{\mathfrak{A}_{cf,\,\alpha\beta}}^{*} cf = c\int_{\mathfrak{A}_{cf,\,\alpha\beta}}^{*} f$$
$$= c\int_{\mathfrak{A}_{cf,\,\alpha\beta}}^{*} f$$

Then $D_{\alpha\beta}$ may be chosen so that $\mathfrak{K}_{\alpha\beta} \doteq 0$.

Now

$$\int_{\mathfrak{A}_{f,\alpha\beta}} f = \int_{\mathfrak{C}_{\alpha\beta}} + \int_{\mathfrak{K}_{\alpha\beta}}$$

since the fields are unmixed. By 536, the second integral on the right $\doteq 0$ as $\alpha, \beta \doteq \infty$. Hence

$$\lim_{\alpha, \beta \doteq \infty} \int_{\mathfrak{A}_{f,\alpha\beta}} f = \lim_{\alpha, \beta \doteq \infty} \int_{\mathfrak{C}_{\alpha\beta}} f.$$

Similar reasoning applies to g and $f + g$.

Again,

$$\int_{\mathfrak{C}_{\alpha\beta}} (f + g) = \int_{\mathfrak{C}_{\alpha\beta}} f + \int_{\mathfrak{C}_{\alpha\beta}} g.$$

Thus, letting $\alpha, \beta \doteq \infty$ we get 1).

2. *When the singular points of f, g are discrete, the relation 1) holds.*

3. *If g is integrable and the conditions 1), 2), 3) are satisfied*

$$\int_{\mathfrak{A}} (f + g) = \int_{\mathfrak{A}} f + \int_{\mathfrak{A}} g.$$

4. *If f, g are integrable and condition 1) is satisfied, $f+g$ is integrable and*

$$\int_{\mathfrak{A}} (f + g) = \int_{\mathfrak{A}} f + \int_{\mathfrak{A}} g.$$

5.
$$\int_{\mathfrak{A}} (f + C) = \int_{\mathfrak{A}} f + C \lim_{\alpha, \beta \doteq \infty} \mathfrak{A}_{\alpha\beta}$$

Example. Let $\mathfrak{A}$ consist of the rational points in $(0,$ [illegible]

Let

$$f = 1 + n, \quad g = 1 - n$$

at the point $x = \frac{m}{n}$. Then

$$f + g = 2 \quad \text{in } \mathfrak{A}.$$

Now

$$\mathfrak{A}_{\ [illegible]} \quad \mathfrak{A}_{\ [illegible]}$$

embrace only a finite number of points for a given α, [illegible]
other hand,

$$\mathfrak{A}_{\ [illegible]} = \mathfrak{A} \quad \text{for } \beta \ [illegible]$$

Thus the upper content of the last set in 1) does [illegible]
$\alpha, \beta \doteq \infty$ and condition 1) is not fulfilled. Also relation [illegible]
not hold in this case. For

$$\overline{\int_{\mathfrak{A}}} (f + g) = 2, \quad \overline{\int_{\mathfrak{A}}} f = 0, \quad \overline{\int_{\mathfrak{A}}} g = 0.$$

59. [illegible]

$$f \geq 0, \quad \text{then} \quad \overline{\int_{\mathfrak{A}}} \ [illegible] = \overline{\int_{\mathfrak{A}}} f,$$

$$g \geq 0, \quad \text{then} \quad \overline{\int_{\mathfrak{A}}} \ [illegible] = \overline{\int_{\mathfrak{A}}} g,$$

provided the integral [illegible]

For

$$\overline{\int_{\mathfrak{A}}} \ [illegible] \quad \overline{\int_{\mathfrak{A}}} \ [illegible] = 0$$

$$\overline{\int_{\mathfrak{A}}} \ [illegible] = 0.$$

Similarly $\qquad \mathfrak{A}_{e\gamma, e\delta} = \mathfrak{A}_{\gamma, \frac{\alpha}{e}, \frac{\beta}{e}} \qquad$ when e

Thus 3), 4) give

$$\int_{\mathfrak{A}_{e\gamma, e\delta}} ef = e\int_{\mathfrak{A}_{\gamma, \frac{\alpha}{e}, \frac{\beta}{e}}} f \qquad e \ldots$$

$$= e\int_{\mathfrak{A}_{\gamma, \frac{\beta}{e}, \frac{\alpha}{e}}} f \qquad e \ldots$$

We now need only to pass to the limit α, β

60. *Let one of the integrals*

$$\int_{\mathfrak{A}} f \quad , \quad \int_{\mathfrak{A}} g$$

converge. If $f = g$, except at a discrete set $\mathfrak{D}$ converge and are equal. A similar theorem integrals.

For let us suppose the first integral in 1) co

$$\mathfrak{A} = A + \mathfrak{D}:$$

then

$$\int_{\mathfrak{A}} f = \int_A f + \int_{\mathfrak{D}} f = \int_A f.$$

Now

$$\int_{\mathfrak{A}} g = \lim_{\alpha, \beta = \infty} \int_{\mathfrak{A}_{\alpha, \alpha\beta}} g = \lim$$

$$= \lim \int_{A_{\alpha, \alpha\beta}} f = \int_A f.$$

Thus the second integral in 1) converges, a the integrals in 1) are equal.

For let $\mathfrak{S}$ [illegible] be defined as in 58, 4. Then

$$\int [illegible] \int [illegible]$$

Let α, β [illegible], we get 2) by the same style of reasoning as in 58.

3. *If the integrals [illegible] converge, and their singular points [illegible]* [illegible]

This follows from [illegible]

4. If the conditions of 1 do not hold, the relation [illegible] be true.

Example. Let $\mathfrak{A}$ denote the rational points in [illegible]

$$[illegible] \text{ in } \mathfrak{A}.$$

$$[illegible] \mathfrak{A}$$

Then [illegible] $\mathfrak{A}$

But

$$\int [illegible] \int [illegible] 1.$$

Relations between the Integrals of Types I, II

62. Let us denote [illegible] the limited [illegible]

$$[illegible] \quad I_{\mathfrak{A}} \quad I'_{\mathfrak{A}}$$

[illegible] the upper and lower integrals [illegible] W [illegible] $\mathfrak{A}$. The [illegible]

Hence
$$P_{\mathfrak{A}} = \lim_{\delta=0} C_{\mathfrak{A}_\delta}$$
$$= C_{\mathfrak{A}}, \quad \text{by definitio}$$

64. If $\bar{C}$ is convergent, we cannot say th similar remark holds for the lower integrals.

Example. For the rational points in $\mathfrak{A}$ (

$$f(x) = \frac{1}{2\sqrt{x}};$$

for the irrational points let

$$f(x) = \frac{1}{x}.$$

Then
$$\bar{C}_{\mathfrak{A}} = \lim_{a=0}\int_a^1 f(x)dx = \lim[\sqrt{x}]$$

On the other hand,
$$P_{\mathfrak{A}} = \lim_{\alpha,\beta=\infty}\int_{\mathfrak{A}_{\alpha\beta}} f$$

does not exist. For however large β is taken

$$\int_{\mathfrak{A}_{\alpha\beta}} f$$

65. *If C is absolutely convergent and $\mathfrak{B}$ is converge and are equal to the corresponding C's*

For let D be any complete division of $\mathfrak{A}$ of

$$\int_{\mathfrak{A}} = \int_{\mathfrak{A}} + \int_{\mathfrak{A}'}$$

Hence [illegible]

$$\int_{\mathfrak{A}} [illegible] \quad \text{for any } \delta < \text{some} [illegible]$$

On the other hand, if ϵ_1 is sufficiently small,

$$[illegible] \quad \text{for } \delta < \delta_0$$

Hence
$$\int_{\mathfrak{A}} [illegible] < \epsilon.$$

Passing to the limit [illegible], we get

$$P = C.$$

66. *If I [illegible] is absolutely convergent, the singular [illegible] discrete.*

For suppose [illegible]. Let $\mathfrak{A}$ denote the points [illegible] Then [illegible] for any d. Hence

$$\int_{\mathfrak{A}} [illegible] \int_{\mathfrak{A}} [illegible]$$

[illegible]

67. *If I [illegible] absolutely convergent, [illegible] is C.*

For let D [illegible] space of norm d.

Then

$$[illegible] \text{ some } d \text{ in } \mathfrak{A}_r$$

For $$\overline{V}_{\mathfrak{A}} f = \int_{\mathfrak{A}} f_\lambda + \epsilon' \quad , \quad 0 < \epsilon'$$

for λ sufficiently large. Let λ be so taken, t

$$\overline{V}_{\mathfrak{A}} f = \int_{\mathfrak{A}'} f_\lambda + \epsilon'' \quad , \quad 0 < \epsilon$$

Also, $$\int_{\mathfrak{A}'} f_\lambda - \lambda \mathfrak{A} \quad \frac{\epsilon}{2}$$

if σ is taken sufficiently small in 2).

From 3), 4) follows 1).

69. *If $\overline{V}_{\mathfrak{A}} f$ is absolutely convergent, both*
equal to the corresponding V integrals.

For by 67, C is absolutely convergent. He

Thus $$C_{\mathfrak{A}} f = \int_{\mathfrak{A}_\iota} f + \alpha \quad , \quad \alpha < \frac{\epsilon}{3}$$

Also $$\overline{V}_{\mathfrak{A}} f = \int_{\mathfrak{A}} f_\mu + \beta \quad , \quad \beta < \frac{\epsilon}{3}$$

Hence $$\eta = C_{\mathfrak{A}} f - \overline{V}_{\mathfrak{A}} f = \int_{\mathfrak{A}_\iota} f - \int_{\mathfrak{A}} f_\mu$$

Now $$\int_{\mathfrak{A}} f_\mu = \int_{\mathfrak{A}_\iota} f_\mu + \int_{\mathfrak{A}_\iota'} f_\mu$$

But $$\left| \int_{\mathfrak{A}_\iota} f_\mu - \int_{\mathfrak{A}_\iota} f_\mu \cdot \mathfrak{A}_\iota \right.$$

Iterated Integrals

70. 1. We consider now the relations which exist betwee
tegrals

$$\int_{\mathfrak{A}} f,$$

d

$$\int_{\mathfrak{B}} \int_{\mathfrak{C}} f,$$

here $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$ lies in a space $\Re_m$, $m = p + q$, and $\mathfrak{B}$ is a proje
$\mathfrak{A}$ in the space $\Re_p$.

It is sometimes convenient to denote the last q coordinate
int $x = (x_1 \cdots x_p\ x_{p+1} \cdots x_{p+q})$ by $y_1 \cdots y_q$. Thus the coordi
$\cdots x_p$ refer to $\mathfrak{B}$ and $y_1 \cdots y_q$ to $\mathfrak{C}$. The section of $\mathfrak{A}$ corres
g to the point x in $\mathfrak{B}$ may be denoted by $\mathfrak{C}_x$ when it is des
indicate which of the sections $\mathfrak{C}$ is meant.

2. Let us set

$$\phi(x_1 \cdots x_p) = \int_{\mathfrak{C}} f;$$

en the integral 2) is

$$\int_{\mathfrak{B}} \phi.$$

It is important to note at once that although the integran
efined for each point in $\mathfrak{A}$, the integrand ϕ in 4) may not b

Example. Let $\mathfrak{A}$ consist of the points (x, y) in the unit sq

$$x = \frac{m}{n}, \quad 0 \leq y \leq \frac{1}{n}$$

At the point (x, y) in $\mathfrak{A}$, let

To provide for the case that ϕ may not
points of $\mathfrak{B}$ we give the symbol 2) the follo

$$\int_{\mathfrak{B}} \int_{\mathfrak{C}} f = \lim_{\alpha, \beta \to \infty} \int_{\mathfrak{B}_{\alpha\beta}} \int$$

where $\Gamma = \mathfrak{C}$ when the integral 3) is conv
trary case Γ is such a part of $\mathfrak{C}$ that

$$\alpha \le \int_{\Gamma} f \le \beta,$$

and such that the integral in 6) is numerica
permit.

Sometimes it is convenient to denote Γ
The points $\mathfrak{B}_{\alpha\beta}$ are the points of $\mathfrak{B}$ at w
be noticed that each $\mathfrak{B}_{\alpha\beta}$ in 5) contains all
the integral 3) is *not* convergent. Thus

$$\mathfrak{B}_{\alpha\beta} \le \mathfrak{B}_{\alpha'\beta'}.$$

Hence when $\mathfrak{B}$ is complete or metric,

$$\lim_{\alpha, \beta \to \infty} \mathfrak{B}_{\alpha\beta} = \mathfrak{B}$$

Before going farther it will aid the re
examples.

71. *Example 1.* Let $\mathfrak{A}$ be as in the exam
at $x = \frac{m}{n}$. We see that

$$\int_{\mathfrak{A}} f = n,$$

the discrete point set used in Ex. 1 by $\mathfrak{D}$. We define f as follows: f shall have in $\mathfrak{D}$ the values assigned to it at the points in Ex. 1. At the other points A of $\mathfrak{A} - \mathfrak{D}$, f shall have the value 1. Then

$$\int_{\mathfrak{A}} = \int_x \int_y = \int_y \int_x = 1.$$

On the other hand $\mathfrak{A}_{x}$ consists of the irrational points and a finite number of other points. Thus

$$\int_{\mathfrak{A}} \int_{\mathfrak{Q}} f = 1.$$

Hence again the two i. i. exist and are equal.

Let us look at the results we get if we use integrals of type I and II. We shall denote them by C and L as in 62.

We see at once that

$$C_{\mathfrak{A}} = L_{\mathfrak{A}} = L'_{\mathfrak{A}} = 1.$$

Let us now calculate the iterated integrals

$$C_{\mathfrak{X}} C_{\mathfrak{Y}},$$

and

$$L_{\mathfrak{X}} L_{\mathfrak{Y}}.$$

We observe that

$$C_{\mathfrak{Y}} = 1 \quad \text{for } x \text{ irrational}$$

$$= x \quad \text{for } x \text{ rational.}$$

Thus the integral ... either is not defined at all since $\mathfrak{A}_x$ does not exist, or if we interpret the definition as broadly as possible, its value is 0. In neither case is

$$C_{\mathfrak{A}} = C_{\mathfrak{X}} C_{\mathfrak{Y}}.$$

Let us now look at the integrals L. We see at once

$$L_{\mathfrak{X}} L_{\mathfrak{Y}}$$

does not exist, as $L_{\mathfrak{Y}} = 1$ for rational x, and $= x$ fo

Example 3. Let $\mathfrak{A}$ be the unit square.

Let

$$f = n \quad \text{for } x = \frac{m}{n} \qquad n \ldots$$

$$= -n \quad \text{for } x = \frac{m}{n}$$

At the other points of $\mathfrak{A}$ let $f = 1$.

Then

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{B}} \int_{\mathfrak{C}} f = 1.$$

Here every point of $\mathfrak{A}$ is a point of …
thus $\mathfrak{J} = \mathfrak{A}$.

Here $C_{\mathfrak{A}}$ is not defined, as $\mathfrak{A}$, does not …
definition its most liberal interpretation,

$$C_{\mathfrak{A}} = n.$$

The same remarks hold for $C_{\mathfrak{B}} C_{\mathfrak{C}}$.

On the other hand

$$I_{\mathfrak{A}} \ldots,$$

while

$$I_{\mathfrak{B}} I_{\mathfrak{C}}$$

does not exist, since

$$I_{\mathfrak{C}} = \pm n \text{ for } \ldots$$

$$= 1 \text{ for } \ldots$$

Moreover

$$I_{\mathfrak{B}} I_{\mathfrak{C}} \ldots, \quad I_{\mathfrak{B}} I_{\mathfrak{C}} \ldots$$

Example 4. Let $\mathfrak{A}$ denote the unit squar…

We see at once that

$$C_{\mathfrak{A}} = U_{\mathfrak{A}} = 1.$$

Again the integral $C_{\mathfrak{B}} C_{\mathfrak{C}}$ does not exist, or on a liberal [interpre-]tation it has the value 0. Also in this example

$$C_{\mathfrak{B}} C_{\mathfrak{C}} \text{ and } C_{\mathfrak{B}} C_{\mathfrak{C}}$$

do not exist or on a liberal interpretation, they $= 0$.

Turning to the U integrals we see that

$$U_{\mathfrak{B}} U_{\mathfrak{C}} = -\infty \ , \quad U_{\mathfrak{B}} U_{\mathfrak{C}} = +\infty,$$

while $U_{\mathfrak{B}} U_{\mathfrak{C}}$ does not exist finite or infinite.

Example 5. Let our field of integration $\mathfrak{A}$ consist of [the] square considered in Ex. 4, let us call it $\mathfrak{C}$, and anothe[r] square $\mathfrak{R}$, lying to its right. Let f be defined over $\mathfrak{C}$ [as] defined in Ex. 4, and let $f = 1$ in $\mathfrak{R}$.

Then

$$\int_{\mathfrak{A}} = \int_{\mathfrak{B}} \int_{\mathfrak{C}} = 2.$$

Also

$$C_{\mathfrak{A}} = U_{\mathfrak{A}} = 2.$$

Then

$$C_{\mathfrak{B}} C_{\mathfrak{C}} = 1,$$

while

$$U_{\mathfrak{B}} U_{\mathfrak{C}} \text{ does not exist,}$$

and

$$U_{\mathfrak{B}} U_{\mathfrak{C}} = -\infty \ , \quad U_{\mathfrak{B}} U_{\mathfrak{C}} = +\infty.$$

72. 1. In the following sections we shall restrict our[selves as] follows:

1° $\mathfrak{A}$ shall be limited and iterable with respect to $\mathfrak{B}$.

2° $\mathfrak{B}$ shall be complete or metric.

3° The singular points $\mathfrak{S}$ of the integrand f shall be di

Let $\mathfrak{A}_n = \mathfrak{B}_n \cdot \mathfrak{C}_n$ denote the points of ... which contain no point of $\mathfrak{Z}$. We observe ... *take without loss of generality*

$$\mathfrak{P}_n \quad \mathfrak{P}.$$

For let us adjoin to $\mathfrak{A}$ a discrete set $\mathfrak{D}$ ... from $\mathfrak{A}$ such that the projection of $\mathfrak{D}$ on $\mathfrak{R}$ ...

Let $$A = \mathfrak{A} + \mathfrak{D} = \mathfrak{P} \cdot C \ldots$$

We now set $$\phi = f \text{ in } \mathfrak{A}$$
$$= 0 \text{ in } \mathfrak{D}$$

Then $$\int_A \phi = \int_{\mathfrak{A}} \phi + \int_{\mathfrak{D}} \phi \ldots$$
$$= \int_{\mathfrak{A}} f.$$

Similarly $$\int_{\mathfrak{C}} \phi = \int \phi + \int \phi \ldots$$

Hence $$\int_{\mathfrak{P}} \int_{\mathfrak{C}} \phi = \int_{\mathfrak{P}} \int_{\mathfrak{C}} f$$

3. The set $\mathfrak{C}_n$ being as in 2, we shall ...

$$\mathfrak{C} \ldots \mathfrak{C}_n \ldots$$

73. *Let $B_{\sigma, n}$ denote the points of $\mathfrak{P}$ at which ... is iterable, with respect to $\mathfrak{P}$.*

$$\lim_{n=\infty} B_{\sigma, n} = 0.$$

For since $\mathfrak{A}$ is iterable,

$$\mathfrak{A} = \int_{\mathfrak{P}} \mathfrak{C} \quad \text{by definition} \ldots$$

Hence $\mathfrak{C}$ considered as a function ...

We have now [illegible]

as [illegible] are [illegible]. Hence [illegible] is an integrable functi[illegible]
But

[illegible]

[illegible]

As the left side [illegible] as n [illegible],

$$\lim \int [illegible] = 0.$$

But

[illegible]

As the left side [illegible], we have for a given [illegible]

$$\lim [illegible] = 0,$$

which is [illegible].

74. *Let* [illegible] *Let the integral*

[illegible]

be convergent and [illegible] *Let* [illegible] *denote* [illegible]

[illegible]

Then [illegible]

For let [illegible]

Let $\mathfrak{A}_n = \mathfrak{B}_n \cdot \mathfrak{C}_n$ denote the points
which contain no point of $\mathfrak{Z}$. We
take without loss of generality

$$\mathfrak{A}_n \quad \mathfrak{A}$$

For let us adjoin to $\mathfrak{A}$ a discrete
from $\mathfrak{A}$ such that the projection of

Let $$A = \mathfrak{A} + \mathfrak{D}$$

We now set

Then

Similarly

Hence

3. The set $\mathfrak{C}_n$ being as in 2, we shall

73. *Let $B_{a,n}$ denote the points of $\mathfrak{A}$*
is iterable, with respect to $\mathfrak{A}$,

$$\lim B$$

We have now

$$\mathfrak{C} = \mathfrak{C}_n + \mathfrak{c}_n \quad [illegible] \quad \mathfrak{c}_n \geq 0$$

as $\mathfrak{C}_n$, $\mathfrak{c}_n$ are measurable. Hence $\mathfrak{c}_n$ is an integrable [illegible]
But

$$\mathfrak{A} - \mathfrak{A}_n = \int_{\mathfrak{B}} (\mathfrak{C}_n + \mathfrak{c}_n) - \int_{\mathfrak{B}} \mathfrak{C}_n$$

$$= \int_{\mathfrak{B}} \mathfrak{c}_n.$$

As the left side [illegible] as n [illegible],

$$\lim \int_{\mathfrak{B}} \mathfrak{c}_n = 0.$$

But

$$\int_{\mathfrak{B}} \mathfrak{c}_n = \int [illegible] \mathfrak{c}_n [illegible] R_{an}.$$

As the left side [illegible] 0, we have for a given a

$$\lim R_{an} = 0,$$

which is 1) [illegible].

74. *Let* $\mathfrak{A}$ [illegible] *Let the integral*

$$\int [illegible] = 0$$

be [illegible] *and* [illegible] *Let* $\mathfrak{C}_n$ [illegible]
[illegible]

$$\int [illegible].$$

Then

$$\lim \mathfrak{C}_n = \mathfrak{A}$$

For let [illegible]

We may now reason on $\mathfrak{B}_1$ as we did on $\mathfrak{B}$, rep[illegible]
We get a complete set $\mathfrak{B}_2 \sim \mathfrak{B}_1$ such that

$$\mathfrak{B}_2 - \mathfrak{B}_1 [illegible]$$

Continuing we get $\quad \mathfrak{B}_n - \mathfrak{B}_n [illegible]$

Thus $\quad \overline{\mathfrak{B}}_n - \mathfrak{B} [illegible]$

$$- \mathfrak{B} \quad \eta.$$

Let now $\quad \mathfrak{b} = Dv \, \mathfrak{B}_n [illegible]$

Then $\quad \mathfrak{b} - \mathfrak{B} \quad \eta$

by 25.

Let $\mathfrak{b}_n$ denote those points of $\mathfrak{b}$ for which [illegible]
$\mathfrak{b} = \{\mathfrak{b}_n\}$. For let b be any point of $\mathfrak{b}$. Since [illegible]
there exists a σ_s such that

$$\text{at } b, \quad \int [illegible] \quad \epsilon,$$

for any c such that c [illegible] σ_s. Thus b [illegible]
$\{\mathfrak{b}_n\}$. Thus $\mathfrak{b}_n \doteq \mathfrak{b}$ as $\mathfrak{b}$ is complete. [illegible]

Hence $\quad \lim \mathfrak{C}_n = \mathfrak{b},$

which with 4), gives 3).

75. *Let $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$ be iterable. Let the integral*

$$\int_{\mathfrak{C}} f [illegible]$$

be convergent and limited in complete $\mathfrak{B}$ [illegible]

Let d_ι'' denote the other cells containing points of

$$S_\iota = \Sigma c_\iota + \Sigma d_\iota'' M,$$

where

$$\int_{\mathfrak{C}} f \quad M.$$

Hence $$S_\iota \quad \epsilon \mathfrak{B}_n \quad M \epsilon \mathfrak{B}_n \quad (\mathfrak{S}_n, \sigma).$$

Letting $d \doteq 0$, we get

$$\int_{\mathfrak{B}_n} \int_{\mathfrak{C}} f \quad \epsilon \mathfrak{B} \quad M \epsilon \mathfrak{B} \quad (\mathfrak{S}_n).$$

Letting now $n \doteq \infty$ and using 3 of 74, we get small at pleasure.

76. *Let $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$ be iterable with respect to $\mathfrak{B}$, wh… plete … Let the singular points $\mathfrak{S}$ of f be di…*

$$\bar{\mathfrak{A}} \quad f \quad \bar{\mathfrak{B}} \quad \int_{\mathfrak{A}} f \quad \int_{\mathfrak{B}} \int_{\mathfrak{C}} f \quad \int_{\mathfrak{A}} f$$

$$\underline{\mathfrak{A}} \quad f \quad \underline{\mathfrak{B}} \quad \int_{\mathfrak{A}} f \quad \int_{\mathfrak{B}} \int_{\mathfrak{C}} f \quad \int_{\mathfrak{A}} f$$

Here any one of the members in 1) may be infin… that follow are also infinite. A similar remark appli…

Let us first suppose

f … $\mathfrak{B}$ is complete, … $\int_{\mathfrak{B}} \int_{\mathfrak{C}} f$ is conv…

We have by 14,

$$\int_{\mathfrak{A}_n} f \quad \int_{\mathfrak{B}} \int_{\mathfrak{C}} f \quad \int_{\mathfrak{A}_n} f.$$

But for a fixed n

$$\int_{\mathfrak{A}}$$

Hence for G_n sufficiently large,

$$\int_{\mathfrak{C}_n} f - \int_{\Gamma} f$$

Then

$$\int_{\mathfrak{B}_n}\int_{\mathfrak{C}_n}$$

where Γ_n, γ_n are points of Γ

Hence

Now $\mathfrak{B}_n$ may not be complete,

As $\mathfrak{B}$ is complete,

We may therefore write

By 75, the last term on the

to the limit,

since $\epsilon > 0$ is small at pleasure.

On the other hand, passing to the

$n = \infty$, we get

Let us now suppose 𝔄 *is metric.* We effect a cub[ical division]
of $\mathfrak{R}_p$ of norm d, and denote by B_d those cells con[taining]
points of 𝔄. Then B_d is complete and

$$\lim_{d=0} B_d = \mathfrak{A}.$$

Let A_d denote those points of 𝔄 whose projection[s] ...
Then A_d is iterable with respect to B_d by 13, 3, an[d]
in the preceding case

$$\int_{A_d} = \int_{B_d}\int_{\mathfrak{C}} = \int_{A_d}.$$

If the middle integral in 11) is divergent, $\int_{\mathfrak{A}}$ is dive[rgent ...]
holds, also if the last integral in 11) is divergent. [Let us sup]pose then that the two last integrals in 11) are [...]
Then by 6,

$$\lim_{d=0}\int_{B_d}\int_{\mathfrak{C}} = \int_{\mathfrak{B}}\int_{\mathfrak{C}},$$

$$\lim_{d=0}\int_{A_d} = \int_{\mathfrak{A}}.$$

Thus passing to the limit $d = 0$ in 11) we get 1).

Let us now suppose $f \geq -G$, $G > 0$.

Then

$$g = f + G \geq 0,$$

and we can apply 1) to the new function g.

Thus

$$\int_{\mathfrak{A}} g = \int_{\mathfrak{B}}\int_{\mathfrak{C}} g = \int_{\mathfrak{A}} g.$$

Now

$$\int_{\mathfrak{A}} g = \int_{\mathfrak{A}} f + G\mathfrak{A},$$

by 18, 5, since 𝔄 is complete. Also by the same theo[rem]

$$\int_{\mathfrak{C}} g = \int_{\mathfrak{C}} f + G \lim \mathfrak{C} = \int_{\mathfrak{C}} f + G\Gamma,$$

But for a fixed n

$$\int_{\mathfrak{C}_n} \quad \text{is limited in } \mathfrak{B}.$$

Hence for G_n sufficiently large,

$$\int_{\mathfrak{C}_n} f \cdot \int_{\Gamma} f \ , \quad \text{at each point of } \mathfrak{B}, \qquad G_n > G. \tag{6}$$

Then

$$\int_{\mathfrak{B}_n}\int_{\mathfrak{C}_n} = \int_{\mathfrak{B}_n}\int_{\Gamma} - \int_{\mathfrak{B}_n}\left\{\int_{\Gamma_n} + \int_{\gamma_n}\right\}, \tag{7}$$

where Γ_n, γ_n are points of Γ in $\mathfrak{C}_n$, c_n.

Hence

$$\int_{\mathfrak{B}_n}\int_{\Gamma} = \int_{\mathfrak{B}_n}\int_{\mathfrak{C}_n} + \int_{\mathfrak{B}_n}\int_{\gamma_n}. \tag{8}$$

Now $\mathfrak{B}_n$ may not be complete; if not let B_n be completed $\mathfrak{B}_n$. As $\mathfrak{B}$ is complete,

$$\int_{\mathfrak{B}_n}\int_{\gamma_n} f = \int_{B_n}\int_{\gamma_n} f.$$

We may therefore write 8), using 5)

$$-\epsilon + \int_{\mathfrak{B}}\int_{\mathfrak{C}} \le \int_{\mathfrak{B}_n}\int_{\mathfrak{C}_n} + \int_{B_n}\int_{\gamma_n} \le \int_{\mathfrak{B}}\int_{\mathfrak{C}_n} + \int_{B_n}\int_{\gamma_n}.$$

By 75, the last term on the right $\doteq 0$ as $n \doteq \infty$. Thus passing to the limit,

$$\int_{\mathfrak{B}}\int_{\mathfrak{C}} f \le \lim_{n=\infty}\int_{\mathfrak{B}_n}\int_{\mathfrak{C}_n} f, \tag{9}$$

since $\epsilon > 0$ is small at pleasure.

On the other hand, passing to the limit $G = \infty$ in 7), and then $n = \infty$, we get

$$\lim_{n=\infty}\int_{\mathfrak{B}_n}\int_{\mathfrak{C}_n} \le \int_{\mathfrak{B}}\int_{\mathfrak{C}}. \tag{10}$$

Thus 8), 10), 9), and 4) give 1).

Let us now suppose that the middle term of 1) is divergent. We have as before

$$\int_{\mathfrak{B}_n}\int_{\Gamma} \ge \lim_{n=\infty}\int_{\mathfrak{B}_n}\int_{\mathfrak{C}_n} = \int_{\mathfrak{A}} f.$$

Hence the integral on the right of 1) is divergent.

Let us now suppose $\mathfrak{B}$ is metric. We effect a cubical division of $\mathfrak{R}_p$ of norm d, and denote by B_d those cells containing only points of $\mathfrak{B}$. Then B_d is complete and

$$\lim_{d=0} B_d = \mathfrak{B}.$$

Let A_d denote those points of $\mathfrak{A}$ whose projections fall on B_d. Then A_d is iterable with respect to B_d by 13, *a*, and we have as in the preceding case

$$\int_{A_d} \le \int_{B_d}\int_{\mathfrak{C}} \le \int_{A_d}. \qquad (11$$

If the middle integral in 11) is divergent, $\int_{\mathfrak{A}}$ is divergent and 1) holds, also if the last integral in 11) is divergent, 1) holds. Suppose then that the two last integrals in 11) are convergent. Then by 57

$$\lim_{d=0}\int_{B_d}\int_{\mathfrak{C}} = \int_{\mathfrak{B}}\int_{\mathfrak{C}},$$

$$\lim_{d=0}\int_{A_d} = \int_{\mathfrak{A}}.$$

Thus passing to the limit $d = 0$ in 11) we get 1).

Let us now suppose $f \geq -G$, $G > 0$.

Then

$$g = f + G \geq 0,$$

and we can apply 1) to the new function g.

Thus
$$\int_{\mathfrak{A}} g = \int_{\mathfrak{B}}\int_{\mathfrak{C}} g \le \int_{\mathfrak{A}} g. \qquad (12$$

Now
$$\int_{\mathfrak{A}} g = \int_{\mathfrak{A}} f + G\mathfrak{A}, \qquad (13$$

by 58, 6, since $\mathfrak{A}$ is discrete. Also by the same theorem,

$$\int_{\mathfrak{C}} g = \int_{\mathfrak{C}} f + G \lim_{\gamma=\infty} \mathfrak{C}_\gamma = \int_{\mathfrak{C}} f + G\Gamma, \qquad (14$$

denoting by $\mathfrak{C}_\gamma$ the points of $\mathfrak{C}$ where

$$-G \le f \le \gamma$$

and setting

$$\Gamma = \lim_{\gamma=\infty} \mathfrak{C}_\gamma.$$

Now for any n

$$\int_{\mathfrak{A}_n} G = \int_{\mathfrak{B}} \int_{\mathfrak{C}_n} G = \int_{\mathfrak{B}} G\,\mathfrak{C}_n .$$

Hence $$G\mathfrak{A} = \lim_{n=\infty} \int_{\mathfrak{B}} \int_{\mathfrak{C}_n} G = G \lim_{n=\infty} \int_{\mathfrak{B}} \mathfrak{C}_n ,$$

or $$\mathfrak{A} = \lim_{n=\infty} \int_{\mathfrak{B}} \mathfrak{C}_n . \qquad (15$$

Now for a fixed n, γ may be taken so large that for all points of $\mathfrak{B}$,

$$\mathfrak{C}_\gamma > \mathfrak{C}_n .$$

Hence

$$\mathfrak{C} \geq \lim_{\gamma=\infty} \mathfrak{C}_\gamma \geq \mathfrak{C}_n .$$

Hence $$\mathfrak{A} = \int_{\mathfrak{B}} \mathfrak{C} \geq \int_{\mathfrak{B}} \Gamma \geq \int_{\mathfrak{B}} \mathfrak{C}_n \doteq \mathfrak{A} .$$

Hence $$\mathfrak{A} = \int_{\mathfrak{B}} \Gamma , \qquad (16$$

and thus Γ is integrable in $\mathfrak{B}$.

This result in 14) gives, on using 58, 3,

$$\int_{\mathfrak{B}} \int_{\mathfrak{C}} g = \int_{\mathfrak{B}} \int_{\mathfrak{C}} f + G\mathfrak{A} . \qquad (17$$

From 12), 13), and 17) follows 1).

77. As corollaries of the last theorem we have, *supposing* $\mathfrak{A}$ *to be as in* 76,

1. *If f is integrable in $\mathfrak{A}$ and $f \geq -G$, then*

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{B}} \int_{\mathfrak{C}} f .$$

If $f \leq G$, then $$\int_{\mathfrak{A}} f = \int_{\mathfrak{B}} \int_{\mathfrak{C}} f .$$

2. *If $f \geq -G$ and $\int_{\mathfrak{A}}$ is divergent, then*

$$\int_{\mathfrak{B}} \int_{\mathfrak{C}} f$$

are divergent.

3. *If $f \geq 0$ and one of the integrals $\int_{\mathfrak{B}}\int_{\mathfrak{C}} f$ is convergent, then*

$$\int_{\mathfrak{A}} f$$

is convergent.

78. *Let $\mathfrak{A} = \mathfrak{B}\cdot\mathfrak{C}$ be iterable with respect to $\mathfrak{B}$, which last is complete or metric. Let the singular points $\mathfrak{S}$ be discrete. If*

$$\int_{\mathfrak{A}} f, \tag{1}$$

$$\int_{\mathfrak{B}}\int_{\mathfrak{C}} f, \tag{2}$$

both converge, they are equal.

For let $D_1, D_2 \cdots$ be a sequence of superimposed cubical divisions as in 72, 2. We may suppose as before that each $\mathfrak{B}_n = \mathfrak{B}$.

Since 1) is convergent

$$\epsilon > 0, \qquad n_0, \qquad \left|\int_{\mathfrak{A}} f - \int_{\mathfrak{A}_n} f\right| < \frac{\epsilon}{2} \qquad n > n_0. \tag{3}$$

Since f is limited in $\mathfrak{A}_n$, which latter is iterable,

$$\int_{\mathfrak{A}_n} f = \int_{\mathfrak{B}}\int_{\mathfrak{C}_n} f. \tag{4}$$

This shows that

$$\int_{\mathfrak{C}_n} f \tag{5}$$

is an integrable function in $\mathfrak{B}$, and hence in any part of $\mathfrak{B}$.

From 3), 4) we have

$$\left|\int_{\mathfrak{A}} - \int_{\mathfrak{B}}\int_{\mathfrak{C}_n}\right| < \frac{\epsilon}{2} \qquad n > n_0. \tag{6}$$

We wish now to show that

$$\left|\int_{\mathfrak{B}}\int_{\mathfrak{C}} - \int_{\mathfrak{B}}\int_{\mathfrak{C}_n}\right| < \frac{\epsilon}{2} \qquad n > n_0. \tag{7}$$

When this is done, 6) and 7) prove the theorem.

To establish 7) we begin by observing that

$$\int_{\mathfrak{B}}\int_{\mathfrak{C}} = \lim_{s, n=\infty}\int_{\mathfrak{B}_{ns}}\int_{\Gamma}.$$

Now for a fixed n, α, β may be taken so that Γ shall embrace all the points of $\mathfrak{E}_n$ for every point of $\mathfrak{A}$. Let us set

$$\Gamma = \mathfrak{E}_n + \gamma_n.$$

Then
$$\int_{\mathfrak{A}_{\alpha\beta}}\int_{\Gamma} = \int_{\mathfrak{A}_{\alpha\beta}}\int_{\mathfrak{E}_n} + \int_{\mathfrak{A}_{\alpha\beta}}\int_{\gamma_n} \tag{8}$$

As
$$\mathfrak{A}_{\alpha\beta} \doteq \mathfrak{A},$$
$$\lim_{\alpha,\beta=\infty}\int_{\mathfrak{A}_{\alpha\beta}}\int_{\mathfrak{E}_n} = \int_{\mathfrak{A}}\int_{\mathfrak{E}_n} \qquad \text{by I, 721.}$$

On the other hand,
$$\left|\int_{\mathfrak{A}_{\alpha\beta}}\int_{\gamma_n} f\right| \le \int_{\mathfrak{A}_{\alpha\beta}}\int_{\gamma_n} |f| \le \int_{\mathfrak{A}_{\alpha\beta}}\int_{\gamma_n} |f| \le \int_{\mathfrak{A}}\int_{\gamma_n} |f|.$$

Thus 7) is established when we show that
$$\int_{\mathfrak{A}}\int_{\gamma_n} |f| < \frac{\epsilon}{2} \qquad n > n_0 \tag{9}$$

To this end we note that $|f|$ is integrable in $\mathfrak{A}$ by 48, 4. Hence by 77, 1,
$$\int_{\mathfrak{A}} |f| = \int_{\mathfrak{A}}\int_{\mathfrak{B}} |f|. \tag{10}$$

Also by I, 734,
$$\int_{\mathfrak{A}_n} |f| = \int_{\mathfrak{A}}\int_{\mathfrak{E}_n} |f|. \tag{11}$$

From 10), 11) we have for $n > n_0$
$$\int_{\mathfrak{A}} |f| - \int_{\mathfrak{A}_n} |f| = \int_{\mathfrak{A}}\int_{\mathfrak{B}} |f| - \int_{\mathfrak{A}}\int_{\mathfrak{E}_n} |f| < \frac{\epsilon}{2} \tag{12}$$

since the left side $\doteq 0$.

But as in 8)
$$\int_{\mathfrak{A}_\beta}\int_{\Gamma} |f| = \int_{\mathfrak{A}_\beta}\int_{\mathfrak{E}_n} |f| + \int_{\mathfrak{A}_\beta}\int_{\gamma_n} |f|$$

Passing to the limit $\beta = \infty$ gives
$$\int_{\mathfrak{A}}\int_{\mathfrak{B}} |f| = \int_{\mathfrak{A}}\int_{\mathfrak{E}_n} |f| + \int_{\mathfrak{A}}\int_{\gamma_n} |f|.$$

This in 12) gives 9).

CHAPTER III

SERIES

Preliminary Definitions and Theorems

79. Let $a_1, a_2, a_3 \cdots$ be an infinite sequence of numbers.

The symbol
$$A = a_1 + a_2 + a_3 + \cdots \tag{1}$$
is called an ***infinite series.*** Let
$$A_n = a_1 + a_2 + \cdots + a_n. \tag{2}$$
If
$$\lim_{n=\infty} A_n \tag{3}$$
is finite, we say the series 1) is ***convergent.*** If the limit 3) is infinite or does not exist, we say 1) is ***divergent.*** When 1) is convergent, the limit 3) is called the ***sum*** of the series. It is customary to represent a series and its sum by the same letter, when no confusion will arise. Whenever practicable we shall adopt the following uniform notation. The terms of a series will be designated by small *Roman* letters, the series and its sum will be denoted by the corresponding capital letter. The ***sum of the first n terms*** of a series as A will be denoted by A_n. The infinite series formed by removing the first n terms, as for example,
$$a_{n+1} + a_{n+2} + a_{n+3} + \cdots \tag{4}$$
will be denoted by A_n, and will be called the ***remainder after n terms.***

The series formed by replacing each term of a series by its numerical value is called the ***adjoint series.*** We shall designate it by replacing the *Roman* letters by the corresponding Greek or German letters. Thus the adjoint of 1) would be denoted by
$$\mathfrak{A} = \mathfrak{a}_1 + \mathfrak{a}_2 + \mathfrak{a}_3 + \cdots = \text{Adj } A \tag{5}$$
where
$$\mathfrak{a}_n = |a_n|.$$

If all the terms of of a series are ≥ 0, it is identical with its adjoint.

A sum of p consecutive terms as

$$a_{n+1} + a_{n+2} + \cdots + a_{n+p}$$

we denote by $A_{n,p}$.

Let $$B = a_{\iota_1} + a_{\iota_2} + a_{\iota_3} + \cdots$$

be the series obtained from A by omitting all its terms that vanish. Then A and B converge or diverge simultaneously, and when convergent they have the same sum.

For $$B_n = A_{\iota_n}.$$

Thus if the limit on either side exists, the limit of the other side exists and both are equal.

This shows that in an infinite series we may omit its zero terms without affecting its character or value. We shall suppose this done unless the contrary is stated.

A series whose terms are all ≥ 0 we shall call a *positive term series*; similarly if its terms are all ≤ 0, we call it a *negative term series*. If $a_n \geq 0$, $n \geq m$ we shall say the series is *essentially a positive term series*. Similarly if $a_n \leq 0$, $n \geq m$ we call it an *essentially negative term series*.

If A is an essentially positive term series and divergent, $\lim A_n = +\infty$; if it is an essentially negative term series and divergent, $\lim A_n = -\infty$.

When $\lim A_n = \pm\infty$, we sometimes say A is $\pm\infty$.

80. 1. *For A to converge, it is necessary and sufficient that*

$$\epsilon > 0, \quad m, \quad |A_{n,p}| < \epsilon, \quad n \geq m, \quad p = 1, 2, \ldots \tag{1}$$

For the necessary and sufficient condition that

$$\lim_{n=\infty} A_n$$

exists is $$\epsilon > 0, \quad m, \quad |A_\nu - A_n| < \epsilon, \quad \nu, n \geq m \tag{2}$$

But if $\nu = n + p$

$$A_\nu - A_n = A_{n,p} = a_{n+1} + a_{n+2} + \cdots + a_{n+p}.$$

Thus 2) is identical with 1).

2. *The two series A, $\bar{A}_n$ converge and diverge simultaneously. When convergent,*

$$A = A_n + \bar{A}_n. \quad (3$$

For obviously if either series satisfies theorem 1, the other must, since the first terms of a series do not enter the relation 1). On the other hand,

$$A_{n+p} = A_n + A_{n,p}.$$

Letting $p \doteq \infty$ we get 3).

3. *If A is convergent, $\bar{A}_n \doteq 0$.*

For

$$\lim \bar{A}_n = \lim (A - A_n)$$
$$= A - \lim A_n = A - A$$
$$= 0.$$

For A to converge it is necessary that $a_n \doteq 0$.

For in 1) take $p = 1$; it becomes

$$|a_{n+1}| < \epsilon \qquad n > m$$

We cannot infer conversely because $a_n \doteq 0$, therefore A is convergent. For as we shall see in 81, 2,

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots$$

is divergent, yet $\lim a_n = 0$.

4. *The positive term series A is convergent if A_n is limited.*

For then $\lim A_n$ exists by I, 109.

5. *A series whose adjoint converges is convergent.*

For the adjoint A of A being convergent,

$$\epsilon > 0, \ m, \ |\mathrm{A}_{n,p}| < \epsilon, \ n > m, \ p = 1, 2, 3 \cdots$$

But

$$\mathrm{A}_{n,p} = |a_{n+1}| + |a_{n+2}| + \cdots + |a_{n+p}| \geq |a_{n+1} + a_{n+2} + \cdots + a_{n+p}| = |A_{n,p}|.$$

Thus

$$|A_{n,p}| < \epsilon$$

and A is convergent.

Definition. A series whose adjoint is convergent is called *absolutely convergent.*

Series which do not converge absolutely may be called, when necessary to emphasize this fact, *simply convergent.*

6. *Let*

$$A = a_1 + a_2 + \cdots$$

be absolutely convergent.

Let

$$B = a_{\iota_1} + a_{\iota_2} + \cdots \quad ; \quad \iota_1 < \iota_2 < \cdots$$

be any series whose terms are taken from A, preserving their relative order. Then B is absolutely convergent and

$$|B| \leq \mathfrak{A}.$$

For

$$|B_m| \leq \mathfrak{B}_m \leq \mathfrak{A}_n \leq \mathfrak{A}, \tag{1}$$

choosing n so large that $\mathfrak{A}_n$ contains every term in $\mathfrak{B}_m$. Moreover for $m >$ some m', $\mathfrak{A}_n - \mathfrak{B}_m =$ some term of $\mathfrak{A}$. Thus passing to the limit in 1), the theorem is proved.

7. *Let* $A = a_1 + a_2 + \cdots$ *The series* $B = ka_1 + ka_2 + \cdots$, $k \neq 0$, *converges or diverges simultaneously with* A. *When convergent,*

$$B = kA.$$

For

$$B_n = kA_n.$$

We have now only to pass to the limit.

From this we see that a negative or an essentially negative term series can be converted into a positive or an essentially positive term series by multiplying its terms by $k = -1$.

8. *If A is simply convergent, the series B formed of its positive terms taken in the order they occur in A, and the series C formed of the negative terms, also taken in the order they occur in A, are both divergent.*

If B and C are convergent, so are $\mathfrak{B}$, $\mathfrak{C}$. Now

$$\mathfrak{A}_n = \mathfrak{B}_{n_1} + \mathfrak{C}_{n_2}, \qquad n = n_1 + n_2.$$

Hence $\mathfrak{A}$ would be convergent, which is contrary to hypothesis. If only one of the series B, C is convergent, the relation

$$A_n = B_{n_1} + C_{n_2}$$

shows that A would be divergent, which is contrary to hypothesis.

9. The following theorem often affords a convenient means of estimating the remainder of an absolutely convergent series.

Let $A = a_1 + a_2 + \cdots$ *be an absolutely convergent series. Let* $B = b_1 + b_2 + \cdots$ *be a positive term convergent series whose sum is known either exactly or approximately. Then if* $|a_n| \leq b_n$, $n \geq m$

$$|\bar{A}_n| \leq \bar{B}_n < B.$$

For

$$\begin{aligned} |A_{n,p}| &\leq \alpha_{n+1} + \cdots + \alpha_{n+p} \\ &\leq b_{n+1} + \cdots + b_{n+p} \\ &< B_n < B. \end{aligned}$$

Letting $p \doteq \infty$ gives the theorem.

EXAMPLES

81. 1. *The geometric series* is defined by

$$G = 1 + g + g^2 + g^3 + \cdots \tag{1}$$

The geometric series is absolutely convergent when $|g| < 1$ *and divergent when* $|g| \overline{>} 1$. *When convergent,*

$$G = \frac{1}{1-g}. \tag{2}$$

When $g \neq 1$,

$$\frac{1}{1-g} = 1 + g + g^2 + \cdots + g^{n-1} + \frac{g^n}{1-g}.$$

Hence

$$G_n = \frac{1}{1-g} - \frac{g^n}{1-g}.$$

When $|g| < 1$, $\lim g^n = 0$, and then

$$\lim G_n = \frac{1}{1-g}.$$

When $|g| \geq 1$, $\lim g^n$ is not 0, and hence by 80, 3, G is not convergent.

2. The series
$$H = 1 + \frac{1}{2^\mu} + \frac{1}{3^\mu} + \frac{1}{4^\mu} + \cdots \tag{3}$$

is called the *general harmonic series of exponent* μ. When $\mu = 1$, it becomes

$$J = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots \qquad (4$$

the *harmonic series*. We show now that

The general harmonic series is convergent when $\mu > 1$ *and is divergent for* $\mu \leq 1$.

Let $\mu > 1$. Then

$$\frac{1}{2^\mu} + \frac{1}{3^\mu} < \frac{1}{2^\mu} + \frac{1}{2^\mu} < \frac{2}{2^\mu} = \frac{1}{2^{\mu-1}} = g \quad ; \quad g < 1.$$

$$\frac{1}{4^\mu} + \frac{1}{5^\mu} + \frac{1}{6^\mu} + \frac{1}{7^\mu} < \frac{1}{4^\mu} + \frac{1}{4^\mu} + \frac{1}{4^\mu} + \frac{1}{4^\mu} = \frac{4}{4^\mu} = g^2.$$

$$\frac{1}{8^\mu} + \cdots + \frac{1}{15^\mu} < \frac{1}{8^\mu} + \frac{1}{8^\mu} + \cdots + \frac{1}{8^\mu} = \frac{8}{8^\mu} = g^3, \text{ etc.}$$

Let $n \leq 2^\nu$. Then

$$H_n < 1 + g + \cdots + g^\nu < \frac{1}{1-g}.$$

Thus $\lim H_n$ exists, by I, 109, and

$$H < \frac{1}{1 - \dfrac{1}{2^{\mu-1}}}. \qquad (5$$

Let $\mu < 1$. Then

$$\frac{1}{n^\mu} > \frac{1}{n}.$$

Thus 3) is divergent for $\mu < 1$, if it is for $\mu = 1$.

But we saw, I, 141, that

$$\lim J_n = \infty,$$

hence J is divergent.

It is sometimes useful to know that

$$\lim \frac{J_n}{\log n} = 1. \qquad (6$$

In fact, by I, 180,

$$\lim \frac{J_n}{\log n} = \lim \frac{J_n - J_{n-1}}{\log n - \log(n-1)} = \lim \frac{\dfrac{1}{n}}{\log\left(\dfrac{n}{n-1}\right)}$$

$$= \lim \frac{1}{\log\left(1 - \dfrac{1}{n}\right)^{-n}} = 1.$$

Since* $n > \log n > l_2 n > \cdots$ we have

$$\lim \frac{l_s}{n} = 0 \quad ; \quad \lim \frac{l_s}{l_r n} = 0 \quad , \quad r > 1. \tag{7}$$

Another useful relation is

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} > \log(n+1). \tag{8}$$

For $\qquad \log(1+m) - \log m = \log\left(1 + \frac{1}{m}\right) < \frac{1}{m}.$

Let $m = 1, 2, \ldots n$. If we add the resulting inequalities we get 8).

3. *Alternating Series.* This important class of series is defined as follows. Let $a_1 \geq a_2 \geq a_3 \geq \cdots \doteq 0$.

Then
$$A = a_1 - a_2 + a_3 - a_4 + \cdots \tag{9}$$
whose signs are alternately positive and negative, is such a series.

The alternating series 9) is convergent and

$$|A_n| \leq a_{n+1}. \tag{10}$$

For let $p > 3$. We have

$$A_{n,p} = (-1)^n \{a_{n+1} - a_{n+2} + \cdots (-1)^{p-1} a_{n+p}\}$$
$$= (-1)^n P.$$

If p is even,

$$P = (a_{n+1} - a_{n+2}) + \cdots + (a_{n+p-1} - a_{n+p}).$$

If p is odd,

$$P = (a_{n+1} - a_{n+2}) + \cdots + (a_{n+p-2} - a_{n+p-1}) + a_{n+p}.$$

Thus in both cases,
$$P \geq a_{n+1} - a_{n+2} \geq 0. \tag{11}$$
Again, if p is even,

$$P = a_{n+1} - (a_{n+2} - a_{n+3}) - \cdots - (a_{n+p-2} - a_{n+p-1}) - a_{n+p}.$$

* In I, 401, the symbol "lim" in the first relation should be replaced by $\underline{\lim}$.

Example 3. $$A = \sum_1^\infty \frac{1}{(x+n-1)(x+n)} \qquad x \neq 0, -1, -2, \cdots$$

$$= \sum \left\{ \frac{1}{x+n-1} - \frac{1}{x+n} \right\}$$

is telescopic.

$$A_n = \frac{1}{x} - \frac{1}{x+n} \doteq \frac{1}{x}.$$

82. *Dini's Series. Let $A = a_1 + a_2 + \cdots$ be a divergent positive term series. Then*

$$D = \frac{a_1}{A_1} + \frac{a_2}{A_2} + \frac{a_3}{A_3} + \cdots$$

is divergent.

For
$$D_{m,p} = \frac{a_{m+1}}{A_{m+1}} + \cdots + \frac{a_{m+p}}{A_{m+p}}$$

$$> \frac{1}{A_{m+p}}(a_{m+1} + \cdots + a_{m+p})$$

$$> \frac{A_{m,p}}{A_m + A_{m,p}} = 1 - \frac{A_m}{A_{m+p}}.$$

Letting m remain fixed and $p \doteq \infty$, we have $\bar{D}_m \geq 1$, since $A_{m+p} \doteq \infty$. Hence D is divergent.

Let
$$A = 1 + 1 + 1 + \cdots \quad \text{Then } A_n = n.$$

Hence
$$D = \tfrac{1}{1} + \tfrac{1}{2} + \tfrac{1}{3} + \cdots \text{ is divergent.}$$

Let
$$A = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots$$

Then
$$D = \frac{1}{1} + \frac{1}{2(1+\frac{1}{2})} + \frac{1}{3(1+\frac{1}{2}+\frac{1}{3})} + \cdots = \sum \frac{1}{nA_n}$$

is divergent, and hence, *a fortiori*,

$$\sum \frac{1}{nA_{n-1}}.$$

But
$$A_{n-1} > \log n.$$

Hence
$$\sum_2^\infty \frac{1}{n \log n} = \frac{1}{2 \log 2} + \frac{1}{3 \log 3} + \cdots$$

is divergent, as *Abel* first showed.

83. 1. *Abel's Series.*

An important class of series have the form

$$B = a_1t_1 + a_2t_2 + a_3t_3 + \cdots \tag{1}$$

As Abel first showed how the convergence of certain types of these series could be established, they may be appropriately called in his honor. The reasoning depends on the simple identity (*Abel's identity*),

$$B_{n,p} = t_{n+1}A_{n,1} + t_{n+2}(A_{n,2} - A_{n,1}) + \cdots + t_{n+p}(A_{n,p} - A_{n,p-1})$$

$$= A_{n,1}(t_{n+1} - t_{n+2}) + \cdots + A_{n,p-1}(t_{n+p-1} - t_{n+p}) + t_{n+p}A_{n,p}, \tag{2}$$

where as usual $A_{n,m}$ is the sum of the first m terms of the remainder series A_n. From this identity we have at once the following cases in which the series 1) converges.

2. *Let the series* $A = a_1 + a_2 + \cdots$ *and the series* $\Sigma|t_{n+1} - t_n|$ *converge. Let the* t_n *be limited. Then* $B = a_1t_1 + a_2t_2 + \cdots$ *converges.*

For since A is convergent, there exists an m such that

$$|A_{n,p}| < \epsilon; \qquad n > m, \qquad p = 1, 2, 3 \ldots$$

Hence

$$|B_{n,p}| < \epsilon \{|t_{n+1} - t_{n+2}| + |t_{n+2} - t_{n+3}| + \cdots + |t_{n+p}|\}.$$

3. *Let the series* $A = a_1 + a_2 + \cdots$ *converge. Let* $t_1, t_2, t_3 \cdots$ *be a limited monotone sequence. Then* B *is convergent.*

This is a corollary of 2.

4. *Let* $A = a_1 + a_2 + \cdots$ *be such that* $|A_n| < G$, $n = 1, 2, \cdots$ *Let* $\Sigma|t_{n+1} - t_n|$ *converge and* $t_n \doteq 0$. *Then* B *is convergent.*

For by hypothesis there exists an m such that

$$|t_{n+1} - t_{n+2}| + |t_{n+2} - t_{n+3}| + \cdots + |t_{n+p}| < \epsilon$$

for any $n > m$.

5. *Let* $|A_n| < G$ *and* $t_1 \geq t_2 \geq t_3 \geq \cdots \doteq 0$. *Then* B *is convergent.*

This is a special case of 4.

6. As an application of 5 we see the alternating series

$$B = t_1 - t_2 + t_3 - \cdots$$

is convergent. For as the A series we may take $A = 1 - 1 + 1 - 1 + \cdots$ as $|A_n| \leq 1$.

84. *Trigonometric Series.*

Series of this type are

$$C = a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \cdots \tag{1}$$

$$S = a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \cdots \tag{2}$$

As we see, they are special cases of Abel's series. Special cases of the series 1), 2) are

$$\Gamma = \tfrac{1}{2} + \cos x + \cos 2x + \cos 3x + \cdots \tag{3}$$

$$\Sigma = \sin x + \sin 2x + \sin 3x + \cdots \tag{4}$$

It is easy to find the sums Γ_n, Σ_n as follows. We have

$$2 \sin mx \sin \tfrac{1}{2} x = \cos \frac{2m-1}{2} x - \cos \frac{2m+1}{2} x.$$

Letting $m = 1, 2, \cdots n$ and adding, we get

$$2 \sin \tfrac{1}{2} x \cdot \Sigma_n = \cos \tfrac{1}{2} x - \cos \frac{2n+1}{2} x. \tag{5}$$

Keeping x fixed and letting $n \doteq \infty$, we see Σ_n oscillates between fixed limits when $x \neq 0, \pm 2\pi, \cdots$

Thus Σ is divergent except when $x = 0, \pm \pi, \cdots$

Similarly we find when $x \neq 2m\pi$,

$$\Gamma_n = \frac{\sin (n - \frac{1}{2})x}{2 \sin \frac{1}{2} x}. \tag{6}$$

Hence for such values Γ_n oscillates between fixed limits. For the values $x = 2m\pi$ the equation 3) shows that $\Gamma_n \doteq +\infty$.

From the theorems 4, 5 we have at once now

If $\Sigma |a_{n+1} - a_n|$ converges and $a_n \doteq 0$, and hence in particular if $a_1 \geq a_2 \geq \cdots \doteq 0$, the series 1) converges for every x, and 2) converges for $x \neq 2m\pi$.

If in 3) we replace x by $x + \pi$, it goes over into

$$\Delta = \tfrac{1}{2} - \cos x + \cos 2x - \cos 3x + \cdots \tag{7}$$

Thus Δ_n oscillates between fixed limits if $x \neq \pm (2m-1)\pi$, when $n \doteq \infty$. Thus

If $\Sigma |a_{n+1} + a_n|$ converges and $a_n \doteq 0$, and hence in particular if $a_1 \geq a_2 \cdots \doteq 0$, the series $a_0 - a_1 \cos x + a_2 \cos 2x - a_3 \cos 3x + \cdots$ converges for $x \neq (2m-1)\pi$.

85. *Power Series.*

An extremely important class of series are those of the type

$$P = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \cdots \qquad (1$$

called power series. Since P reduces to a_0 if we set $x = a$, we see that every power series converges for at least one point. On the other hand, there are power series which converge at but one point, *e.g.*

$$a_0 + 1!(x-a) + 2!(x-a)^2 + 3!(x-a)^3 + \cdots \qquad (2$$

For if $x \neq a$, $\lim n!|x-a|^n = \infty$, and thus 2) is divergent.

1. *If the power series P converges for $x = b$, it converges absolutely within*

$$D_\lambda(a) \quad , \quad \lambda = |a - b|.$$

If P diverges for $x = b$, it diverges without $D_\lambda(a)$.

Let us suppose first that P converges at b. Let x be a point in D_λ, and set $|x - a| = \xi$. Then the adjoint of P becomes for this point

$$\begin{aligned} \Pi &= \alpha_0 + \alpha_1\xi + \alpha_2\xi^2 + \alpha_3\xi^3 + \cdots \\ &= \alpha_0 + \alpha_1\lambda \cdot \frac{\xi}{\lambda} + \alpha_2\lambda^2 \cdot \left(\frac{\xi}{\lambda}\right)^2 + \alpha_3\lambda^3 \cdot \left(\frac{\xi}{\lambda}\right)^3 + \cdots \end{aligned}$$

But

$$\lim \alpha_n\lambda^n = 0,$$

since series P is convergent for $x = b$.

Hence

$$\alpha_n\lambda^n < M \qquad n > 1, 2, \cdots$$

Thus

$$\Pi_n < M\left(1 + \frac{\xi}{\lambda} + \cdots + \frac{\xi^n}{\lambda^n}\right) < \frac{M}{1 - \dfrac{\xi}{\lambda}}$$

and Π is convergent.

If P diverges at $x = b$, it must diverge for all b' such that $|b' - a| > \lambda$. For if not, P would converge at b by what we have just proved, and this contradicts the hypothesis.

2. Thus we conclude that the set of points for which P converges form an interval $(a-\rho,\ a+\rho)$ about the point a, called the *interval of convergence*; ρ is called its norm. We say P is *developed about the point a*. When $a=0$, the series 1) takes on the simpler form

$$a_0+a_1x+a_2x^2+\cdots$$

which for many purposes is just as general as 1). We shall therefore employ it to simplify our equations.

We note that the geometric series is a simple case of a power series.

86. *Cauchy's Theorem on the Interval of Convergence.*

The norm ρ of the interval of convergence of the power series,

$$P=a_0+a_1x+a_2x^2+\cdots$$

is given by

$$\frac{1}{\rho}=\overline{\lim}\sqrt[n]{\alpha_n}\qquad \alpha_n=|a_n|.$$

We show Π diverges if $\xi>\rho$. For let

$$\frac{1}{\rho}>\beta>\frac{1}{\xi}.$$

Then by I, 338, 1, there exist an infinity of indices $\iota_1,\ \iota_2\cdots$ for which

$$\sqrt[\iota_n]{\alpha_{\iota_n}}>\beta.$$

Hence

$$\alpha_{\iota_n}>\beta^{\iota_n},$$

and thus

$$\alpha_{\iota_n}\xi^{\iota_n}>(\xi\beta)^{\iota_n}>1,$$

since $\xi\beta>1$. Hence

$$\sum_n \alpha_{\iota_n}\xi^{\iota_n}$$

is divergent and therefore Π.

We show now that Π converges if $\xi<\rho$. For let

$$\xi<\frac{1}{\beta}<\rho.$$

Then there exist only a finite number of indices for which

$$\sqrt[n]{\alpha_n}>\beta.$$

Let m be the greatest of these indices. Then

$$\sqrt{\alpha_n}<\beta\qquad n>m.$$

Hence

$$a_n < \beta^n,$$

and

$$a_n\xi^n < (\beta\xi)^n.$$

Thus

$$a_{m+1}\xi^{m+1} + \cdots a_{m+p}\xi^{m+p} < (\beta\xi)^{m+1}\{1 + (\beta\xi) + \cdots + (\beta\xi)^{m+p-1}\}$$

$$< \frac{(\beta\xi)^{m+1}}{1-\beta\xi},$$

and H is convergent.

Example 1.

$$1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Here

$$\sqrt[n]{a_n} = \frac{1}{\sqrt[n]{n!}} \doteq 0 \qquad \text{by I, 185, 4.}$$

Hence $\rho = \infty$ and the series converges absolutely for every x.

Example 2.

$$\frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

Here

$$\sqrt[n]{a_n} = \frac{1}{\sqrt[n]{n}} \doteq 1 \qquad \text{by I, 185, 3.}$$

Hence $\rho = 1$, and the series converges absolutely for $|x| < 1$.

Tests of Convergence for Positive Term Series

87. To determine whether a given positive term series

$$A = a_1 + a_2 + \cdots$$

is convergent or not, we may compare it with certain standard series whose convergence or divergence is known. Such comparisons enable us also to establish criteria of convergence of great usefulness.

We begin by noting the following theorem which sometimes proves useful.

1. *Let A, B be two series which differ only by a finite number of terms. Then they converge or diverge simultaneously.*

This follows at once from 80, 2. Hence if a series A whose convergence is under investigation has a certain property only

after the mth term, we may replace A by A_m, which has this property from the start.

2. The *fundamental theorem* of comparison is the following:

Let $A = a_1 + a_2 + \cdots$, $B = b_1 + b_2 + \cdots$ *be two positive term series. Let* $r > 0$ *denote a constant. If* $a_n \leq rb_n$, A *converges if* B *does and* $A \leq rB$. *If* $a_n \geq rb_n$, A *diverges if* B *does.*

For on the first hypothesis

$$A_n \leq rB_n.$$

On the second hypothesis

$$A_n \geq rB_n.$$

The theorem follows on passing to the limit.

3. From 2 we have at once:

Let $A = a_1 + a_2 + \cdots$, $B = b_1 + b_2 + \cdots$ *be two positive term series. Let* r, s *be positive constants. If*

$$r \leq \frac{a_n}{b_n} \leq s \qquad n = 1, 2, \cdots$$

or if

$$\lim \frac{a_n}{b_n}$$

exists and is $\neq 0$, A *and* B *converge or diverge simultaneously. If* B *converges and* $\frac{a_n}{b_n} \doteq 0$, A *also converges. If* B *diverges and* $\frac{a_n}{b_n} \doteq \infty$, A *also diverges.*

4. *Let* $A = a_1 + a_2 + \cdots$, $B = b_1 + b_2 + \cdots$ *be positive term series. If* B *is convergent and*

$$\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n} \qquad n = 1, 2, 3 \cdots$$

A *converges. If* B *is divergent and*

$$\frac{a_{n+1}}{a_n} \geq \frac{b_{n+1}}{b_n},$$

A *diverges.*

For on the first hypothesis

$$\frac{a_{n+1}}{b_{n+1}} \leq \frac{a_n}{b_n} \leq \cdots \leq \frac{a_1}{b_1}.$$

We may, therefore, apply 3. On the second hypothesis, we have

$$\frac{a_n}{b_n} < \frac{a_1}{b_1}$$

and we may again apply 3.

Example 1. $$A = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots$$
is convergent. For

$$a_n = \frac{1}{n\cdot \overline{n+1}} < \frac{1}{n^2}$$

and $\sum \frac{1}{n^2}$ is convergent. The series A was considered in 81, 4, Ex. 1.

Example 2. $$A = e^{-x}\cos x + e^{-2x}\cos 2x + \cdots$$
is absolutely convergent for $x > 0$.

For $$|a_n| < \frac{1}{e^{nx}}$$

which is thus $<$ the nth term in the convergent geometric series

$$\frac{1}{e^x} + \frac{1}{e^{2x}} + \frac{1}{e^{3x}} + \cdots$$

Example 3. $$A = \sum \frac{1}{n}\log\frac{n+1}{n}$$
is convergent.

For $$\log\left(1+\frac{1}{n}\right) = \frac{1}{n} - \frac{\theta_n}{n^2}, \qquad 0 < \theta_n < 1.$$

Hence $$0 < a_n = \frac{1}{n^2}\left(1 - \frac{\theta_n}{n}\right) < \frac{1}{n^2}.$$

Thus A is comparable with the convergent series $\sum \frac{1}{n^2}$.

88. We proceed now to deduce various tests for convergence and divergence. One of the simplest is the following, obtained by comparison with the hyperharmonic series.

Let $A = a_1 + a_2 + \cdots$ be a positive term series. It is convergent if

$$\overline{\lim}\, a_n n^\mu < \infty \quad , \quad \mu > 1,$$

and divergent if

$$\underline{\lim}\, n a_n > 0.$$

For on the first hypothesis there exists, by I, 338, a constant $G > 0$ such that

$$a_n < \frac{G}{n^\mu} \qquad n = 1, 2, \cdots$$

Thus each term of A is less than the corresponding term of the convergent series $G \sum \frac{1}{n^\mu}$.

On the second hypothesis there exists a constant c such that

$$a_n > \frac{c}{n} \qquad n = 1, 2, \cdots$$

and each term of A is greater than the corresponding term of the divergent series $c \sum \frac{1}{n}$.

Example 1. $$A = \sum \frac{1}{\log^m n} \qquad m > 0.$$

Here $$na_n = \frac{n}{\log^m n} \doteq +\infty, \quad \text{by I, 463.}$$

Hence A is divergent.

Example 2. $$A = \sum \frac{1}{n \log n}.$$

Here $$na_n = \frac{1}{\log n} \doteq 0.$$

Thus the theorem does not apply. The series is divergent by 82.

Example 3.

$$L = \Sigma l_n = \Sigma \log\left(1 + \frac{\mu}{n} + \frac{\theta_n}{n^r}\right) \quad , \quad r > 1,$$

where μ is a constant and $|\theta_n| < G$.

From I, 413, we have, setting $r = 1 + s$,

$$l_n = \frac{1}{n}\left(\mu + \frac{\theta_n}{n^s}\right) - \frac{\alpha_n}{n^2}\left(\mu + \frac{\theta_n}{n^s}\right)^2 \qquad 0 < \alpha_n < 1.$$

Hence $$nl_n \doteq \mu \quad , \quad \text{if } \mu \neq 0,$$

and L is divergent. If $\mu > 0$, L is an essentially positive term series. Hence $L = +\infty$. If $\mu < 0$, $L = -\infty$.

Let $\mu = 0$. Then

$$l_n = \frac{\theta_n}{n^r}\left(1 - \frac{\alpha_n \theta_n}{n^r}\right) \qquad 0 < \alpha_n < 1,$$

which is comparable with the convergent series

$$\sum \frac{1}{n^r} \quad , \quad r > 1.$$

Thus L is convergent in this case.

Example 4. The harmonic series

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots$$

is divergent. For

$$\lim n a_n = 1.$$

Example 5.

$$A = \sum \frac{1}{n^\alpha \log^\beta n} \qquad \beta \text{ arbitrary.}$$

Here

$$n a_n = \frac{n^{1-\alpha}}{\log^\beta n} \doteq \infty \quad , \quad \alpha < 1$$

by I, 463, 1. Hence A is divergent for $\alpha < 1$.

Example 6.

$$A = \sum \frac{1}{\sqrt[n]{n^{n+1}}}.$$

Here

$$n a_n = \frac{1}{\sqrt[n]{n}} \doteq 1 \qquad \text{by I, 185, Ex. 8.}$$

Example 7.

$$A = \sum \left(\frac{\log(n+1)}{\log n} - 1 \right).$$

Here, if $\mu > 0$,

$$n^{1+\mu} a_n = n^{1+\mu} \frac{\log\left(1 + \frac{1}{n}\right)}{\log n} = \frac{n^\mu}{\log n} \log\left(1 + \frac{1}{n}\right)^n \doteq \infty,$$

since $n^\mu > \log n$ and $\left(1 + \frac{1}{n}\right)^n \doteq e$.

Hence A is divergent.

89 *D'Alembert's Test. The positive term series $A = a_1 + a_2 + \cdots$ converges if there exists a constant $r < 1$ for which*

$$\frac{a_{n+1}}{a_n} \leq r, \qquad n = 1, 2, \cdots$$

It diverges if

$$\frac{a_{n+1}}{a_n} \geq 1.$$

This follows from 87, 4, taking for B the geometric series $1 + r + r^2 + r^3 + \cdots$

Corollary. Let $\frac{a_{n+1}}{a_n} \doteq l$. *If* $l < 1$, A *converges. If* $l > 1$, *it diverges.*

Example 1. The Exponential Series.

Let us find for what values of x the series

$$E = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{1}$$

is convergent. Applying D'Alembert's test to its adjoint, we find

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^n}{n!} \cdot \frac{n-1!}{x^{n-1}}\right| = \frac{|x|}{n}.$$

Thus E converges absolutely for every x.

Let us employ 80, 9 to estimate the remainder $\overline{E}_n$. *Let* $x > 0$. The terms of E are all > 0. Since

$$\frac{x^{n+p}}{(n+p)!} = \frac{x^n}{n!} \cdot \frac{x^p}{n+1 \cdot n+2 \cdot \cdots n+p} \leq \frac{x^n}{n!}\left(\frac{x}{n+1}\right)^p,$$

we have

$$\frac{x^n}{n!} < \overline{E}_n < \frac{x^n}{n!} \sum_0^\infty \left(\frac{x}{n+1}\right)^p. \tag{2}$$

However large x may be, we may take n so large that $x < n + 1$. Then the series on the right of 2) is a convergent geometric series.

Let $x < 0$. Then however large $|x|$ is, $\overline{E}_n$ is alternating for some m. Hence by 81, 3 for $n \geq m$,

$$|\overline{E}_n| < \frac{|x|^n}{n!} \tag{3}$$

Example 2. The Logarithmic Series.

Let us find for what values of x the series

$$L = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

is convergent. The adjoint gives

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n}{n+1} \cdot |x| \doteq |x|.$$

Thus L converges absolutely for any $|x| < 1$, and diverges for $|x| > 1$.

When $x = 1$, L becomes

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots$$

which is simply convergent by 81, 4.

When $x = -1$, L becomes

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots$$

which is the divergent harmonic series.

Example 3. $A = \dfrac{1}{1^\mu} + \dfrac{1}{2^\mu} + \dfrac{1}{3^\mu} + \cdots$

$$\frac{a_{n+1}}{a_n} = \left(\frac{n}{n+1}\right)^\mu \doteq 1.$$

As A is convergent when $\mu > 1$ and divergent if $\mu \leq 1$, we see that D'Alembert's test gives us no information when $l = 1$. It is, however, convergent for this case by 81, 2.

Example 4.

$$\sum_1^\infty \frac{n!}{(1+x)\cdots(n+x)} \qquad x > 0.$$

Here

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{n+1+x} \doteq 1,$$

and D'Alembert's test does not apply.

Example 5.

$$A = \Sigma n^\mu x^n.$$

Here

$$\left|\frac{a_{n+1}}{a_n}\right| = \left(\frac{n+1}{n}\right)^\mu |x| \doteq |x|.$$

Thus A converges for $|x| < 1$ and diverges for $|x| > 1$. For $|x| = 1$ the test does not apply. For $x = 1$ we know by 81, 2 that A is convergent for $\mu < -1$, and is divergent for $\mu > -1$.

For $x = -1$, A is divergent for $\mu \geq 0$, since a_n does not $\doteq 0$. A is an alternating series for $\mu < 0$, and is then convergent.

90. *Cauchy's Radical Test. Let $A = a_1 + a_2 + \cdots$ be a positive term series. If there exists a constant $r < 1$ such that*

$$\sqrt[n]{a_n} < r \qquad n = 1, 2, \cdots$$

A is convergent. If, on the other hand,

$$\sqrt[n]{a_n} \geq 1$$

A is divergent.

For on the first hypothesis,

$$a_n < r^n$$

so that each term of A is $<$ the corresponding term in $r + r^2 + r^3 + \cdots$ a convergent geometric series. On the second hypothesis, this geometric series is divergent and $a_n \geq r^n$.

Corollary. If $\lim \sqrt[n]{a_n} = l$, *and $l < 1$, A is convergent. If $l > 1$, A is divergent.*

Example 1. The series

$$\sum_2^\infty \frac{1}{\log^n n} = a_2 + a_3 + \cdots$$

is convergent. For

$$\sqrt[n]{a_n} = \frac{1}{\log n} \doteq 0.$$

Example 2.

$$A = \sum \frac{n^{n^2}}{(n+1)^{n^2}}$$

is convergent. For

$$\sqrt[n]{a_n} = \frac{n^n}{(n+1)^n} = \frac{1}{\left(1 + \frac{1}{n}\right)^n} \doteq \frac{1}{e} < 1.$$

Example 3. In the elliptic functions we have to consider series of the type

$$\theta(v) = 1 + 2 \sum_1^\infty q^{n^2} \cos 2\pi n v \qquad 0 < q < 1.$$

This series converges absolutely if

$$q + q^4 + q^9 + \cdots$$

does. But here

$$\sqrt[n]{a_n} = \sqrt[n]{q^{n^2}} = q^n \doteq 0.$$

Thus $\theta(v)$ converges absolutely for every v.

Example 4. Let $0 < a < b < 1$. The series

$$A = a + b^2 + a^3 + b^4 + \cdots$$

is convergent. For if

$$n = 2m$$

$$\sqrt[n]{a_n} = \sqrt[2m]{b^{2m}} = b.$$

If $n = 2m + 1$,

$$\sqrt[n]{a_n} = \sqrt[2m+1]{a^{2m+1}} = a.$$

Thus for all n

$$\sqrt[n]{a_n} \leq b < 1.$$

Let us apply D'Alembert's test. Here

$$\frac{a_{n+1}}{a_n} = b\left(\frac{b}{a}\right)^{2m-1} \doteq \infty \qquad n = 2m + 1,$$

$$= a\left(\frac{a}{b}\right)^{2m} \doteq 0 \qquad n = 2m.$$

Thus the test gives us no information.

91. *Cauchy's Integral Test.*

Let $\phi(x)$ be a positive monotone decreasing function in the interval (a, ∞). The series

$$\Phi = \phi(1) + \phi(2) + \phi(3) + \cdots$$

is convergent or divergent according as

$$\int_a^\infty \phi(x)\,dx$$

is convergent or divergent.

For in the interval $(n, n + 1)$, $\quad n > m \geq a$,

$$\phi(n + 1) \leq \phi(x) \leq \phi(n).$$

Hence

$$\phi(n+1) \leq \int_n^{n+1} \phi dx \leq \phi(n).$$

Letting $n = m, \ m+1, \cdots m+p$, and adding, we have

$$\Phi_{m,\,p+1} \leq \int_m^{m+p} \phi dx \leq \Phi_{m-1,\,p+1}.$$

Passing to the limit $p = \infty$, we get

$$\overline{\Phi}_m \leq \int_m^\infty \phi dx \leq \overline{\Phi}_{m-1}, \tag{1}$$

which proves the theorem.

Corollary. When Φ is convergent

$$\overline{\Phi}_m \leq \int_m^\infty \phi dx.$$

Example 1. We can establish at once the results of 81, 2. For, taking $\phi(x) = \frac{1}{x^\mu}$,

$$\int_1^\infty \phi dx = \int_1^\infty \frac{dx}{x^\mu}$$

is convergent or divergent according as $\mu > 1$, or $\mu \leq 1$, by I, 635, 636.

We also note that if

$$A = \frac{1}{1^{1+\mu}} + \frac{1}{2^{1+\mu}} + \frac{1}{3^{1+\mu}} + \cdots$$

then

$$\overline{A} < \int_n^\infty \frac{1}{x^{1+\mu}} = \frac{1}{\mu} \cdot \frac{1}{n^\mu}.$$

Example 2. The *logarithmic series*

$$\sum_n \frac{1}{nl_1nl_2n \cdots l_{s-1}nl_s^\mu n} \qquad s = 1, 2, \cdots$$

are convergent if $\mu > 1$; divergent if $\mu \leq 1$.

We take here

$$\phi(x) = \frac{1}{xl_1x \cdots l_{s-1}xl_s^\mu x}$$

and apply I, 637, 638.

92. 1. One way, as already remarked, to determine whether a given positive term series $A = a_1 + a_2 + \cdots$ is convergent or divergent is to compare it with some series whose convergence or divergence is known. We have found up to the present the following *standard series* S:

The geometric series

$$1 + g + g^2 + \cdots \tag{1}$$

The general harmonic series

$$\frac{1}{1^\mu} + \frac{1}{2^\mu} + \frac{1}{3^\mu} + \cdots \tag{2}$$

The logarithmic series

$$\sum \frac{1}{nl_1^\mu n}, \tag{3}$$

$$\sum \frac{1}{nl_1 nl_2^\mu n}, \tag{4}$$

$$\sum \frac{1}{nl_1 nl_2 nl_3^\mu n} \tag{5}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

We notice that none of these series could be used to determine by comparison the convergence or divergence of the series following it.

In fact, let a_n, b_n denote respectively the nth terms in 1), 2). Then for $g < 1$, $\mu > 0$,

$$\frac{b_n}{a_{n+1}} = \frac{1}{n^\mu g^n} = \frac{e^{-n \log g}}{n^\mu} \doteq \infty \qquad \text{by I, 464,}$$

or using the *infinitary notation* of I, 461,

$$b_n \succ a_n.$$

Thus the terms of 2) converge to 0 infinitely slower than the terms of 1), so that it is useless to compare 2) with 1) for convergence. Let $g \geq 1$. Then

$$\frac{a_{n+1}}{b_n} = n^\mu g^n \doteq \infty,$$

or

$$a_n \succ b_n.$$

This shows we cannot compare 2) with 1) for divergence.

Again, if a_n, b_n denote the nth terms of 2).
have, if $\mu > 1$,

$$\frac{b_n}{a_n} = \frac{n^{\mu-1}}{\log^\mu n} \doteq \infty \qquad \text{by I, 4}$$

or

$$b_n \succ a_n.$$

If $\mu = 1$,

$$\frac{a_n}{b_n} = \log n \doteq \infty,$$

or

$$a_n \succ b_n.$$

Thus the convergence or divergence of
from 2) by comparison. In the same way w
the others.

2. These considerations lead us to intro
notions. Let $A = a_1 + a_2 + \cdots$, $B = b_1 + b_2 +$
series. Instead of considering the behavior
eralize and consider the ratios $A_n : B_n$ for di
for convergent series. These ratios obviously
of the rate at which A_n and B_n approach thei
B are divergent and

$$A_n \sim B_n,$$

we say A, B diverge equally fast; if

$$A_n \prec B_n,$$

A diverges slower than B, and B diverges fas
I, 180, we have:

Let A, B be divergent and

$$\lim \frac{a_n}{b_n} = l.$$

B converges faster than A, and A converges slower than B.
I, 184, we have:

Let A, B be convergent and

$$\lim \frac{a_n}{b_n} = l.$$

According as l is 0, $\neq 0$, ∞, A converges faster, equally fast, or than B.

Returning now to the set of standard series S, we see that converges (diverges) slower than any preceding series of th Such a set may therefore appropriately be called a *scale* of vergent (divergent) series. Thus if we have a decreasing po term series, whose convergence or divergence is to be ascerta we may compare it successively with the scale S, until we at one which converges or diverges equally fast. In practice series may always be found. It is easy, however, to show that exist series which converge or diverge slower than any in the scale S or indeed any other scale.

For let

$$A, \quad B, \quad C, \dots$$

be any scale of positive term convergent or divergent series.

Then, if convergent,

$$A_n{}^1 \succ B_n{}^1 \succ C_n{}^1 \succ \cdots;$$

if divergent,

$$A_n \succ B_n \succ C_n \succ \cdots$$

Thus in both cases we are led to a sequence of functions type

$$f_1(n) \succ f_2(n) \succ f_3(n) \succ \cdots$$

Thus to show the existence of a series Ω which converges

Then there exist positive increasing functions … *any* f_n.

For as $f_1 > f_2$ there exists an $a_1 > a$ such … $x \geq a_1$. Since $f_2 > f_3$, there exists an $a_2 > a_1$ … for $x \geq a_2$. And in general there exists an … $f_n > f_{n+1} + n$ for $x > a_n$. Let now

$$g(x) = f_n(x) + n - 1 \qquad \text{in } (a_n \ldots$$

Then g is an increasing unlimited functi… finally remains below any $f_m(x) + m - 1$, m a…

Thus $$0 \leq \lim_{x=\infty} \frac{g(x)}{f_m(x)} = \lim \frac{g(x)}{f_m + m - 1} < \lim \ldots$$

Hence $g < f_m$.

93. From the logarithmic series we can … tests, for example, the following:

1. (*Bertram's Tests.*) *Let* $A = a_1 + a_2 + \ldots$ *series.*

Let

$$Q_s(n) = \frac{\log \dfrac{1}{a_n n l_1 n \cdots l_{s-1} n}}{l_{s+1} n} \qquad s = \ldots$$

If for some s *and* m,

$$Q_s(n) > \mu > 1 \qquad n > m,$$

A is convergent. If, however,

$$Q_s(n) < 1,$$

A is divergent.

For multiplying 1) by $l_{s+1}n$, we get

Thus A is convergent.
The rest of the theorem follows similarly.

2. *For the positive term series* $A = a_1 + a_2 + \cdots$ *to conve necessary that, for* $n = \infty$,

$$\underline{\lim}\, a_n = 0, \qquad \underline{\lim}\, na_n = 0, \qquad \underline{\lim}\, na_n l_1 n = 0, \qquad \underline{\lim}\, na_n l_1 n l_2 n$$

We have already noted the first two. Suppose now tha

$$\underline{\lim}\, na_n l_1 n \cdots l_s n > 0.$$

Then by I, 338 there exists an m and a $c > 0$, such that

$$na_n l_1 n \cdots l_s n > c \quad , \quad n > m,$$

or

$$a_n > \frac{c}{nl_1 n \cdots l_s n}.$$

Hence A diverges.

Example 1.
$$A = \sum \frac{1}{n^\alpha \log^\beta n}.$$

We saw, 88, Ex. 5, that A is divergent for $\alpha < 1$. Fo A is convergent for $\beta > 1$ and divergent if $\beta \gtreqless 1$, accor 91, Ex. 2.

If $\alpha > 1$, let
$$\alpha = \alpha' + \alpha'' \quad , \quad \alpha'' > 1.$$

Then if $\beta \geq 0$,
$$a_n = \frac{1}{n^\alpha \log^\beta n} \leq \frac{1}{n^\alpha} \quad , \quad n > 2,$$

and A is convergent since $\sum \frac{1}{n^\alpha}$ is. If $\beta < 0$, let

Example 2.
$$A = \sum \frac{1}{n^\mu e^{\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)}} = \sum \frac{1}{n^\mu e^{H_n}}.$$

Here
$$Q_1 = \frac{\log \frac{1}{a_n n}}{l_2 n} = \frac{-\log n + \mu \log n + H_n}{l_2 n}$$
$$= \frac{\log n}{l_2 n}\left\{\mu - 1 + \frac{H_n}{\log n}\right\} \doteq \mu \infty \qquad \text{by 81, 6)}.$$

Hence A is convergent for $\mu > 0$ and divergent for $\mu <$
test for $\mu = 0$.

But for $\mu = 0$,
$$Q_2 = \frac{\log \frac{1}{a_n n l_1 n}}{l_3 n} = \frac{H_n - l_1 n - l_2 n}{l_3 n}$$
$$= \frac{l_1 n}{l_3 n}\left\{-1 - \frac{l_2 n}{l_1 n} + \frac{H_n}{l_1 n}\right\}$$
$$\doteq -\infty,$$
since $l_2 n > l_3 n$. Thus A is divergent for $\mu = 0$.

94. A very general criterion is due to *Kummer*, viz.:

Let $A = a_1 + a_2 + \cdots$ be a positive term series. Let k_1, k
set of positive numbers chosen at pleasure. A is converge
some constant $k > 0$.
$$K_n = k_n \frac{a_n}{a_{n+1}} - k_{n+1} \geq k \qquad n = 1, 2, \cdots$$
A is divergent if
$$R = \frac{1}{k_1} + \frac{1}{k_2} + \cdots$$
is divergent and
$$K_n \leq 0 \qquad n = 1, 2, \cdots$$

For on the first hypothesis

$$A_n < a_1 + \frac{1}{k}(a_1k_1 - a_nk_n) < a_1\Big($$

and A is convergent by 80, 4.

On the second hypothesis,

$$\frac{a_n}{a_{n+1}} \le \frac{k_{n+1}}{k_n},$$

or

$$\frac{a_{n+1}}{a_n} \ge \frac{k_{n+1}^{-1}}{k_n^{-1}}.$$

Hence A diverges since B is divergent.

95. 1. From Kummer's test we may deduce
at once. For take

$$k_1 = k_2 = \cdots = 1.$$

Then $A = a_1 + a_2 + \cdots$ converges if

$$K_n = \frac{a_n}{a_{n+1}} - 1 \ge k > 0,$$

i.e. if

$$\frac{a_{n+1}}{a_n} \le \rho < 1.$$

Similarly A diverges if $\frac{a_{n+1}}{a_n} \ge 1.$

2. To derive Raabe's test we take

$$k_n = n.$$

Then A converges if

$$K_n = n\frac{a_n}{a_{n+1}} - (n+1) \ge$$

i.e. if

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) \ge l > 1.$$

Similarly A diverges if

96. 1. *Let* $A = a_1 + a_2 + \cdots$ *be a positive term*

$$\lambda_0(n) = n\left(\frac{a_n}{a_{n+1}} - 1\right)$$

$$\lambda_1(n) = l_1 n \left\{ n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1 \right\}$$

$$\lambda_2(n) = l_2 n \left\{ l_1 n \left[n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1 \right] \right.$$

$$\cdots\cdots\cdots\cdots$$

Then A *converges if there exists an* s *such that*

$$\lambda_s(n) \geq \delta > 1 \qquad \textit{for some } n > m$$

it diverges if

$$\lambda_s(n) \leq 1 \qquad \textit{for } n > m.$$

We have already proved the theorem for $\lambda_0(n$
how to prove it for $\lambda_1(n)$. The other cases foll
For the Kummer numbers k_n we take

$$k_n = n \log n.$$

Then A converges if

$$k_n = n \log n \cdot \frac{a_n}{a_{n+1}} - (n+1) \log (n+1)$$

As

$$n + 1 = n\left(1 + \frac{1}{n}\right),$$

$$K_n = \lambda_1(n) - \log\left(1 + \frac{1}{n}\right)^n - \log\left(1 + \right.$$

$$= \lambda_1(n) - \log\left(1 + \frac{1}{n}\right)^{n+1}$$

For if this upper limit is not $+\infty$,

$$n\left\{n\left(\frac{a_n}{a_{n+1}}-1\right)-1\right\}<G$$

for all n. Hence

$$\lambda_1(n) < \frac{\log n}{n} \cdot G.$$

But the right side $\doteq 0$. Hence $\lambda_1(n) < 1$ for $n >$ some m. A is divergent by 1.

Example. We note that Raabe's test does apply to the harmonic series

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots$$

Here

$$n\left(\frac{a_n}{a_{n+1}}-1\right)=1.$$

Hence

$$P_n = 0, \qquad \text{and}$$

$$\lim P_n = 0.$$

Hence the series 1) is divergent.

97. *Gauss' Test.* *Let $A = a_1 + a_2 + \cdots$ be a positive term series such that*

$$\frac{a_n}{a_{n+1}} = \frac{n^s + \alpha_1 n^{s-1} + \cdots + \alpha_s}{n^s + b_1 n^{s-1} + \cdots + b_s},$$

where $s, \alpha_1 \cdots b_1 \cdots$ do not depend on n. Then A is convergent if $\alpha_1 - b_1 > 1$, and divergent if $\alpha_1 - b_1 \leq 1$.

Using the identity I, 91, 2), we have

$$\lambda_0(n) = n\left(\frac{a_n}{a_{n+1}}-1\right) = \frac{\alpha_1 - b_1 + \frac{1}{n}\{\alpha_2 - b_2 + \cdots\}}{1 + \frac{1}{n}\{b_1 + \cdots\}}.$$

98. *Let* $A = a_1 + a_2 + \cdots$ *be a positive*

$$\frac{a_n}{a_{n+1}} = 1 + \frac{\alpha}{n} + \frac{\beta_n}{n^\mu} \qquad \mu > 1$$

Then A *is convergent if* $\alpha > 1$ *and diverge*

For

$$\lambda_0(n) = n\left(\frac{a_n}{a_{n+1}} - 1\right) = \alpha -$$

and A converges if $\alpha > 1$ and diverges i

$$\lambda_1(n) = l_1 n \{\lambda_0(n) - 1\} =$$

and A is divergent.

EXAMPLES

99. *The Binomial Series.* Let us fin

μ the series

$$B = 1 + \mu x + \frac{\mu \cdot \mu - 1}{1 \cdot 2} x^2 + \frac{\mu \cdot \mu -}{1}$$

converges. If μ is a positive integer, B is

For $\mu = 0$, $B = 1$. We now exclude the

Applying D'Alembert's test to its adjoin

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\mu - n + 1}{n}\right| |x$$

Thus B converges absolutely for $|x| <$

Let $x = 1$. Then

$$B = 1 + \mu + \frac{\mu \cdot \mu - 1}{1 \cdot 2} + \frac{\mu \cdot \mu -}{1 \cdot}$$

Here D'Alembert's test applied to its

Thus B converges absolutely if $\mu > 0$, and its adjoint dive $\mu < 0$. Thus B does not converge absolutely for $\mu < 0$.

But in this case we note that the terms of B are alterna ositive and negative. Also

$$\frac{a_{n+1}}{a_n} = \left| 1 - \frac{1+\mu}{n} \right|,$$

that the $|a_n|$ form a decreasing sequence from a certain t Ve investigate now when $a_n \doteq 0$. Now

$$a_n = (-1)^n \frac{(-\mu)(-\mu+1)\cdots(-\mu+n-1)}{1\cdot 2\cdot\cdots n} = (-1)^n Q_n.$$

In I, 143, let $\alpha = -\mu$, $\beta = 1$. We thus find that $\lim a_n = 0$ hen $\mu > -1$. Thus B converges when $\mu > -1$ and dive hen $\mu < -1$.

Let $x = -1$. Then

$$B = 1 - \mu + \frac{\mu\cdot\mu-1}{1\cdot 2} - \frac{\mu\cdot\mu-1\cdot\mu-2}{1\cdot 2\cdot 3} + \cdots$$

If $\mu > 0$, the terms of B finally have one sign, and

$$n\left(\left|\frac{a_n}{a_{n+1}}\right| - 1\right) \doteq 1 + \mu.$$

Hence B converges absolutely.

If $\mu < 0$, let $\mu = -\lambda$. Then B becomes

$$1 + \lambda + \frac{\lambda\cdot\lambda+1}{1\cdot 2} + \frac{\lambda\cdot\lambda+1\cdot\lambda+2}{1\cdot 2\cdot 3} + \cdots$$

Here

100. *The Hypergeometric Series*

$$F(\alpha, \beta, \gamma, x) = 1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} x + \frac{\alpha \cdot \alpha+1 \cdot \beta \cdot \beta+1}{1 \cdot 2 \cdot \gamma \cdot \gamma+1} x^2$$

$$+ \frac{\alpha \cdot \alpha+1 \cdot \alpha+2 \cdot \beta \cdot \beta+1 \cdot \beta+2}{1 \cdot 2 \cdot 3 \cdot \gamma \cdot \gamma+1 \cdot \gamma+2} x$$

Let us find for what values of x this series converges. to the adjoint series, we find

$$\left|\frac{a_{n+2}}{a_{n+1}}\right| = \left|\frac{(\alpha+n)(\beta+n)}{(n+1)(\gamma+n)}\right| \cdot |x| \doteq |x|.$$

Thus F converges absolutely for $|x| < 1$ and diverges for

Let $x = 1$. The terms finally have one sign, and

$$\frac{a_{n+1}}{a_{n+2}} = \frac{n^2 + n(1+\gamma) + \gamma}{n^2 + n(\alpha+\beta) + \alpha\beta}.$$

Applying Gauss', test we find F converges when and o

$$\alpha + \beta - \gamma < 0.$$

Let $x = -1$. The terms finally alternate in sign. L when $a_n \doteq 0$. We have

$$|a_{n+2}| = \frac{\alpha\beta}{\gamma} \cdot \frac{(\alpha+1)\cdots(\alpha+n)(\beta+1)\cdots(\beta+}{(1+1)\cdots(1+n)(\gamma+1)\cdots(\gamma+}$$

Now

$$\alpha + m = m\left(1 + \frac{\alpha}{m}\right), \quad \beta + m = m\left(1 + \frac{\beta}{m}\right),$$

$$1 + m = m\left(1 + \frac{1}{m}\right), \quad \gamma + m = m\left(1 + \frac{\gamma}{m}\right).$$

Thus

$$\left(1 + \frac{\alpha}{\ }\right)\left(1 + \frac{\beta}{\ }\right)$$

Hence

$$|a_{n+2}| = \prod_1^n \left(1+\frac{\alpha}{m}\right)\left(1+\frac{\beta}{m}\right)\left(1-\frac{1}{m}+\frac{\sigma_m}{m^2}\right)\left(1-\frac{\gamma}{m}+\right.$$

$$= \prod_1^n \left(1+\frac{\alpha+\beta-\gamma-1}{m}+\frac{\eta_m}{m^2}\right).$$

Hence

$$\log|a_{n+2}| = \sum_1^n \log\left(1+\frac{\alpha+\beta-\gamma-1}{m}+\frac{\eta_m}{m^2}\right) = \sum_1^n l_m =$$

and thus

$$L = \lim \log|a_{n+2}| = \sum_1^\infty l_m.$$

Now for a_n to $\doteq 0$ it is necessary that $L_n \doteq -\infty$. In
we saw this takes place when and only when $\alpha+\beta-\gamma-$

Let us find now when $|a_{n+1}|<|a_n|$. Now 1) gives

$$\left|\frac{a_{n+2}}{a_{n+1}}\right| = 1+\frac{\alpha+\beta-\gamma-1}{n}+\frac{\delta_m}{n^2}.$$

Thus when $\alpha+\beta-\gamma-1<0$, $|a_{n+2}|<|a_{n+1}|$. Henc
case F is an alternating series. We have thus the i
theorem:

The hypergeometric series converges absolutely when $|x$
diverges when $|x|>1$. *When* $x=1$, *F converges only wh*
$-\gamma<0$ *and then absolutely. When* $x=-1$, *F conve*
when $\alpha+\beta-\gamma-1<0$, *and absolutely if* $\alpha+\beta-\gamma<0$.

Pringsheim's Theory

101. 1. In the 35th volume of the *Mathematische*

A_n^μ, where A is any positive term divergent se

$\bar{B}_n{}^{-\mu}$ where B is any positive term convergent

$$e^{\mu n}\quad,\quad \left(e^{e^n}\right)^\mu\quad,\quad \left(e^{e^{e^n}}\right)^\mu\quad,\quad \cdots e^\mu$$

It will be convenient to denote in general a co term series by the symbol

$$C = c_1 + c_2 + \cdots$$

and a divergent positive term series by

$$D = d_1 + d_2 + \cdots$$

2. *The series*

$$C = \sum_1^\infty \frac{M_{n+1} - M_n}{M_n M_{n+1}} = \sum_1^\infty \left(\frac{1}{M_n} - \frac{1}{M_{n+1}}\right) =$$

is convergent, and conversely every positive term may be brought into this form.

For

$$C_m = \sum_1^m \left(\frac{1}{M_n} - \frac{1}{M_{n+1}}\right)$$

$$= \frac{1}{M_1} - \frac{1}{M_{m+1}} \doteq \frac{1}{M_1}$$

and C is convergent.

Let now conversely $C = c_1 + c_2 + \cdots$ be a g positive term series. Let

$$\bar{C}_{n-1} = \frac{1}{M_n}.$$

Then

$$c_n = \frac{1}{M_n} - \frac{1}{M_{n+1}}.$$

3. *The series*

Let now conversely $D = d_1 + d_2 + \cdots$ be a given positive te vergent series. Let

$$M_n = D_{n-1}.$$

Then

$$d_n = M_{n+1} - M_n.$$

102. Having now obtained a general form of all converg d divergent series, we now obtain another general form o nvergent or divergent series, but which converges slower th or diverges slower than 101, 2). Let us consider first c rgence. Let $M'_n < M_n$, then

$$\sum\left(\frac{1}{M'_n} - \frac{1}{M'_{n+1}}\right)$$

convergent, and if M'_n is properly chosen, not only is ea rm of 1) greater than the corresponding term of 101, 1), but ll converge slower than 101, 1). For example, for M'_n let ke M^μ_n, $0 < \mu < 1$. Then denoting the resulting series $= \Sigma c'_n$, we have

$$\frac{c'_n}{C_n} = \frac{M^\mu_{n+1} - M^\mu_n}{M^\mu_n M^\mu_{n+1}} \frac{M_n M_{n+1}}{M_{n+1} - M_n}$$

$$= \frac{1 - r^\mu}{1 - r} M_n^{1-\mu} \quad , \quad r = \frac{M_n}{M_{n+1}} < 1.$$

Thus C' converges slower than C. But the preceding a ows that C' and

$$\sum \frac{M_{n+1} - M_n}{M_{n+1} M^\mu_n}$$

nverge equally fast. In fact 2) states that

$$c' \sim c_n M_n^{1-\mu}.$$

Since M_n is any positive increasing function of n whose lir

Now by I, 413, for sufficiently large n,

$$\log M_{n+1} - \log M_n = -\log\left(1 - \frac{M_{n+1} - M_n}{M_{n+1}}\right) >$$

Replacing here M_n by $\log M_n$, we get

$$l_2 M_{n+1} - l_2 M_n > \frac{\log M_{n+1} - \log M_n}{\log M_{n+1}} > \frac{M_{n+}}{M_{n+1}}$$

and in general

$$l_2 M_{n+1} - l_2 M_n > \frac{M_{n+1} - M_n}{M_{n+1} l_1 M_{n+1} \cdots l_{r-1} M_{n+}}$$

Thus the series

$$\sum \frac{M_{n+1} - M_n}{M_{n+1} l_1 M_{n+1} \cdots l_{r-1} M_{n+1} l_r^{1+\mu} M_{n+1}}$$

converges as is seen by comparing with 4). We
the theorem:

The series

$$\sum \frac{M_{n+1} - M_n}{M_n M_{n+1}}, \quad \sum \frac{M_{n+1} -}{M_{n+1} M}$$

$$\sum_1^\infty \frac{M_{n+1} - M_n}{M_{n+1} l_1 M_{n+1} \cdots l_{r-1} M_{n+1} l_r^{1+\mu} M_{n+1}} \qquad r = 1,$$

form an infinite set of convergent series; each s
slower than any preceding it.

The last statement follows from I, 463, 1, 2.

Corollary 1 (Abel). Let $D = d_1 + d_2 + \cdots$ *denot*
divergent series. Then

$$\sum \frac{d_n}{D^{1+\mu}} \qquad \mu > 0$$

For by 101, 2 we may bring C to the form

$$\sum \frac{M_{n+1} - M_n}{M_n M_{n+1}}.$$

Then any of the series 7) converges slower than C.

103. 1. Let us consider now divergent series. Here
oblem is simpler and we have at once the theorem:

The series

$$D = \sum_1^\infty \frac{M_{n+1} - M_n}{M_n} = \Sigma d_n$$

verges slower than

$$\Sigma (M_{n+1} - M_n) = \Sigma d_n'.$$

That 1) is divergent is seen thus: Consider the product

$$P_n = \prod_1^n \left(1 + \frac{M_{m+1} - M_m}{M_m}\right) = \prod_1^n \frac{M_{m+1}}{M_m}$$

$$= \frac{M_{m+1}}{M_1}$$

hich obviously $\doteq \infty$.

Now

$$P_n = (1 + d_1)(1 + d_2) \cdots (1 + d_n)$$

$$= 1 + (d_1 + \cdots + d_n) + (d_1 d_2 + d_1 d_3 + \cdots)$$

$$+ (d_1 d_2 d_3 + \cdots) + \cdots + d_1 d_2 \cdots d_n$$

$$< 1 + D_n + \frac{1}{2} D_n^2 + \cdots + \frac{1}{n!} D_n^n < e^{D_n} < e^D.$$

Hence $D_n \doteq \infty$ and D is divergent.

As

$$\frac{d_n}{d_n'} = \frac{1}{M_n} \doteq 0$$

Then $M_{n+1} > M_n$ and

$$d_n = \frac{M_{n+1} - M_n}{M_n}.$$

Moreover $M_n \doteq \infty$. For

$$\frac{M_{n+1}}{M_1} = (1 + d_1) \cdots (1 + d_n)$$

$$> 1 + D_n \qquad \text{by I, 90, 1.}$$

But $D_n \doteq \infty$.

3. *The series*

$$\sum_1^\infty (M_{n+1} - M_n) = \sum d_n$$

$$\sum_1^\infty \frac{d_n}{M_n l_1 M_n \cdots l_r M_n} \qquad r =$$

form an infinite set of divergent series, each series
than any preceding it. $l_0 M_n = M_n$.

For
$$\log M_{n+1} - \log M_n = \log\left(1 + \frac{M_{n+1}}{M_n}\right.$$

$$< \frac{M_{n+1} - M_n}{M_n}.$$

This proves the theorem for $r = 0$. Hence as
replacing repeatedly M_n by $\log M_n$,

$$l_{r+1} M_{n+1} - l_{r+1} M_n < \frac{M_{n+1} - M_n}{M_n l_1 M_n \cdots l_r M_n}.$$

Corollary 1. If we take $M_n = n$, we get the seri

Corollary 2 (Abel). *Let* $D = d_1 + d_2 + \cdots$ *be a*
term series. Then

$$\sum \frac{d_n}{D_{n-1}}$$

4. In Ex. 3 of I, 454, we have seen that M_{n+1} is not always ~
In case it is we have

The series

$$\sum \frac{M_{n+1}-M_n}{M_n^{1+\mu}} \qquad \mu>0 \quad , \quad M_{n+1}\sim M_n$$

nvergent.

ollows from 102, 3).

The series

$$\sum \frac{M_{n+1}-M_n}{e^{\mu M_n}} \qquad M_{n+1}\sim M_n$$

nvergent if $\mu>0$; it is divergent if $\mu<0$.

or $$e^{\mu M_n} > \tfrac{1}{2}\mu^2 M_n^2 \sim M_n^2 \qquad \mu>0.$$

us $$\frac{M_{n+1}-M_n}{e^{\mu M_n}} \precsim \frac{M_{n+1}-M_n}{M_n^2}.$$

f $\mu<0$ $$\frac{M_{n+1}-M_n}{e^{\mu M_n}} \succsim M_{n+1}-M_n.$$

. *If $M_{n+1}\sim M_n$, we have*

$$l_{r+1}M_{n+1}-l_{r+1}M_n \sim \frac{M_{n+1}-M_n}{M_n l_1 M_n \cdots l_r M_n} \sim \frac{M_{n+1}-M_n}{M_{n+1} l_1 M_{n+1} \cdots l_r M_{n+1}}.$$

For by 102, 5), 103, 3),

$$l_{r+1}M_{n+1}-l_{r+1}M_n < \frac{M_{n+1}-M_n}{M_n l_1 M_n \cdots l_r M_n},$$

$$\frac{M_{n+1}-M_n}{M_{n+1} l_1 M_{n+1} \cdots l_r M_n} < l_{r+1}M_{n+1}-l_{r+1}M_n.$$

Now since $M_{n+1}\sim M_n$, we have also obviously

$$l_m M_n \sim l_m M_{n+1} \qquad m=1, 2, \cdots r.$$

It diverges if

$$\frac{a_n}{d_n} > G.$$

In the second place, A converges if

$$\frac{a_{n+1}}{a_n} - \frac{c_{n+1}}{c_n} \leq 0,$$

and diverges if

$$\frac{a_{n+1}}{a_n} - \frac{d_{n+1}}{d_n} \geq 0.$$

The tests 1), 2) involve only a single term of
and the comparison series, while the tests 3),
terms. With *Du Bois Reymond* such tests we n
tively tests of the *first* and *second kinds.* And
relation between p terms

$$a_n, a_{n+1}, \cdots a_{n+p-1}$$

of the given series and p terms of a comparison se

$$c_n, c_{n+1}, \cdots c_{n+p-1}, \quad \text{or } d_n, d_{n+1} \quad \cdots d_n$$

which serves as a criterion of convergence or div
called a *test of the p^{th} kind.*

Let us return now to the tests 1), 2), 3), 4),
are testing A for convergence. If for a cert
series C

$$\frac{a_n}{c_n} \quad \text{not always} \leq G \quad , \quad n > m$$

it might be due to the fact that $c_n \doteq 0$ too fast.
take another comparison series $C' = \Sigma c'_n$ which co
than C. As there always exist series which conve
any given positive term series, the test 1) must
vergence of A if a proper comparison series is f

It diverges if $$\lim \frac{M_n}{M_{n+1}-M_n} a_n > 0.$$

This follows at once from 105, 1), 2); and 101, 2; 103, 1.

2. To get tests of greater power we have only to replac
series
$$\sum \frac{M_{n+1}-M_n}{M_{n+1}M_n} \quad , \quad \sum \frac{M_{n+1}-M_n}{M_n}$$
just employed in 1), 2) by the series of 102 and 103, 3 whic
verge (diverge) slower. We thus get from 1:

The positive term series A converges if
$$\lim \frac{M_{n+1}M_n^\mu}{M_{n+1}-M_n} a_n \qquad \text{or} \lim \frac{M_n l_1 M_n \cdots l_{r-1} M_{n+1} l_r^{1+\mu} M_n}{M_{n+1}-M_n} a_n <$$

It diverges if $$\lim \frac{M_n l_1 M_n \cdots l_r M_n}{M_{n+1}-M_n} a_n > 0.$$

Bonnet's Test. The positive term series A converges if
$$\lim n l_1 n \cdots l_{r-1} n l_r^{1+\mu} n \cdot a_n < \infty \quad , \quad \mu > 0.$$

It diverges if $$\lim n l_1 n \cdots l_r n \cdot a_n > 0.$$

Follows from the preceding setting $M_n = n$.

3. *The positive term series A converges or diverges accordin*
$$\frac{e^{\mu M_n} a_n}{M_{n+1}-M_n} \begin{array}{l} < 1 \quad , \quad \mu > 0, \\ > 1 \quad , \quad \mu < 0. \end{array} \qquad M_{n+1} \sim M_n.$$

For in the first case
$$a_n < \frac{M_{n+1}-M_n}{e^{\mu M_n}} \quad ; \quad \mu > 0,$$
and in the second case

It diverges if

$$\overline{\lim}\,\frac{\dfrac{M_{n+1}-M_n}{a_n}}{M_n}<0 \qquad \text{or } \overline{\lim}\,\frac{\log\dfrac{M_n}{M_n l_1 M_n \cdots}}{l_{r+1} \cdots}$$

Here $r = 0, 1, 2, \cdots$ *and as before* $l_0 M_n = M$

For taking the logarithm of both sides of
vergence

$$q = \frac{\log\dfrac{M_{n+1}-M_n}{a_n}}{M_n} > \mu.$$

As μ is an arbitrarily small but fixed posi
verges if $\underline{\lim}\, q > 0$. Making use of 104, 3 v
of the theorem. The rest follows similarly.

Remark. If we take $M_n = n$ we get Cau
and Bertram's tests 93.

For if

$$\frac{\log\dfrac{1}{a_n}}{n} = \log\sqrt[n]{\frac{1}{a_n}} = -\log\sqrt[n]{a_n} \geq$$

it is necessary that

$$\sqrt[n]{a_n} \leq \mu < 1.$$

Also if

$$\frac{\log\dfrac{1}{a_n n l_1 n \cdots l_r n}}{l_{r+1} n} = \frac{\log\dfrac{1}{a_n n l_1 n \cdots l_{r-1} n}}{l_{r+1} n}$$

$$= -1 + \frac{\log\dfrac{}{a_n n l_1 n}}{l_{r+1}}$$

it is necessary that

$$\frac{\log\dfrac{1}{a_n n l_1 n \cdots l_{r-1} n}}{l_{r+1} n} > \mu + 1$$

Pringsheim's Criterion. Let $p_1, p_2 \cdots$ be a set of posi chosen at pleasure, and let $P_n = p_1 + \cdots + p_n$. The series A converges if

$$\underline{\lim} \frac{\log \frac{p_n}{a_n}}{P_n} > 0.$$

For A converges if

$$\underline{\lim} \frac{\log \frac{M_{n+1} - M_n}{a_n}}{M_n} > 0 \quad , \quad \text{by } 106$$

But $M_{n+1} - M_n = d_n$ is the general term of the dive $D = d_1 + d_2 + \cdots$

Thus 2) may be written

$$\underline{\lim} \frac{\log \frac{d_n}{a_n}}{D_n} > 0.$$

Moreover A converges if

$$\frac{c_n}{a_n} \geq r > 1,$$

that is, if

$$\underline{\lim} \frac{c_n}{a_n} > 0,$$

where as usual $C = c_1 + c_2 + \cdots$ is a convergent series.

Hence A converges if

$$\underline{\lim} \frac{\frac{c_n}{a_n}}{C_n} > 0.$$

But now the set of numbers $p_1, p_2 \cdots$ gives rise $P = p_1 + p_2 + \cdots$ which must be either convergent o Thus 3), 4) show that in either case 1) holds.

Or in less general form:

The positive term series A converges if

$$\underline{\lim}\left(c_{n+1}\frac{a_n}{a_{n+1}} - c_n\right) > 0.$$

It diverges if

$$\overline{\lim}\left(d_{n+1}\frac{a_n}{a_{n+1}} - d_n\right) < 0.$$

Here as usual $C = c_1 + c_2 + \cdots$ is a convergent,
a divergent series.

2. Although we have already given on
Kummer's theorem we wish to show here its p
general theory, and also to exhibit it under a
Let us replace c_n, c_{n+1} in 1) by their valu
We get

$$\frac{M_{n+2} - M_{n+1}}{M_{n+2}} \cdot \frac{a_n}{a_{n+1}} - \frac{M_{n+1} - M_n}{M_n}$$

or since

$$M_{n+2} > M_{n+1},$$

$$\frac{M_{n+2} - M_{n+1}}{M_{n+1}} \cdot \frac{a_n}{a_{n+1}} - \frac{M_{n+1} - M_n}{M_n}$$

or by 103, 2

$$d_{n+1}\frac{a_n}{a_{n+1}} - d_n < 0,$$

where $D = d_1 + d_2 + \cdots$ is *any* divergent p
Since any set of positive numbers $k_1, k_2, \cdots$ g
$k_1 + k_2 + \cdots$ which must be either convergent
from 1) that 5) holds when we replace the
have therefore:

Since the k's are entirely arbitrary positive numbers, the re-
(6) also gives

converges if

$$k_n \frac{a_n}{a_{n+1}} - k_{n+1} > 0\ ;$$

seen by writing

$$k_n = \frac{1}{k'_n}$$

cing, and then dropping the accent.

. From Kummer's theorem we may at once deduce a set
s of increasing power, viz.:

The positive term series A is convergent or divergent according

$$\frac{M_{n+2} - M_{n+1}}{M_{n+1} l_1 M_{n+1} \cdots l_r M_{n+1}} \cdot \frac{a_n}{a_{n+1}} - \frac{M_{n+1} - M_n}{M_n l_1 M_n \cdots l_r M_n}$$

> 0 or is ≤ 0.

For $k_1, k_2 \cdots$ we have used here the terms of the diverg-
es of 103, 3.

Arithmetic Operations on Series

09. 1. Since an infinite series

$$A = a_1 + a_2 + a_3 \cdots$$

not a true sum but the limit of a sum

$$A = \lim_{n=\infty} A_n,$$

now inquire in how far the properties of polynomials hold
infinite polynomial 1). The *associative property* is expre-
the theorem :

This theorem relates to grouping the terms (
The following relate to removing them.

2. *Let* $B = b_1 + b_2 + \cdots$ *be convergent and le*
$b_2 = a_{m_1+1} + \cdots + a_{m_2}, \cdots$ *If* 1° $A = a_1 + a_2$
$A = B$. 2° *If the terms* $a_n > 0$, A *is conve*
$m_n - m_{n-1} \leq p$ *a constant, and* $a_n \doteq 0$, A *is conve*

On the first hypothesis we have only t
$A = B$. On the second hypothesis

$$\epsilon > 0, \qquad m, \qquad B_n < \epsilon, \qquad n >$$

Then
$$B - A_s < \epsilon, \qquad s > m_n.$$

On the third hypothesis we may set

$$A_s = B_r + b'_{r+1}$$

where b'_{r+1} denotes a part of the a-terms in b_r
tains at most p terms of A, $b'_{r+1} \doteq 0$.

Hence
$$\lim A_s = \lim B_r \quad , \quad \text{or} \quad A =$$

Example 1. The series

$$B = (1 - 1) + (1 - 1) + (1 - 1)$$

is convergent. The series obtained by removi

$$A = 1 - 1 + 1 - 1 +$$

is divergent.

Example 2.

$$A = 1 - \frac{1}{1+x} + \frac{1}{2} - \frac{1}{2+x} + \frac{1}{3} - \frac{1}{3+x} + \cdots;$$

$$B = \sum\left(\frac{1}{} - \frac{1}{}\right) = \sum \frac{x}{}$$

The terms of a simply convergent series $A = a_1 + a_2 + \cdots$ *c*
arranged to form a series S, *for which* $\lim S_n$ *is any prese*
number, or $\pm\infty$.

For let

$$B = b_1 + b_2 + \cdots$$
$$C = c_1 + c_2 + \cdots$$

be the series formed respectively of the positive and neg
terms of A, the relative order of the terms in A being prese

To fix the ideas let l be a positive number; the demonstr
of the other cases is similar. Since $B_n \doteq +\infty$, there exists a
such that

$$B_{m_1} > l.$$

Let m_1 be the least index for which 1) is true. Since $C_n \doteq$
there exists an m_2 such that

$$B_{m_1} + C_{m_2} < l.$$

Let m_2 be the least index for which 2) is true. Contin
we take just enough terms, say m_3 terms of B, so that

$$B_{m_1} + C_{m_2} + B_{m_1, m_3} > l.$$

Then just enough terms, say m_4 terms of C, so that

$$B_{m_1} + C_{m_2} + B_{m_1, m_3} + C_{m_2, m_4} < l,$$

etc. In this way we form the series

$$S = B_{m_1} + C_{m_2} + B_{m_1, m_3} + \cdots$$

whose sum is l. For

$$|a_s| < e \qquad s > \sigma;$$

also

We may now take n so large that $A_n - I$
whose index is $< m$. Thus the terms of .
positive sign are a part of A_m and hence

$$A_n - B_n \mid < \mathrm{A}_m < \epsilon \qquad n$$

Thus B is convergent and $B = A$.

The same reasoning shows that B is con
absolutely convergent.

3. *If* $A = a_1 + a_2 + \cdots$ *enjoys the commut*
absolutely convergent.

For if only simply convergent we could arr
to have any desired sum. But this contradict

Addition and Subtraction

111. *Let* $A = a_1 + a_2 + \cdots \quad , \qquad B = b_1 + b_2$
The series

$$C = (a_1 \pm b_1) + (a_2 \pm b_2) + \cdot$$

are convergent and $C = A \pm B$.

For obviously $C_n = A_n \pm B_n$. We have now
limit.

Example. We saw, 81, 3, Ex. 1, that

$$A = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots$$

We note now that

$$\frac{3}{2}A = A + \frac{1}{2}A$$

$$= \sum\left(\frac{1}{4n-3} - \frac{1}{4n-2} + \frac{1}{4n-1} - \frac{1}{4n}\right) + \frac{1}{2}\sum\left(\frac{1}{2n-1} - \frac{1}{2n}\right)$$

$$= \sum\left(\frac{1}{4n-3} - \frac{1}{4n-2} + \frac{1}{4n-1} - \frac{1}{4n}\right) + \sum\left(\frac{1}{4n-2} - \frac{1}{4n}\right)$$

$$= \sum\left(\frac{1}{4n-1} + \frac{1}{4n-3} - \frac{1}{2n}\right)$$

$= B$ by 109, 2.

Thus $$B = \tfrac{3}{2}A.$$

This example, due to *Dirichlet*, illustrates the non-commutati operty of simply convergent series. We have shown the co rgence of B by actually determining its sum. As an exercise proceed directly as follows :

The series 1) may be written :

$$\sum\frac{8n-3}{2n(4n-1)(4n-3)} = \sum\frac{1}{n^2}\cdot\frac{8-\dfrac{3}{n}}{2\left(4-\dfrac{1}{n}\right)\left(4-\dfrac{3}{n}\right)}.$$

Comparing this with

$$\sum\frac{1}{n^2}$$

e see that it is convergent by 87, 3. Since 1) is convergent, also by 109, 2.

112. 1. *Multiplication.* We have already seen, 80, 7, that

responds a lattice point ι, κ and conversely.
it a great help here and later to keep this corr

Let $A = a_1 + a_2 + \cdots$, $B = b_1 + b_2 + \cdots$ *be a*
Then $C = \Sigma a_\iota b_\kappa$ *is absolutely convergent and* A

Let m be taken large at pleasure; we may
$\Gamma_n - \mathrm{A}_m \cdot \mathrm{B}_m$ contains no term both of whos

Then
$$\Gamma_n - \mathrm{A}_m\mathrm{B}_m < \alpha_1\overline{\mathrm{B}}_m + \alpha_2\overline{\mathrm{B}}_m + \cdots +$$
$$+ \beta_1\overline{\mathrm{A}}_m + \beta_2\overline{\mathrm{A}}_m + \cdots +$$
$$< \mathrm{A}_m\overline{\mathrm{B}}_m + \mathrm{B}_m\overline{\mathrm{A}}_m$$
$$< \epsilon \text{ for } m \text{ sufficiently}$$

Hence
$$\lim \Gamma_n = \mathrm{A} \cdot \mathrm{B}$$

and C is absolutely convergent.

To show that $C = A \cdot B$, we note that

$$| C_n - A_mB_m | \leq \Gamma_n - \mathrm{A}_m\mathrm{B}_m < \epsilon$$

2. We owe the following theorem to *Merte*

If A *converges absolutely and* B *converges* (
lutely), *then*

$$C = a_1b_1 + (a_1b_2 + a_2b_1) + (a_1b_3 + a_2b_2 +$$

is convergent and $C = A \cdot B$.

We set
$$C = c_1 + c_2 + c_3 + \cdots$$

where
$$c_1 = a_1b_1$$

But
$$B_m = B - \bar{B}_m \qquad m = 1, 2, \cdots$$

Hence
$$C_n = a_1(B - \bar{B}_n) + a_2(B - \bar{B}_{n-1}) + \cdots + a_n(B - \bar{B}_1)$$
$$= B(a_1 + \cdots + a_n) - (a_1\bar{B}_n + \cdots + a_n\bar{B}_1)$$
$$= A_n B - d_n,$$
where
$$d_n = a_1\bar{B}_n + a_2\bar{B}_{n-1} + \cdots + a_n\bar{B}_1.$$

The theorem is proved when we show $d_n \doteq 0$. To this
us consider the two sets of remainders

$$\bar{B}_1, \quad \bar{B}_2, \quad \cdots \bar{B}_{n_1}$$
$$\bar{B}_{n_1+1}, \quad \bar{B}_{n_1+2}, \quad \cdots \bar{B}_{n_1+n_2} \qquad n_1 + n_2 = n.$$

Let * each one in the first set be $| < | M_1$, and each in the
set $| < | M_2$. Then since

$$d_n = (a_1\bar{B}_n + \cdots + a_{n_2}\bar{B}_{n_1+1}) + (a_{n_2+1}\bar{B}_{n_1} + \cdots + a_n\bar{B}_1)$$
$$|d_n| < M_2(\alpha_1 + \cdots + \alpha_{n_2}) + M_1(\alpha_{n_2+1} + \cdots + \alpha_n)$$
$$< M_2\mathrm{A}_{n_2} + M_1\bar{\mathrm{A}}_{n_2} < M_2\mathrm{A} + M_1\bar{\mathrm{A}}_{n_2}.$$

Now for each $\epsilon > 0$ there exists an n_1 such that
$$M_2 < \frac{\epsilon}{2\mathrm{A}},$$
also a ν, such that
$$\bar{\mathrm{A}}_{n_2} < \frac{\epsilon}{2M_1} \qquad n_2 > \nu.$$

Thus 1) shows that
$$|d_n| < \epsilon.$$

2. When neither A nor B converges absolutely, the

The series A being alternating is converge
adjoint is divergent by 81, 2, since here $\mu = \frac{1}{2}$.

$$C = \frac{1}{\sqrt{1}}\frac{1}{\sqrt{1}} - \left(\frac{1}{\sqrt{1}\sqrt{2}} + \frac{1}{\sqrt{2}\sqrt{1}}\right)$$

$$+\left(\frac{1}{\sqrt{1}}\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \cdots\right.$$

$$= c_2 + c_3 + c_4 + \cdots$$

and

$$|c_n| = \frac{1}{\sqrt{1}}\frac{1}{\sqrt{n-1}} + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{n-2}} + \cdots$$

By I, 95,

$$\sqrt{m(n-m)} \leq \frac{n}{2}.$$

Hence

$$\frac{1}{\sqrt{m(n-m)}} \geq \frac{2}{n}, \quad |c_n| > \frac{2(n-}{n}$$

Hence C is divergent since c_n does not $\doteq 0$,
were convergent, by 80, 3.

4. In order to have the theorems on multip
we state here one which we shall prove later.

If all three series A, B, C are convergent, then

113. We have seen, 109, 1, that we may grou
convergent series $A = a_1 + a_2 + \cdots$ into a series
each term b_n containing but a finite number of t
easy to arrange the terms of A into a finite o
number of infinite series, B', B'', $B''' \cdots$ For ex

$$B' = a_1 + a_{p+1} + a_{2p+1} + \cdots$$

$^{(m)}$ all terms whose index is the product of m primes. We a
ow what is the relation between the original series A and t
ries B', $B''\cdots$

If $A = a_1 + a_2 + \cdots$ *is absolutely convergent, we may break it to a finite or infinite number of series* B', B'', B''', $\cdots$ *Each ese series converges absolutely and*

$$A = B' + B'' + B''' + \cdots$$

That each $B^{(m)}$ converges absolutely was shown in 80, 6. I suppose first that there is only a finite number of these seri y p of them. Then

$$A_n = B'_{n_1} + B''_{n_2} + \cdots + B^{(p)}_{n_p} \qquad n = n_1 + \cdots + n_p.$$

As $n \doteq \infty$, each n_1, $n_2 \cdots \doteq \infty$. Hence passing to the lin $= \infty$, the above relation gives

$$A = B' + B'' + \cdots + B^{(p)}.$$

Suppose now there are an infinite number of series $B^{(m)}$.

Set

$$B = B' + B'' + B''' + \cdots$$

We take ν so large that $A - B_n$, $n > \nu$, contains no term a_n dex $\leq m$, and m so large that

$$\bar{\mathrm{A}}_m < \epsilon.$$

Then

$$|A - B_n| \leq \bar{\mathrm{A}}_m < \epsilon. \quad n > \nu.$$

Two-way Series

114. 1. Up to the present the terms of our infinite series ha tended to infinity only one way. It is, however, convenie

Such series we called *two-way series.*
if

$$\lim_{r,s=\infty} \sum_{n=-r}^{n=s} a_n$$

is finite. If the limit 2) does not exist,
tension of the other terms employed in
present case are too obvious to need any
$n=0$ is excluded in 1); the fact may

thus $\sum' a_n$.

2. Let m be an integer; then while n r

$$\dots -3, -2, -1, 0, 1,$$

$\nu = n+m$ will range over the same set w
will be m units ahead or behind n acc
shows that

$$\sum_{n=-\infty}^{\infty} a_n = \sum_{n=-\infty}^{\infty} a_{n+m}.$$

Similarly,

$$\sum_{n=-\infty}^{\infty} a_n = \sum_{n=-\infty}^{\infty} a_{-n}.$$

3. *Example 1.* $\Theta = \sum_{-\infty}^{\infty} e^{nx+an^2}$

$$= 1 + e^{x+a} + e^{2x+4a} +$$

$$+ e^{-x+a} + e^{-2x+4a}$$

This series is fundamental in the ellip

Example 2. $1 + \sum{}' \Big(\frac{1}{} - \frac{1}{} \Big)$

115. *For a two-way series A to converge, it is necessa[ry and] sufficient that the series B formed with the terms with negative [indices] and the series C formed with the terms with non-negative in[dices be] convergent. If A is convergent, $A = B + C$.*

It is necessary. For A being convergent,

$$|A - B_r - C_s| < \epsilon/2 \quad , \quad |A - B_r - C_{s'}| < \epsilon/2$$

if s, $s' >$ some σ and $r >$ some ρ. Hence adding,

$$|C_s - C_{s'}| < \epsilon,$$

which shows C is convergent. Similarly we may show th[at B is] convergent.

It is sufficient. For B, C being convergent,

$$|B - B_r| < \epsilon/2 \quad , \quad |C - C_s| < \epsilon/2$$

for r, $s >$ some p. Hence

$$|B + C - (B_r + C_s)| < \epsilon,$$

or

$$|B + C - \sum_{n=-r}^{n=s} a_n| < \epsilon.$$

Thus

$$\lim \sum_{n=-r}^{n=s} a_n = B + C.$$

Example 1. The series

$$\frac{1}{x} + \sum_{-\infty}^{\infty}{}' \left(\frac{1}{x+n} - \frac{1}{n} \right)$$

is absolutely convergent if $x \neq 0, \pm 1, \pm 2, \cdots$

For

$$|a_n| = \left| \frac{1}{x+n} - \frac{1}{n} \right| = \frac{|x|}{|n^2 + nx|}.$$

Here
$$n > 0, \qquad \sqrt[n]{a_n} = e^{x}e^{an}$$
$$\doteq x$$
$$n = -n',\ n' > 0 \qquad \sqrt[n']{a_n} = e^{-x}e^{an}$$
$$\doteq x$$

The case $a = 0$ is obvious.

Thus the series defines a one-valued

As an exercise in manipulation let us pr

1° $\Theta(x)$ *is an even function.*

For
$$\Theta(-x) = \sum e^{-nx+a}$$

If we compare this series with 2) we

sponding to $n = m$ and $n = -m$ have sim

reader will see if he actually writes ou

Cf. 114, 2.

2° $\Theta(x + 2\,ma) = e^{-m(x+ma)}\Theta(x)$. $m =$

For we can write 2) in the form
$$\Theta(x) = e^{-\frac{x^2}{4a}} \sum_n e^{\frac{(x+2na)^2}{4a}}$$

Thus
$$\Theta(x + 2\,ma) = e^{-\frac{(x+2ma)^2}{4a}} \sum_n e$$
$$= e^{-m(x+ma)} \cdot e$$

which with 4) gives 3).

CHAPTER IV

MULTIPLE SERIES

116. Let $x = x_1, \cdots x_m$ be a point in m-way space $\Re_m$. I
coördinates of x are all integers or zero, x is called a *lattice*
and any set of lattice points a *lattice system*. If no coördina
any point in a lattice system is negative, we call it a *non-ne*
lattice system, etc. Let $f(x_1 \cdots x_m)$ be defined over a l
system $\iota = \iota_1, \cdots \iota_m$. The set $\{f(\iota_1 \cdots \iota_m)\}$ is called an m
sequence. It is customary to set

$$f(\iota_1 \cdots \iota_m) = a_{\iota_1 \cdots \iota_m}.$$

Then the sequence is represented by

$$A = \{a_{\iota_1 \cdots \iota_m}\}.$$

The terms

$$\lim A \quad , \quad \lim A \quad , \quad \lim A$$

as $\iota_1 \cdots \iota_m$ converges to an ideal point have therefore been de
and some of their elementary properties given in the discu
of I, 314–328; 336–338.

Let $x = x_1 \cdots x_m$ $y = y_1 \cdots y_m$ be two points in $\Re_m$.
$y_1 \geq x_1 \cdots y_m \geq x_m$ we shall write more shortly $y \geq x$.
ranges over a set of points $x' \geq x'' \geq x''' \cdots$ we shall say tha

117. A very important class of multiple sequ
with multiple series as we now show. Let $a_{\iota_1 \cdots}$
a non-negative lattice system. The symbol

$$\Sigma a_{\iota_1 \cdots \iota_m} \qquad \iota_1 = 0, 1, \cdots \nu_1 \quad , \quad \cdots \iota_m =$$

or

$$\sum_0^{\nu_1 \cdots \nu_m} a_{\iota_1 \cdots \iota_m} \quad , \quad \text{or } A_{\nu_1 \cdots \nu_m}$$

denotes the sum of all the a's whose lattice po
tangular cell $\quad 0 \leq x_1 \leq \nu_1 \quad \cdots 0 \leq x_m \leq \nu_m$.

Let us denote this cell by $R_{\nu_1 \cdots \nu_m}$ or by R_ν. T
effected in a variety of ways. To fix the ideas

$$A_{\nu_1 \nu_2 \nu_3} = \sum_0^{\nu_1} \sum_0^{\nu_2 \nu_3} a_{\iota_1 \iota_2 \iota_3} = \sum_0^{\nu_2 \nu_3} \sum_0^{\nu_1} a_{\iota_1 \iota_2 \iota_3} = \sum_0^{\nu_1} \sum_0^{\nu_2} \Sigma$$

etc. In the first sum, we sum up the terms
then add these results. In the second sum, w
parallel lines and then add the results. In th
the terms on the parallel lines lying in a given
results; we then sum over the different planes.

Returning now to the general case, the symb

$$A = \Sigma a_{\iota_1 \cdots \iota_m} \qquad \iota_1, \cdots \iota_m = 0, 1, \cdot$$

or

$$A = \sum_0^{\infty} a_{\iota_1 \cdots \iota_m}$$

is called an m-tuple infinite series. For $m =$
out more fully thus

$$a_{00} + a_{01} + a_{02} + \cdots$$

$$+ a_{10} + a_{11} + a_{12} + \cdots$$

$$+ a_{20} + a_{21} + a_{22} + \cdots$$

is finite, A is *convergent* and the limit 2) is called the sum of series A. When no confusion will arise, we may denote the series and its sum by the same letter. If the limit 2) is infinite or does not exist, we say A is *divergent*.

Thus every m-tuple series gives rise to an m-tuple sequence $\{A_{\nu_1 \dots \nu_m}\}$. Obviously if all the terms of A are ≥ 0 and A is divergent, the limit 2) is $+\infty$. In this case we say A is *infinite*.

Let us replace certain terms of A by zeros, the resulting series may be called the *deleted series*. If we delete A by replacing the terms of the cell $R_{\nu_1 \dots \nu_m}$ by zero, the resulting series is called the *remainder* and is denoted by $\bar{A}_{\nu_1 \dots \nu_m}$ or by $\bar{A}_\nu$. Similarly if the cell R_ν contains the cell R_μ, the terms lying in R_ν and not in R_μ may be denoted by $A_{\mu,\nu}$.

The series obtained from A by replacing each term of A by its numerical value is called the *adjoint series*. In a similar manner most of the terms employed for simple series may be carried over to m-tuple series. In the series $\Sigma a_{\iota_1 \dots \iota_m}$ the indices ι all begin with 0. There is no necessity for this; they may each begin with any integer at pleasure.

118. *The Geometric Series.* We have seen that

$$\frac{1}{1-a} = 1 + a + a^2 + \cdots \qquad |a| < 1,$$

$$\frac{1}{1-b} = 1 + b + b^2 + \cdots \qquad |b| < 1.$$

Hence

$$\frac{1}{(1-a)(1-b)} = \sum_0^\infty a^m b^n$$

119. 1. It is important to show how any te

be expressed by means of the $A_{\nu_1\dots\nu_m}$.

Let $\qquad D_{\nu_1\nu_2\cdots\nu_{m-1}} = A_{\nu_1\nu_2\cdots\nu_m} - A_{\nu_1\nu_2\cdots\nu_{r}}$

Then $\qquad D_{\nu_1\nu_2\cdots\nu_{m-1}-1} = A_{\nu_1\nu_2\cdots\nu_{m-1}-1\,\nu_m} -$

Let $\qquad D_{\nu_1\nu_2\cdots\nu_{m-2}} = D_{\nu_1\nu_2\cdots\nu_{m-1}} - D_{\nu_1\nu_2\cdots}$

Similarly

$$D_{\nu_1\nu_2\cdots\nu_{m-3}} = D_{\nu_1\nu_2\cdots\nu_{m-2}} - D_{\nu_1\nu_2\cdots}$$

$$D_{\nu_1\nu_2\cdots\nu_{m-4}} = D_{\nu_1\nu_2\cdots\nu_{m-3}} - D_{\nu_1\nu_2\cdots}$$

. . . .

Finally $\qquad D_{\nu_1} = D_{\nu_1\nu_2} - D_{\nu_1\nu_2-1},$

and $\qquad a_{\nu_1\nu_2\cdots\nu_m} = D_{\nu_1} - D_{\nu_1-1}.$

If now we replace the D's by their values

the relation 7) shows that $a_{\nu_1\dots\nu_m}$ may be

terms of a number of $A_{\mu_1\dots\mu_m}$ where each μ_r

For $m = 2$ we find

$$a_{\nu_1\nu_2} = A_{\nu_1\nu_2} + A_{\nu_1-1,\nu_2-1} - A_{\nu_1,\nu_2-1}$$

2. From 1 it follows that we may take a

to form a multiple series

$$A = \Sigma a_{\iota_1\dots\iota_m}.$$

This fact has theoretic importance in stud

that multiple series present.

120. We have now the following theorem

3. *For A to converge it is necessary and sufficient that*

$$\lim_{\nu=\infty} A_\nu = 0.$$

4. *A series whose adjoint converges is convergent.*

5. *Let A be absolutely convergent. Any deleted series B of* *absolutely convergent and* $|B| < \mathrm{A}$.

6. *If $A = \Sigma a_{\iota_1\cdots\iota_m}$ is convergent, so is $B = \Sigma k a_{\iota_1\cdots\iota_m}$ and*

$$B = kA, \qquad k \text{ a constant.}$$

121. 1. *For A to converge it is necessary that*

$$D_{\nu_1\nu_2\cdots\nu_{m-1}}\,, \quad D_{\nu_1\nu_2\cdots\nu_{m-2}}\,, \quad \cdots D_{\nu_1}\,, \quad a_{\nu_1\nu_2\cdots\nu_m} \doteq 0, \text{ as } \nu \doteq$$

For by 120, 1

$$|A_{\lambda_1\cdots\lambda_m} - A_{\mu_1\cdots\mu_m}| < \epsilon$$

if $\lambda_1\cdots\lambda_m$, $\mu_1\cdots\mu_m > p$.

Thus by 119, 1)

$$|D_{\nu_1\nu_2\cdots\nu_{m-1}}| < \epsilon \qquad \nu > p.$$

Hence passing to the limit $p = \infty$,

$$\lim_{\nu=\infty} D_{\nu_1\cdots\nu_{m-1}} \leq \epsilon.$$

As ϵ is small at pleasure, this shows that $D_{\nu_1\cdots\nu_{m-1}} \doteq 0$. In way we may continue.

2. Although

$$\lim_{\nu_1\cdots\nu_m=\infty} a_{\nu_1\cdots\nu_m} = 0$$

when A converges, we must guard against the error of supp

As
$$\lim_{r, s = \infty} A_{r, s} = 0$$
A is convergent. But
$$\lim_{r = \infty} | a_{r, s} | = \frac{1}{a^s} \quad , \quad \lim_{s = \infty} | a$$

That is when the point (r, s) conver
(∞, s), or to the ideal point (r, ∞), a_{rs} de

3. However, we do have the theorem :

Let
$$A = \Sigma a_{\iota_1 \cdots \iota_m}$$
converge. Then for each $\epsilon > 0$ *there exists*
for any ι *outside the rectangular cell* R_λ.

This follows at once from 120, 1, since
$$a_\iota \leq A_{\mu, \nu}.$$

122. 1. *Let* $f(x_1 \cdots x_m)$ *be monotone.*
$$\lim_{x = a} f(x_1 \cdots x_m) = l \qquad x_1 < a_1, \cdots x_m <$$
exists, finite or infinite. If f *is limited,* l *i*
ited, $l = +\infty$ *when* f *is monotone increasin*
monotone decreasing.

For, let f be limited. Let $A = \alpha_1 < \alpha_2$

Then
$$\lim_{n = \infty} f(\alpha_n) = l$$
is finite by I, 109.

Let now $B = \beta_1, \beta_2, \cdots \doteq a$ be any othe

But for each α_n there exists a $\gamma_{\iota_n} \geq \alpha_n$;
hence
$$f(\gamma_{\iota_n}) \geq f(\alpha_n)$$
and therefore
$$l \geq l.$$

Similarly, for each d_n there exists an $\alpha_{\kappa_n} > \delta_n$;
hence
$$f(\delta_n) \leq f(\alpha_{\kappa_n})$$
and therefore
$$l \leq l.$$
Thus 2), 3) give
$$\lim_{n} f(x) = l.$$

Hence by I, 316, 2 the relation 1) holds.
The rest of the theorem follows along the same lines.

2. As a corollary we have

The positive term series $A = \Sigma a_{\iota_1 \cdots \iota_m}$ is convergent if $A_{\nu_1 \cdots}$ limited.

123. 1. *Let $A = \Sigma a_{\iota_1 \ldots \iota_s} = \Sigma a_\iota$, $B = \Sigma b_{\iota_1 \ldots \iota_s} = \Sigma b_\iota$ be two negative term series. If they differ only by a finite numb terms, they converge or diverge simultaneously.*

This follows at once from 120, 2.

2. *Let A, B be two non-negative term series. Let $r > 0$ a constant. If $a_\iota \leq rb_\iota$, A converges if B is convergent and A If $a_\iota \geq rb_\iota$, A diverges if B is divergent.*

For on the first hypothesis
$$A_\lambda \leq rB_\lambda,$$
and on the second
$$A_\lambda \geq rB_\lambda.$$

3. *Let A, B be two positive term series. Let r, s be po*

4. *The infinite non-negative term series*

$$\Sigma a_{\iota_1 \cdots \iota_s} \quad \textit{and} \quad \Sigma \log (1 + a_{\iota_1 \cdots \iota_s})$$

converge or diverge simultaneously.

This follows from 2.

5. *Let the power series*

$$P = \Sigma c_{m_1 m_2 \cdots m_s} x_1^{m_1} x_2^{m_2} \cdots x_s^{m_s}$$

converge at the point $a = (a_1, \cdots a_s)$, then it conver *all points x within the rectangular cell R whose cen* *and one of whose vertices is a; that is for $| x_\iota | < |$*

For since P converges at a,

$$\lim_{m=\infty} c_{m_1 m_2 \cdots} a_1^{m_1} \cdots a_s^{m_s} = 0.$$

Thus there exists an M such that each term

$$| c_{m_1 \cdots} a_1^{m_1} \cdots a_s^{m_s} | \leq M.$$

Hence

$$| c_{m_1 \cdots} x_1^{m_\iota} \cdots x_s^{m_s} | = | c_{m_1 \cdots} a_1^{m_1} \cdots a_s^{m_s} | \cdot \left| \frac{x_1}{a_1} \right|^{m_1}$$

$$\leq M \left| \frac{x_1}{a_1} \right|^{m_1} \cdots \left| \frac{x_s}{a_s} \right|^{m_s}.$$

Thus each term of P is numerically $\leq$ than responding term in the convergent geometric seri

$$\sum \left| \frac{x_1}{a_1} \right|^{m_1} \cdots \left| \frac{x_s}{a_s} \right|^{m_s}.$$

We apply now 2.

We shall call R a *rectangular cell of convergenc*

124. 1. Associated with any m-tuple series

the terms of A arranged in order lying in $R_{\lambda_2} - R_{\lambda_1}$, and so
efinitely.

Then $$\mathfrak{A} = a_1 + a_2 + \cdots + a_{s_1} + a_{s_1+1} + \cdots$$
n associate simple series of A.

2. Conversely associated with any simple series $\mathfrak{A} = \Sigma a_n$ are
nity of *associate m-tuple series.* In fact we have only to arran
terms of $\mathfrak{A}$ over the non-negative lattice points, and call n
term a_n which lies at the lattice point $\iota_1 \cdots \iota_m$ the term $a_{\iota_1 \cdots \iota_m}$

3. *Let* $\mathfrak{A}$ *be an associate series of* $A = \Sigma a_{\iota_1 \cdots \iota_m}$. *If* $\mathfrak{A}$ *is converge*
is A *and*
$$A = \mathfrak{A}.$$

For $$A_{\nu_1 \cdots \nu_m} = \mathfrak{A}_n.$$

Let now $\nu \doteq \infty$, then $n \doteq \infty$. But $\mathfrak{A}_n \doteq \mathfrak{A}$, hence $A_{\nu_1 \cdots \nu_m} \doteq \mathfrak{A}$.

4. *If the associate series* $\mathfrak{A}$ *is absolutely convergent, so is* A.

Follows from 3.

5 *If* $A = \Sigma a_{\nu_1 \cdots \nu_m}$ *is a non-negative term convergent series, all*
sociate series $\mathfrak{A}$ *converge.*

For, any $\mathfrak{A}_{m, p}$ lies among the terms of some $A_{\mu, \nu}$. But fo
fficiently large
$$A_{\mu, \nu} < \epsilon \qquad \lambda < \mu < \nu.$$
Hence
$$\mathfrak{A}_{m, p} < \epsilon \qquad m > m_0.$$

6. *Absolutely convergent series are commutative.*

For let B be the series resulting from rearranging the gi
ries A.

Then any associate $\mathfrak{B}$ of B is simply a rearrangement of
sociate series $\mathfrak{A}$ of A. But $\mathfrak{A} = \mathfrak{B}$, hence $A = B$.

7. *A simply convergent m-tuple series A can be rearran*

Then either $\mathfrak{B}'_n \doteq +\infty$ or $\mathfrak{B}''_n \doteq -\infty$, or botl
suppose the former. Then we can arrange
form a series $\mathfrak{C}$ such that $\mathfrak{C}_n \doteq +\infty$. Let n
series of C. Then

$$C_\nu = C_{\nu_1 \nu_2 \cdots \nu_m} = \mathfrak{C}_n$$

and thus

$$\lim C_\nu = \lim \mathfrak{C}_n = +\infty.$$

Hence C is divergent.

8. *If the multiple series A is commutative*
vergent.

For if simply convergent, we can rearrange
resulting series divergent, which contradicts

9. In 121, 2 we exhibited a convergent
$a_{\iota_1 \cdots \iota_m}$ does not need to converge to 0 if $\iota_1 \cdots \iota_m$
point some of whose coördinates are finite.
have the following:

Let A be absolutely convergent. Then for e
a λ, such that any finite set of terms B lying
relation

$$|B| < \epsilon;$$

and conversely.

For let $\mathfrak{A}$ be an associate simple series of
convergent there exists an n, such that

$$\overline{\mathfrak{A}}_n < \epsilon.$$

But if λ is taken sufficiently large, each
which proves 1).

Suppose now A were simply convergent.
there exists an associate series $\mathfrak{D}$ which is inf

extended over a lattice system $\mathfrak{M}$ in $\mathfrak{R}_m$ is a simple series i
We can generalize as follows. Let $\mathfrak{M} = \{\iota\}$ be associated w
lattice system $\mathfrak{N} = \{j\}$ in $\mathfrak{R}_n$ such that to each ι corresponds a j
conversely.

If $\iota \sim j$ we set $$a_{\iota_1 \cdots \iota_m} = a_{j_1 \cdots j_n}.$$

Then A gives rise to an infinity of n-tuple series as

$$B = \Sigma a_{j_1 \cdots j_n}.$$

We say B is a *conjugate n-tuple series.*

We have now the following:

Let A be absolutely convergent. Then the series B is abso convergent and $A = B$.

For let A', B' be associate simple series of A, B. Then are absolutely convergent and hence $A' = B'$. But $A = A'$, B
Hence $A = B$, and B is absolutely convergent.

11. Let $A = \Sigma a_{\iota_1 \cdots \iota_m}$ be absolutely convergent. Let $B = \Sigma$
be any p-tuple series formed of a part or all the terms of A.
B is absolutely convergent and

$$|B| \leq \text{Adj } A.$$

For let A', B' be associate simple series of A and B. Th
converges absolutely and $|B'| \leq \text{Adj } A$.

125. 1. Let $$A = \Sigma a_{\iota_1 \cdots \iota_m}.$$

Set $$f(x_1 \cdots x_m) = a_{\iota_1 \cdots \iota_m}$$
in the cell
$$\iota_1 - 1 < x_1 \leq \iota_1, \quad \cdots \quad \iota_m - 1 < x_m \leq \iota_m.$$

by employing a sequence of rectangular cell
$a_\nu \geq 0$ we may, and we have

For the non-negative term series 1) *to converge i*
sufficient that the integral 3) *converges.*

2. *Let* $f(x_1 \cdots x_m) \geq 0$ *be a monotone decre*
x *in* R, *the aggregate of points all of whose coö*
negative. Let

$$a_{\iota_1 \cdots \iota_m} = f(\iota_1 \cdots \iota_m).$$

The series

$$A = \Sigma a_{\iota_1 \cdots \iota_m}$$

is convergent or divergent with

$$J = \int_R f dx_1 \cdots dx_m.$$

For let $R_1, R_2, \cdots$ be a sequence of rectangul
contained in R_{n+1}.
Let

$$R_{n,s} = R_s - R_n \qquad s > n.$$

Then λ, μ being taken at pleasure but $>$ some
l, m such that

$$A_{\lambda\mu} < \int_R f_{lm}.$$

But the integral on the right can be made small
is convergent on taking $l > m >$ some n. Hence
if J is. Similarly the other half of the theorem

Iterated Summation of Multiple

126. Consider the finite sum

$$\Sigma a_{\iota_1 \cdots \iota_m} \qquad \iota_1 = 0, 1, \cdots n_1 \quad \cdots \quad \iota_m = 0,$$

and so on arriving finally at

$$\sum_{\iota_m=0}^{m_n} \cdots \sum_{\iota_1=0}^{m_1} a_{\iota_1 \cdots \iota_m}$$

whose value is that of 1). We call this process *iterated* *tion.* We could have taken the indices $\iota_1 \cdots \iota_m$ in a instead of the one just employed; in each case we wo arrived at the same result, due to the commutative pr finite sums.

Let us see how this applies to the infinite series,

$$A = \Sigma a_{\iota_1 \cdots \iota_m}, \qquad \iota_1 \cdots \iota_m = 0, 1, \cdots \infty.$$

The corresponding process of iterated summation woul to a series

$$\mathfrak{A} = \sum_{\iota_m=0}^{\infty} \sum_{\iota_{m-1}=0}^{\infty} \cdots \sum_{\iota_1=0}^{\infty} a_{\iota_1 \cdots \iota_m},$$

which is an *m-tuple iterated series.* Now by definition

$$\mathfrak{A} = \lim_{\nu_m = \infty} \sum_{\iota_m=0}^{\nu_m} \lim_{\nu_{m-1}=\infty} \sum_{\iota_{m-1}=0}^{\nu_{m-1}} \cdots \lim_{\nu_1=\infty} \sum_{\iota_1=0}^{\nu_1} a_{\iota_1 \cdots \iota_m}$$

$$= \lim_{\nu_m=\infty} \lim_{\nu_{m-1}=\infty} \cdots \lim_{\nu_1=\infty} A_{\nu_1 \cdots \nu_m},$$

while

$$A = \lim_{\nu_1 \cdots \nu_m} A_{\nu_1 \cdots \nu_m}.$$

Thus A is defined by a general limit while $\mathfrak{A}$ is defin iterated limit. These two limits may be quite different. in 6) we have passed to the limit in a certain order. C this order in 6) would give us another iterated series of 4) with a sum which may be quite different. However i

Let
$$A = a_{00} + a_{01} + a_{02} + \cdots + a_{10} + a_{11} + a$$

be a double series. The m^{th} row forms a series

$$R^{(m)} = a_{m,0} + a_{m1} + \cdots = \sum_{n=0}^{\infty} a_{mn}$$

and the n^{th} column, the series

$$C^{(n)} = a_{0n} + a_{1n} + \cdots = \sum_{m=0}^{\infty} a_{mn}.$$

Then
$$R = \sum_{m=0}^{\infty} R^{(m)} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn},$$

$$C = \sum_{n=0}^{\infty} C^{(n)} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{mn}$$

are the series formed by *summing by rows* anc
tively.

2. *A double series may converge although eve column is divergent.*

This is illustrated by the series considered i

is convergent while $\sum_{r=0}^{\infty} a_{rs}$, $\sum_{s=0}^{\infty} a_{rs}$ are divergent, sin
not evanescent.

3. *A double series A may be divergent althoug tained by summing A by rows or the series C obta by columns is convergent.*

For let
$$A_{rs} = 0 \qquad \text{if } r \text{ or } s = 0$$

$$= \frac{r}{r+s} \qquad \text{if } r, s > 0.$$

4. In the last example R and C converged but their sums w
fferent. We now show:

*A double series may diverge although both R and C converge a
ve the same sum.*

For let
$$A_{r,s} = 0 \qquad \text{if } r \text{ or } s = 0$$
$$= \frac{rs}{r^2 + s^2} \qquad \text{if } r, s > 0.$$

Then by I, 319, $\lim A_{rs}$ does not exist and A is divergent. O
e other hand,
$$R = \lim_{r=\infty} \lim_{s=\infty} A_{rs} = 0,$$
$$C = \lim_{s=\infty} \lim_{r=\infty} A_{rs} = 0.$$

Then R and S both converge and have the same sum.

128. We consider now some of the cases in which iterated su
tion is permissible.

*Let $A = \sum_0^\infty a_{\iota_1 \cdots \iota_m}$ be convergent. Let $\iota_1', \iota_2', \cdots \iota_m'$ be any permutat
the indices $\iota_1, \iota_2, \cdots \iota_m$. If all the $m-1$-tuple series*
$$\sum_{\iota_2'=0}^\infty \sum_{\iota_3'=0}^\infty \cdots \sum_{\iota_m'=0}^\infty a_{\iota_1 \cdots \iota_m}.$$
e convergent,
$$A = \sum_{\iota_1'=0}^\infty \cdots \sum_{\iota_m'=0}^\infty a_{\iota_1 \cdots \iota_m}.$$

This follows at once from I, 324. For simplicity the theor
there stated only for two variables; but obviously the dem
ation applies to any number of variables.

For if f is limited, $\lim_{x_{\iota_s}=a_{\iota_s}} f \quad , \quad x_{\iota_s} < a_{\iota_s}$

exists by 122, 1. Moreover 3) is a monotone fu
maining $m-1$ variables.

Hence similarly $\lim_{x_{\iota_{s-1}}=a_{\iota_{s-1}}} \lim_{x_{\iota_s}=a_{\iota_s}} f$

exists and is a monotone function of the remai
ables, etc. The rest of the theorem follows as i

2. As a corollary we have

*Let A be a non-negative term m-tuple series. I
its m-tuple iterated series is convergent, A and a
m-tuple series are convergent and have the same sun
series is divergent, they all are.*

3. *Let a be a non-negative term m-tuple series.
the s-tuple iterated series of A are convergent if A
these iterated series is divergent, so is A.*

130. 1. *Let $A = \Sigma a_{\iota_1 \cdots \iota_m}$ be absolutely converg
s-tuple iterated series $s = 1, 2 \cdots m$, converge a
m-tuple iterated series all $= A$.*

For as usual let $\alpha_{\iota_1 \cdots \iota_m} = | a_{\iota_1 \cdots \iota_m} |$. Since A
vergent, all the s-tuple iterated series of A
Thus $s_1 = \sum_{\iota_1=0}^{\infty} a_{\iota_1 \cdots \iota_m}$ is convergent since $\sum_{\iota_1=0}^{\infty} \alpha_{\iota_1 \cdots \iota_m}$

$| s_1 | < \sigma_1$. Similarly $\sum_{\iota_2=0}^{\infty} \sum_{\iota_1=0}^{\infty} a_{\iota_1 \cdots \iota_m} = \sum_{\iota_2} s_1$ is c

$\sum_{\iota_2=0}^{\infty} \sum_{\iota_1=0}^{\infty} \alpha_{\iota_1 \cdots \iota_m} = \sum_{\iota_2} \sigma_1$ is convergent; etc. Thus e

Consider the series

$$A = 1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + \cdots$$

$$+ 1 + 2a + \frac{(2a)^2}{2!} + \frac{(2a)^3}{3!} + \cdots$$

$$+ 1 + 3a + \frac{(3a)^2}{2!} + \frac{(3a)^3}{3!} + \cdots$$

$$+ \cdots$$

Here $$R^{(n)} = 1 + \frac{na}{1!} + \frac{(na)^2}{2!} + \cdots = e^{na}$$

and $$R = e^a + e^{2a} + e^{3a} + \cdots$$

This is a geometric series and converges *absolutely* for $a <$ us one of the double iterated series of A is absolutely conv nt. We cannot, however, infer from this that A is converge the theorem of 2 requires that one of the iterated series form m the *adjoint* of A should converge. Now both those ser divergent. The series A is divergent, for $|a_{rs}| \doteq \infty$, $s \doteq \infty$.

131. 1. Up to the present the series

$$\Sigma a_{\iota_1 \cdots \iota_m}$$

ve been extended only over non-negative lattice points. T striction was imposed only for convenience; we show now h may be removed. Consider the signs of the coördinates o int $x = (x_1, \cdots x_m)$. Since each coördinate can have two sig ere are 2^m combinations of signs. The set of points x wh ördinates belong to a given one of these combinations form *adrant* for $m = 2$, an *octant* for $m = 3$, and a 2^m-*tant* or *poly*

If
$$\lim_{\lambda \doteq \infty} A_\lambda$$

exists, we say A is *convergent*, otherwise A
similar manner the other terms employed in r
be extended to the present case. The rectang
figures in the above definition may without lo
replaced by the cube

$$|x_1| \leq \lambda_0 \quad \cdots \quad |x_m| \leq \lambda_0.$$

Moreover the condition necessary and suffic
ence of the limit 3) is that

$$|A_\lambda - A_\mu| < \epsilon \qquad \lambda, \mu > \lambda_0$$

132. The properties of series lying in the
may be readily extended to series lying in sev
the convenience of the reader we bring the
omitting the proof when it follows along the s

1. *For A to converge it is necessary and suffic*

$$\lim_{\lambda \doteq \infty} \bar{A}_\lambda = 0.$$

2. *A series whose adjoint converges is converg*

3. *Any deleted series B of an absolutely co*
absolutely convergent and

$$|B| < \text{Adj}\, A.$$

4. *If $A = \Sigma a_{\iota_1 \ldots \iota_m}$ is convergent, so is $B = \Sigma$*

5. *The non-negative term series A is converg*
$\lambda \doteq \infty$.

Then
$$A = \Sigma a_{\iota_1 \cdots \iota_m}$$
converges or diverges with
$$\int_{\mathfrak{R}} f dx_1 \cdots dx_m,$$

the integration extended over all space containing terms of A.

133. 1. *Let* B, C, $D \cdots$ *denote the series formed of the term lying in the different polyants. For* A *to converge it is su although not necessary that* B, C, $\cdots$ *converge. When they do*

$$A = B + C + D + \cdots$$

For if B_λ, $C_\lambda \cdots$ denote the terms of B, $C \cdots$ which l rectangular cell R_λ,
$$A_\lambda = B_\lambda + C_\lambda + \cdots$$

Passing to the limit we get 1).

That A may converge when B, C, $\cdots$ do not is shown following example. Let all the terms of $A = \Sigma a_{\iota_1 \cdots \iota_m}$ vani cept those lying next to the coördinate axes. Let these ha value $+1$ if $\iota_1, \iota_2 \cdots \iota_m > 0$ and let two a's lying on opposit of the coördinate planes have the same numerical value but op signs. Obviously, $A_\lambda = 0$, hence A is convergent. On th hand, every B, $C \cdots$ is divergent.

2. Thus when B, $C \cdots$ converge, the study of the given A may be referred to series whose terms lie in a single p But obviously the theory of such series is identical with the series lying in the first polyant.

3. The preceding property enables us at once to exte theorems of 129, 130 to series lying in more than one p The iterated series will now be made up, in general of tw

CHAPTER V

SERIES OF FUNCTIONS

134. 1. Let $\iota = (\iota_1, \iota_2 \cdots \iota_p)$ run over an infi
Let the one-valued functions

$$f_{\iota_1 \ldots \iota_p}(x_1 \cdots x_m) = f_\iota(x) = f_\iota$$

be defined over a domain $\mathfrak{A}$, finite or infinite.

$$F = F(x) = F(x_1 \cdots x_m) = \Sigma f_{\iota_1 \ldots \iota_p}($$

extended over the lattice system $\mathfrak{L}$ is converge
valued function $F(x_1 \cdots x_m)$ over $\mathfrak{A}$. We p
properties of this function with reference to
tiation and integration.

2. Here, as in so many parts of the theory
ing on changing the order of an iterated limit,
is fundamental.

We shall therefore take this opportunity to
properties in an entirely general manner so t
not only to infinite series, but to infinite pro
grals, etc.

3. In accordance with the definition of I, 3
1) is *uniformly convergent in* $\mathfrak{A}$ when F_μ conve
limit F. Or in other words when for each ϵ

135. 1. Let

$$\lim_{t=\tau} f(x_1 \cdots x_m, t_1 \cdots t_n) = \phi(x_1 \cdots x_m)$$

in $\mathfrak{A}$. Here $\mathfrak{A}$, τ may be finite or infinite. If there ex $\eta > 0$ such that $f \doteq \phi$ uniformly in $V_\eta(a)$, a finite or infin shall say f converges uniformly at a; if there exists no η say f *does not* converge uniformly at a.

2. Let now a range over $\mathfrak{A}$.* Let $\mathfrak{B}$ denote the points c which no η exists or those points, they may lie in $\mathfrak{A}$ or whose vicinity the minimum of η is 0. Let D denote a division of space of norm d. Let $\mathfrak{B}_D$ denote as usual the D containing points of $\mathfrak{B}$. Let $\mathfrak{C}_D$ denote the points of $\mathfrak{A}$ $\mathfrak{B}^D$. Then $f \doteq \phi$ uniformly in $\mathfrak{C}_D$ however small d is tak then fixed. The converse is obviously true.

3. *If f converges uniformly in $\mathfrak{A}$, and if moreover it conver finite number of other points $\mathfrak{B}$, it converges uniformly in $\mathfrak{A}$ -*

For if $f \doteq \phi$ uniformly in $\mathfrak{A}$,

$$|f - \phi| < \epsilon \qquad x \text{ in } \mathfrak{A}, \qquad t \text{ in } V_{\delta_0}^*(\tau).$$

Then also at each point b_s of $\mathfrak{B}$,

$$|f - \phi| < \epsilon \qquad x = b_s \qquad t \text{ in } V_{\delta_s}^*(\tau).$$

If now $\delta < \delta_0, \delta_1, \delta_2 \cdots$ these relations hold for any x in and any t in $V_\delta^*(\tau)$.

4. *Let $f(x_1 \cdots x_m, t_1 \cdots t_n) \doteq \phi(x_1 \cdots x_m)$ uniformly in f be limited in $\mathfrak{A}$ for each t in $V_\delta^*(\tau)$. Then ϕ is limited i*

For $$\phi = f(x, t) + \epsilon' \qquad |\epsilon'| < \epsilon$$

2. *If the power series $P = a_0 + a_1x + a_2x^2 +$ end point of its interval of convergence, it con this point.*

Suppose P converges at the end point $x = R$

$$| a_{m+1}R^{m+1} + \cdots + a_nR^n | < \epsilon$$

however large n is taken. But for $0 < x \leq R$

$$| a_{m+1}x^{m+1} + \cdots + a_nx^n |$$

$$= \left| a_{m+1}R^{m+1}\left(\frac{x}{R}\right)^{m+1} + \cdot\right.$$

$$< \epsilon \qquad \text{by Abel's identi}$$

Thus the convergence is uniform at $x =$ manner we may treat $x = -R$.

3. Let $f_n(x_1 \cdots x_m)$, $n = 1, 2 \cdots$ be defined ov $|f_n| <$ some constant c_n in $\mathfrak{A}$, f_n is limited in $\mathfrak{A}$ c_n are all $<$ some constant C, we say the f *limited* in $\mathfrak{A}$. In general if each function in $\{f\}$ defined over at point set $\mathfrak{A}$ satisfy the rela

$$|f| < \text{a fixed constant } C, x \text{ in}$$

we say the f's are uniformly limited in $\mathfrak{A}$.

The series $F = \Sigma g_n h_n$ is uniformly convergent in $\mathfrak{A}$ is uniformly convergent in $\mathfrak{A}$, while $\Sigma | h_{n+1} -$ uniformly limited in $\mathfrak{A}$.

This follows at once from Abel's identity as

4. *The series $F = \Sigma g_n h_n$ is uniformly conver $\Sigma | h_{n+1} - h_n |$ is uniformly convergent, h_n is un and the G_n uniformly limited.*

6. *The series $F = \Sigma g_n h_n$ is uniformly convergent in $\mathfrak{A}$ if $G_2 = g_1 + g_2, \cdots$ are uniformly limited in $\mathfrak{A}$ and if $h_1, h_2, \cdots$ form a monotone decreasing sequence for x in $\mathfrak{A}$ but also formly evanescent.*

For by 83, 1,
$$| F_{n,p} | < | h_{n+1} | G.$$

Example. Let $A = a_1 + a_2 + \cdots$ be convergent. Let b_1, b be a limited monotone sequence. Then

$$F(x) = \sum \frac{a_n}{1 - b_n x}$$

converges uniformly in any interval $\mathfrak{A}$ which does not c point of $\left\{ \frac{1}{b_n} \right\}$.

For obviously the numbers

$$h_n = \frac{1}{1 - b_n x}$$

form a monotone sequence at each point of $\mathfrak{A}$. We now a

7. As an application of these theorems we have, using sults of 84,

The series
$$a_0 + a_1 \cos x + a_2 \cos 2x + \cdots$$

converges uniformly in any complete interval not containing the points $\pm 2m\pi$ provided $\Sigma | a_{n+1} - a_n |$ is convergent and and hence in particular if $a_1 \geq a_2 \geq \cdots \doteq 0$.

8. *The series*
$$a_0 - a_1 \cos x + a_2 \cos 2x - \cdots$$

10. *The series* $a_1 \sin x - a_2 \sin 2x + a^3 \sin 3$

converges uniformly in any complete interval no the points $\pm(2m-1)\pi$ *provided* $\Sigma \mid a_{n-1} + a_n$ $a_n \doteq 0$, *and hence in particular if* $a_1 > a_2 > \cdots$

138. 1. *Let*

$$F = \Sigma f_{\iota_1 \cdots \iota_s}(x_1 \cdots x_m)$$

be uniformly convergent in $\mathfrak{A}$. *Let* A, B *be two*

$$Af_\iota(x) \leq g_\iota(x) \leq Bf_\iota(x) \qquad \text{in}$$

Then

$$G = \Sigma g_{\iota_1 \cdots \iota_s}(x_1 \cdots x_m)$$

is uniformly convergent in $\mathfrak{A}$.

For then

$$AF_{\lambda,\mu} \leq G_{\lambda,\mu} \leq BF_{\lambda,\mu}.$$

But F being uniformly convergent,

$$| F_{\lambda,\mu} | < \epsilon.$$

2. *Let*

$$F = \Sigma f_{\iota_1 \cdots \iota_s}(x_1 \cdots x_m) \qquad f_\iota \geq$$

converge uniformly in $\mathfrak{A}$. *Then*

$$L = \Sigma \log(1 + f_\iota)$$

is uniformly convergent in $\mathfrak{A}$. *Moreover if* F *is* L.

For $f_\iota \geq 0$ in $\mathfrak{A}$, hence

$$|f_\iota| < \epsilon$$

for any ι outside some rectangular cell R_λ.

Thus for such i

$$Af_\iota \leq \log(1 + f_\iota) \leq Bf_\iota \qquad \text{in}$$

For if $f \doteq \phi$ uniformly in $\mathfrak{A}$,

$$\epsilon > 0, \qquad \delta > 0 \qquad |f - \phi| < \epsilon$$

any x in $\mathfrak{A}$ and any t in $V_\delta^*(\tau)$, δ independent of x.

But then $\qquad |\Delta| < \epsilon \qquad t$ in $V_\delta^*(\tau)$.

2. As a corollary we have :

Let $a_1, a_2, \cdots \doteq a$. *Let* $F = \Sigma f_s$ *be uniformly convergent at*

en

$$\overline{F}_n(a_n) \doteq 0.$$

140. *Example 1.*

$$\lim_{u=0} f = \lim_{u=0} \frac{\sin u \sin 2u}{\sin^2 u + x \cos^2 u} = \phi(x) = \begin{cases} 2 \text{ for } x = 0, \\ 0 \text{ for } x \neq 0. \end{cases}$$

The convergence is not uniform at $x = 0$. For

$$f = \frac{2 \cos u}{1 + x \cot^2 u}.$$

Hence if we set $x = u^2$

$$\lim_{u=0} f = 1, \qquad \text{since } u^2 \cot^2 u \doteq 1.$$

Thus on this assumption

$$\lim |f - g| = |1 - 2| = 1.$$

Example 2. $\quad F = 1 - x + x(1-x) + x^2(1-x) + x^3(1-x) + \cdots$

Here

$$F = \sum_0^\infty (1-x) \cdot x^n.$$

Hence F is uniformly convergent in any $(-r, r)$, $0 < r < 1$,

6, 2.

We can see this directly. For

We show now that F does not converge unif
For let

$$a_n = 1 - \frac{1}{n}.$$

Then

$$|\bar{F}_n(a_n)| = \left(1 - \frac{1}{n}\right)^n \doteq \frac{1}{e}$$

and F does not converge uniformly at $x = 1$, by 1

Example 3.

$$F(x) = \sum_1^\infty \frac{x^2}{(1 + nx^2)(1 + (n+1)x^2)}$$

Here

$$f_n = \frac{1}{1 + nx^2} - \frac{1}{1 + (n+1)x^2}$$

and F is telescopic. Hence

$$F_n = \frac{1}{1 + x^2} - \frac{1}{1 + (n+1)x^2}$$

$$\doteq \frac{1}{1 + x^2} \quad , \quad x \neq 0$$

$$\doteq 0 \quad , \quad x = 0.$$

Thus

$$|\bar{F}_n| = \frac{1}{1 + (n+1)x^2} \quad , \quad x \neq 0.$$

Let us take

$$a_n = \frac{1}{\sqrt{n+1}}.$$

Then

$$\bar{F}_n(a_n) = \tfrac{1}{2}$$

and F is not uniformly convergent at $x = 0$. It
$(-\infty, \infty)$ except at this point. For let us tak

Here

$$f_n = x\left\{\frac{n}{1+n^2x^2} - \frac{n+1}{1+(n+1)^2x^2}\right\}$$

F is telescopic. Hence

$$F_n = \frac{x}{1+x^2} - \frac{(n+1)x}{1+(n+1)^2x^2}$$

$$\doteq \frac{x}{1+x^2} \quad \text{in } \mathfrak{A} = (-R,\ R).$$

The convergence is not uniform at $x = 0$.

For set $\quad a_n = \dfrac{1}{n+1}$. Then

$$|F_n(a_n)| = \tfrac{1}{2}, \text{ does not } = 0.$$

It is, however, uniformly convergent in $\mathfrak{A}$ except at 0. F $|x| > \delta$,

$$|F_n(x)| = \left|\frac{(n+1)x}{1+(n+1)^2x^2}\right| < \frac{(n+1)R}{1+(n+1)^2\delta^2}.$$

$$< \epsilon \qquad \text{for } n > \text{some } m.$$

141. Let us suppose that the series F converges absolutely a formly in $\mathfrak{A}$. Let us rearrange F, obtaining the series ce F is absolutely convergent, so is G and $F = G$. We ca however, state that G is uniformly convergent in $\mathfrak{A}$, as *Bôch* shown.

Example.

$$F = \frac{1-x}{x}\{1 - 1 + x - x + x^2 - x^2 + x^3 - x^3 + \cdots$$

Here

$$F_{2n} = 0.$$

$$F_{2n+1} = x^{n-1}(1-x).$$

Let

$$a_n = 1 - \frac{1}{n}.$$

Then

$$G_{2n+2}(a_n) = \left(1 - \frac{1}{n}\right)^n \left\{ 1 - \left(1 - \right.\right.$$

$$\doteq \frac{1}{e}\left(1 - \frac{1}{e}\right) \quad \text{as } n$$

Hence G does not converge uniformly at x

142. 1. *Let $f \doteq \phi$ uniformly in a finite*
$\mathfrak{A}_2, \cdots \mathfrak{A}_p$. *Then f converges uniformly in thei*

For by definition

$$\epsilon > 0, \ \delta_s > 0, \ |f - \phi| < \epsilon \qquad x \text{ in } \mathfrak{A}_s,$$

Since there are only p aggregates, the mi
is > 0. Then 1) holds if we replace δ_s by δ.

2. The preceding theorem may not be tr
of aggregates $\mathfrak{A}_1, \mathfrak{A}_2 \cdots$ is infinite. For consi

$$F = \Sigma(1 - x)x^n,$$

which converges uniformly in $\mathfrak{A} = (0, 1)$ e

$$\mathfrak{A}_s = \left(\frac{s-1}{s}, \frac{s}{s+1}\right) \qquad s = 1, 2,$$

Then F is uniformly convergent in each $\mathfrak{A}$
union, which is $\mathfrak{A}$.

3. *Let $f \doteq \phi$, $g \doteq \psi$ uniformly in $\mathfrak{A}$.*

Then $\qquad f \pm g \doteq \phi \pm \psi \qquad$ *uniform*

4. To show that 1), 2) may be false if ϕ, ψ are not limite

Let

$$f = g = \frac{1}{x} + t, \qquad \mathfrak{A} = (0^*, 1), \qquad \tau = 0.$$

Then $\phi = \psi = \frac{1}{x}$ and the convergence is uniform.

But

$$\Delta = fg - \phi\psi = \frac{2t}{x} + t^2.$$

Let $x = t$. Then $\Delta \doteq 2$ as $t \doteq 0$, and fg does not uniformly.

Again, let

$$f = \frac{1}{x} + t, \qquad g = x + t,$$

the rest being as before.

Then

$$\phi = \frac{1}{x}, \quad \psi = x.$$

But setting $x = t$,

$$|\Delta| = \begin{vmatrix} f & \phi \\ g & \psi \end{vmatrix} = \frac{t^2 - 1}{2t^2} \doteq -\infty \text{ as } t \doteq 0$$

and $\frac{f}{g}$ does not converge uniformly to $\frac{\phi}{\psi}$.

143. 1. As an extension of I, 317, 2 we have:

Let

$$\lim_{y=\eta} f(x_1 \cdots x_m, y_1 \cdots y_p) = \phi(x_1 \cdots x_m)$$

uniformly in $\mathfrak{A}$. *Let*

$$\lim_{t=\tau} y_1(t_1 \cdots t_n) = \eta_1 \cdots \lim_{t=\tau} y_p(t_1 \cdots t_n) = \eta_p.$$

form a limited set $\mathfrak{B}$. *Let* $F(u_1 \cdots u_p)$ *be continu* *set containing* $\mathfrak{B}$. *Then*

$$\lim_{t=\tau} F(u_1 \cdots u_p) = F(v_1 \cdots v_p)$$

uniformly in $\mathfrak{A}$.

For F, being continuous in the complete set uniformly continuous. Hence for a given $\epsilon >$ fixed $\sigma > 0$, such that

$$| F(u) - F(v) | < \epsilon \qquad u \text{ in } V_\sigma(v) \quad ,$$

But as $u_\iota \doteq v_\iota$ uniformly there exists a fixed $\delta > ($

$$| u_\iota - v_\iota | < \epsilon' \quad , \quad x \text{ in } \mathfrak{A} \quad , \quad t \text{ in } V_\delta^*$$

Thus if ϵ' is sufficiently small, $u = (u_1, \cdots u_p$ when x is in $\mathfrak{A}$ and t in $V_\delta^*(\tau)$.

144. 1. *Let* $\lim_{t=\tau} f(x_1 \cdots x_m, t_1 \cdots t_n) = \phi(x_1 \cdots x_m$

uniformly in $\mathfrak{A}$. *Then* $\lim_{t=\tau} e^f = e^\phi$

uniformly in $\mathfrak{A}$, *if* ϕ *is limited.*

This is a corollary of 143, 2.

2. *Let* $\lim_{t=\tau} f(x_1 \cdots x_m, t_1 \cdots t_n) = \phi(x_1 \cdots x_m)$

uniformly in $\mathfrak{A}$. *Let* ϕ *be greater than some positi* *Then*

$$\lim_{t=\tau} \log f = \log \phi,$$

t by 2), $\log f \doteq \log \phi$ uniformly in $\mathfrak{A}$; and by 142,
$f \doteq \psi \log \phi$, uniformly in $\mathfrak{A}$. Hence 2) gives 1) by 1.

5. 1. The definition of uniform convergence may be given
tly different form which is sometimes useful. The function

$$f(x_1 \cdots x_m, t_1 \cdots t_n)$$

function of two sets of variables x and t, one ranging in an $\mathfrak{R}$,
other in an $\mathfrak{R}_n$.

t us set now $w = (x_1 \cdots x_m, t_1 \cdots t_n)$ and consider w as a point i
p way space.

s x ranges over $\mathfrak{A}$ and t over $V_\delta^*(\tau)$, let w range over $\mathfrak{B}_\delta$

$$\lim_{t=\tau} f = \phi$$

ormly in $\mathfrak{A}$ when and only when

$$\epsilon > 0, \quad \delta > 0 \qquad |f - \phi| < \epsilon \qquad w \text{ in } \mathfrak{B}_\delta, \quad \delta \text{ fixed.}$$

means of this second definition we obtain at once the follow
heorem:

Instead of the variables $x_1 \cdots x_m, t_1 \cdots t_n$ *let us introduce th*
ables $y_1 \cdots y_m, u_1 \cdots u_n$ *so that as* w *ranges over* $\mathfrak{B}_\delta$,

$$z = (y_1 \cdots y_m, u_1 \cdots u_n)$$

es over $\mathfrak{C}_\delta$, *the correspondence between* $\mathfrak{B}_\delta$, $\mathfrak{C}_\delta$ *being uniform*
$f \doteq \phi$ *uniformly in* $\mathfrak{A}$ *when and only when*

$$\epsilon > 0, \qquad \delta > 0 \qquad |f - \phi| < \epsilon \quad , \quad z \text{ in } \mathfrak{C}_\delta, \quad \delta \text{ fixed.}$$

Example. Let $f(x, n) = \dfrac{n^\lambda x^\mu}{e^{n^\nu x^\beta}}$

As the term on the right $\doteq 0$ as $n \doteq \infty$, we
in (a, b).

When, however, $a = 0$, or $b = \infty$, this reason
In this case we set

$$t = e^{n^\mu x^\beta},$$

which gives

$$x = \frac{\log^{1/\beta} \cdot t}{n^{\mu/\beta}}.$$

As the point (x, n) ranges over $\mathfrak{X}$ defined by

$$x \geq 0 \quad , \quad n > 1,$$

the point (t, n) ranges over a field $\mathfrak{T}$ defined b

$$t \geq 1 \quad , \quad n \geq 1,$$

and the correspondence between $\mathfrak{X}$ and $\mathfrak{T}$ is un

$$|f - \phi| = \frac{1}{n^{\frac{a\mu}{\beta} - \lambda}} \cdot \frac{\log^{a/\beta} \cdot t}{t}.$$

The relation 2) shows that when $x > 0$, t
when $x = 0$, $t = 1$ for any n. Thus the conv
uniform when

$$\frac{a}{\beta} > \frac{\lambda}{\mu}.$$

The convergence is not uniform at $x = 0$ when
For take

$$x = \frac{1}{n^{\lambda/a}} \quad , \quad n = 1, 2, \cdots$$

For these values of x

$$|f - \phi| = e^{n^{\frac{\beta\lambda}{a}} - n^\mu},$$

which does not $\doteq 0$ as $n \doteq \infty$.

146. 1. (*Moore, Osgood.*) *Let*

[W]e first show Φ exists. To this end we show that

$$[\epsilon > 0] \quad , \quad \delta > 0 \quad , \quad |\phi(x') - \phi(x'')| < \epsilon \qquad x', x'' \text{ in } V_\delta^*(a). \quad (1)$$

[N]ow since $f(x, t)$ converges uniformly, there exists an $\eta >$ [0 suc]h that for any x', x'' in $\mathfrak{A}$

$$\phi(x') = f(x', t) + \epsilon' \qquad t \text{ in } V_\eta^*(\tau) \quad (2)$$

$$\phi(x'') = f(x'', t) + \epsilon''. \qquad |\epsilon'|, |\epsilon''| < \frac{\epsilon}{4}. \quad (3)$$

[O]n the other hand, since $f \doteq \psi$ there exists a $\delta > 0$ such that

$$f(x', t) = \psi(t) + \epsilon''' \quad (4)$$

$$f(x'', t) = \psi(t) + \epsilon^{\text{iv}} \qquad |\epsilon'''|, |\epsilon^{\text{iv}}| < \frac{\epsilon}{4} \quad (5)$$

[for] any x', x'' in $V_\delta^*(a)$; t fixed.

[F]rom 2), 3), 4), 5) we have at once 1). Having establishe[d the] existence of Φ, we show now that $\Phi = \Psi$. For since f con[ver]ges uniformly to ϕ, we have

$$|f(x, t) - \phi(x)| < \frac{\epsilon}{3} \qquad x \text{ in } \mathfrak{A} \quad , \quad t \text{ in } V_\eta^*(\tau). \quad (6)$$

[Sin]ce $f \doteq \psi$, we have

$$[|f(]x, t) - \psi(t)| < \frac{\epsilon}{3} \qquad x \text{ in } V_{\delta'}^*(a) \quad , \quad t \text{ fixed in } V_\eta^*(\tau). \quad (7)$$

[Sin]ce $\phi \doteq \Phi$,

$$|\phi(x) - \Phi| < \frac{\epsilon}{3} \qquad x \text{ in } V_{\delta''}^*(a). \quad (8)$$

[Th]us 7), 8) hold simultaneously for $\delta < \delta'$, δ''.

Hence $$|\psi(t) - \Phi| < \epsilon \qquad t \text{ in } V_\eta^*(\tau),$$

3. The theorem in 1 obviously holds when we replace the stricted limits, by limits which are subjected to some condit. the variables are to approach their limits along some curv

4. As a corollary we have :

Let $F = \Sigma f_s(x_1 \cdots x_m)$ be uniformly convergent in $\mathfrak{A}$, of which x a limiting point. Let $\lim_{x=a} f_s = l_s$, and set $L = \Sigma l_s$. Then

$$\lim_{x=a} F = L; \qquad \textit{a finite or infinite,}$$

in other words

$$\lim \Sigma f_s = \Sigma \lim f_s.$$

Example 1.

$$F(x) = \sum \frac{e^{nx}-1}{2^n e^{nx}}$$

nverges uniformly in $\mathfrak{A} = (0, \infty)$ as we saw 136, 2, Ex. 1. F

$$\lim_{x=\infty} f_n = \frac{1}{2^n} = l_n,$$

d

$$L = \Sigma l_n = \sum \frac{1}{2^n} = 1.$$

Hence

$$\lim_{x=\infty} F(x) = 1.$$

Also

$$R \lim_{x=0} f_n = 0;$$

nce

$$R \lim_{x=0} F(x) = 0.$$

Example 2.

$$F(x) = 1 + \sum_1^\infty \frac{1}{n!}\left(\frac{\sin x}{x}\right)^n$$

Example 3.

$$F(x) = \sum_1^{\infty} \frac{x^2}{(1+nx^2)(1+(n+1)x^2)} = \Sigma f_n($$

$$= \frac{1}{1+x^2} \qquad \text{for } x \neq 0 \quad .$$

$$= 0 \qquad \text{for } x = 0.$$

Here $$\lim_{x=0} F(x) = 1,$$

while $$\Sigma \lim_{x=0} f_n(x) = \Sigma 0 = 0.$$

Thus here $$\lim_{x=0} \Sigma f_n(x) \neq \Sigma \lim_{x=0} f_n(x),$$

But F does not converge uniformly at $x = 0$.
hand, it does converge uniformly at $x = \pm\infty$.

Now $$\lim_{x=\pm\infty} F(x) = 0 \quad , \quad \lim_{x=\pm\infty} f_n(x) = 0,$$

and $$\lim_{x=\pm\infty} \Sigma f_n(x) = \Sigma \lim_{x=\pm\infty} f_n(x),$$

as the theorem requires.

Example 4. $$F(x) = \sum_1^{\infty} \left\{ \frac{nx^2}{e^{nx^2}} - \frac{(n+1)x^2}{e^{(n+1)x^2}} \right\} = \frac{x^2}{e^{x^2}},$$

which converges about $x = 0$ but not uniformly.

However, $$\lim_{x=0} \Sigma f_n(x) = \Sigma \lim_{x=0} f_n(x) = 0.$$

Thus the uniform convergence is not a necessary

147. 1. *Let* $\lim f(x_1 \cdots x_m, t_1 \cdots t_n) = \phi(x_1 \cdots x_m$

A direct proof may be given as follows:

$$f(x, t) = \phi(x) + \epsilon' \qquad |\epsilon'| < \epsilon, x \text{ in } V$$

$$\phi(x') - \phi(x'') = f(x', t) - f(x'', t) + \epsilon$$

But $\qquad |f(x'', t) - f(x', t)| < \epsilon \quad , \quad \text{if } |x' - x$

2. *Let* $F = \Sigma f_{s_1 \cdots s_p}(x_1 \cdots x_m)$ *be uniformly conve* *Let each* $f_{s_1 \cdots s_p}$ *be continuous at* a. *Then* $F(x_1 \cdots x$ *at* $x = a$.

Follows at once from 1).

3. In Ex. 3 of 140 we saw that

$$F = \sum \frac{x^2}{(1 + nx^2)(1 + (n+1)x^2)}$$

is discontinuous at $x = 0$ and does not converge un

In Ex. 4 of 140 we saw that

$$F = \sum x \frac{n(n+1)x^2 - 1}{(1 + n^2x^2)(1 + (n+1)^2x^2)}$$

does not converge uniformly at $x = 0$ and yet is co We have thus the result: *The condition of uniforn* 1, *is sufficient but not necessary.*

Finally, let us note that

$$F(x) = \sum \left\{ \frac{nx^a}{e^{nx^\beta}} - \frac{(n+1)x^a}{e^{(n+1)x^\beta}} \right\} \qquad 0 < \alpha$$

$$= \frac{x^a}{e^{x^\beta}}, \qquad x \geq 0$$

is a series which is not uniformly convergent at $F(x)$ is continuous at this point.

5. *The power series* $P = \sum a_{s_1 \cdots s_m} x_1^{s_1} \cdots x_m^{s_m}$ *is continuous at*
ner point of its rectangular cell of convergence.

For we saw P converges uniformly at this point.

6. *The power series* $P = a_0 + a_1 x + a_2 x^2 + \cdots$ *is a contin*
nction of x *in its interval of convergence.*

For we saw P converges uniformly in this interval. In p
ular we note that if P converges at an end point $x = e$ of
terval of convergence, P is continuous at e.

This fact enables us to prove the theorem on multiplication
o series which we stated 112, 4, viz.:

148. *Let*
$$A = a_0 + a_1 + a_2 + \cdots \quad , \quad B = b_0 + b_1 + b_2 + \cdots$$
$$C = a_0 b_0 + (a_0 b_1 + a_1 b_0) + (a_0 b_2 + a_1 b_1 + a_2 b_0) + \cdots$$
nverge. Then $AB = C$.

For consider the auxiliary series

$$F(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$
$$G(x) = b_0 + b_1 x + b_2 x^2 + \cdots$$
$$H(x) = a_0 b_0 + (a_0 b_1 + a_1 b_0) x + \cdots$$

Since A, B, C converge, F, G, H converge for $x = 1$, and he
solutely for $|x| < 1$. But for all $|x| < 1$,

$$H(x) = F(x) G(x).$$

Thus
$$L \lim_{x=1} H(x) = L \lim_{x=1} F(x) \cdot L \lim_{x=1} G(x),$$

$$C = A \cdot B.$$

149. 1. We have seen that we cannot say that $f \doteq \phi$ uniform

Suppose now $|\psi(x, t')| \leq |\psi(x, t)|$ for any t' i
lar cell one of whose vertices is t and whose cente
then that the convergence of f to ϕ is *steady or*
If for each x in $\mathfrak{A}$, there exists a rectangular cel
above inequality holds, we say the convergence
steady in $\mathfrak{A}$.

The modification in this definition for the case t
point is obvious. See I, 314, 315.

2. We may now state *Dini's* theorem.

Let $f(x_1 \cdots x_m, t_1 \cdots t_n) \doteq \phi(x_1 \cdots x_m)$ *steadily in*
plete field $\mathfrak{A}$ *as* $t \doteq \tau$; τ *finite or ideal. Let* f *and*
functions of x *in* $\mathfrak{A}$. *Then* f *converges uniformly t*

For let x be a given point in $\mathfrak{A}$, and

$$f(x, t) = \phi(x) + \psi(x, t).$$

We may take t' so near τ that $|\psi(x, t')| < \frac{\epsilon}{3}$.

Let x' be a point in $V_\eta(x)$. Then

$$f(x', t') = \phi(x') + \psi(x', t').$$

As f is continuous in x,

$$|f(x', t') - f(x, t')| < \frac{\epsilon}{3}.$$

Similarly,

$$|\phi(x') - \phi(x)| < \frac{\epsilon}{3}.$$

Thus

$$|\psi(x', t')| < \epsilon \qquad x' \text{ in } V_\eta(x).$$

Hence

$$|\psi(x', t)| < \epsilon \qquad \text{for any } x' \text{ in } V_\eta(x$$

and for any t in the rectangular cell determined by

uous at a. Then G and a fortiori $F = \Sigma f_{\iota_1 \cdots \iota_s}$ *converge uniforn a.*

5. *Let* $G = \Sigma\,|f_{\iota_1 \cdots \iota_s}(x_1 \cdots x_m)|$ *converge in the limited compl main* $\mathfrak{A}$, *having a as limiting point. Let* $\lim_{x=a} G$ *and each* $\lim_{x=a}$ *st. Moreover, let*

$$\lim G = \Sigma \lim f_\iota.$$

Then G is uniformly convergent at a.

For if in 4 the function had values assigned them at $x = a$ d ent from their limits, we could redefine them so that they a ntinuous at a.

150. 1. *Let* $\lim_{t=\tau} f(x_1 \cdots x_m, t_1 \cdots t_n) = \phi(x_1 \cdots x_m)$ *uniformly limited field* $\mathfrak{A}$. *Let* ϕ *be limited in* $\mathfrak{A}$. *Then*

$$\lim_{t=\tau} \underline{\overline{\int}}_{\mathfrak{A}} f = \underline{\overline{\int}}_{\mathfrak{A}} \phi = \underline{\overline{\int}}_{\mathfrak{A}} \lim_{t=\tau} f.$$

For let
$$f = \phi + \psi.$$

Since $f \doteq \phi$ uniformly
$$|\psi| < \epsilon$$

any t in some $V^*(\tau)$ and for any x in $\mathfrak{A}$.

Thus
$$\left| \underline{\overline{\int}}_{\mathfrak{A}} f - \underline{\overline{\int}}_{\mathfrak{A}} \phi \right| < \epsilon \overline{\mathfrak{A}}.$$

Remark. Instead of supposing ϕ to be limited we may supp at $f(x, t)$ is limited in $\mathfrak{A}$ for each t near τ.

2. As corollary we have

Let $\lim_{t=\tau} f(x_1 \cdots x_m, t_1 \cdots t_n) = \phi(x_1 \cdots x_m)$ *uniformly in the limi ld* $\mathfrak{A}$. *Let f be limited and integrable in* $\mathfrak{A}$ *for each t in* $V_\delta^*($ *en* ϕ *is integrable in* $\mathfrak{A}$ *and*

If the $f_{\iota_1 \cdots \iota_s}$ are not integrable, we have

$$\int_{\mathfrak{A}}^{\bullet} F = \Sigma \int_{\mathfrak{A}}^{\bullet} f_{\iota_1 \cdots \iota_s}.$$

Example.
$$F = \sum_0^\infty \frac{x^2}{(1+nx^2)(1+(n+1)x^2)}$$

does not converge uniformly at $x = 0$. Cf. 140, Ex. 3.

Here
$$F_n = 1 - \frac{1}{1+nx^2}$$

and
$$F(x) = \begin{cases} 1 & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

Hence
$$\int_0^1 F dx = 1,$$

$$\int_0^1 F_n dx = 1 - \int_0^1 \frac{dx}{1+nx^2}$$

$$= 1 - \frac{\operatorname{arctg} \sqrt{n}}{\sqrt{n}} \doteq 1.$$

Thus we can integrate F termwise although F does no
uniformly in $(0, 1)$.

151. That uniform convergence of the series

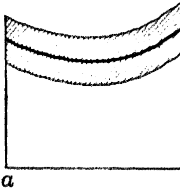
$$F(x) = f_1(x) + f_2(x) + \cdots$$

with integrable terms, in the interval $\mathfrak{A} = (a^* b)$ is
condition for the validity of the relation

$$\int_a^b F dx = \int_a^b f_1 dx + \int_a^b f_2 dx + \cdots$$

In the figure, the graph of $F(x)$ is drawn heavy. O
side of it are drawn the curves $F-\epsilon$, $F+\epsilon$ giving the
band which we call the *ε-band*.

From 2), 3) we see that the graph of each F_n, $n \geq m$ lies in the ε-band. The figure thus shows at once that



$$\int_a^b F dx \qquad \text{and} \qquad \int_a^b F_n dx$$

can differ at most by the area of the ε-band, *i.e.* by at most

$$\int_a^b 2\,\epsilon dx = 2\,\epsilon(b-a).$$

152. 1. Let us consider a case where the convergenc
uniform, as

$$F(x) = \sum_1^\infty \left\{ \frac{nx}{e^{nx^2}} - \frac{(n-1)x}{e^{(n-1)x^2}} \right\} = 0.$$

Here
$$F_n(x) = \frac{nx}{e^{nx^2}}.$$

If we plot the curves $y = F_n(x)$, we observe that they
out more and more as $n \doteq \infty$, and approach the x-axis

near the origin, they have peaks increase indefini height. The $F_n(x)$, $n > m$, and ficiently large, lie

$$\int_0^a F_n dx = \frac{1}{2}\int_0^a \frac{d}{dx}\left[-\frac{1}{e^{nx^2}}\right]dx = \frac{1}{2}\left[-\frac{1}{e^{nx^2}}\right]_0^a = \frac{1}{2}\left(1 - \frac{1}{e^{na^2}}\right)$$

$$\doteq \frac{1}{2} \qquad \text{as } n \doteq \infty.$$

hus we cannot integrate the F series termwise.

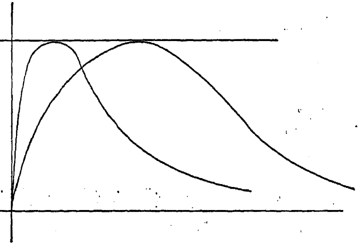
As another example in which the convergence is not uniforn
s consider

$$F(x) = \sum_0^\infty \left\{ \frac{(n+1)x}{e^{(n+1)x}} - \frac{nx}{e^{nx}} \right\} = 0.$$

ere

$$F_n = \frac{nx}{e^{nx}}.$$

he convergence of F is uniform in $\mathfrak{A} = (0, 1)$ except at $x = 0$
peaks of the curves $F_n(x)$ all have the height e^{-1}.



Obviously the area of the region under the peaks can be made small at pleasure if m is taken sufficiently large. Thus in this case we can obviously integrate termwise, although the convergence is not uniform in $\mathfrak{A}$.

e may verify this analytically. For

$$\int_0^x F_n dx = \int_0^x \frac{nx}{e^{nx}} dx = \frac{1}{n} - \frac{1+nx}{ne^{nx}} \doteq 0 \qquad \text{as } n \doteq \infty.$$

Finally let us consider

$$F(x) = \sum^\infty \left\{ \frac{(n+1)^2 x}{\quad} - \frac{n^2 x}{\quad} \right\} = 0$$

Their height thus increases indefinitely with n. But same time they become so slender that the area under then In fact

$$\int_0^a F_n(x)dx = \int_0^a \frac{1}{2n} d\log(1+n^3x^2)$$

$$= \frac{1}{2n}\Big[\log(1+n^3x^2)\Big]_0^a = \frac{1}{2}\,\frac{\log(1+n^3a^2)}{n} \doteq$$

We can therefore integrate termwise in $(0 < a)$.

153. 1. *Let* $\lim_{t=\tau} G(x, t_1 \cdots t_n) = g(x)$ *in* $\mathfrak{A} = (a, a+\delta)$, *or infinite. Let each* $G'_x(x, t)$ *be continuous in* $\mathfrak{A}$; *also let converge to a limit uniformly in* $\mathfrak{A}$ *as* $t \doteq \tau$. *Then*

$$\lim_{t=\tau} G'_x(x, t) = g'(x) \qquad \text{in } \mathfrak{A},$$

and $g'(x)$ *is continuous.*

For by 150, 2,

$$\lim_{t=\tau} \int_a^x G'_x dx = \int_a^x \lim_{t=\tau} G'_x dx.$$

By I, 538,

$$\int_a^x G'_x dx = G(x, t) - G(a, t).$$

Also by hypothesis, $\lim_{t=\tau} \{G(x, t) - G(a, t)\} = g(x) - g($

Hence

$$g(x) - g(a) = \int_a^x \lim_{t=\tau} G'_x(x, t)dx.$$

But by 147, 1, the integrand is continuous in $\mathfrak{A}$.

Hence by I, 537, the derivative of the right side of 2) is

3. The more general case that the terms $f_{\iota_1 \cdots \iota_s}$ are functions of several variables $x_1, \cdots x_m$ follows readily from 2.

154. *Example.*

$$F(x) = \sum_0^\infty \left\{ \frac{n^\lambda x^\alpha}{e^{n^\mu x^\beta}} - \frac{(n+1)^\lambda x^\alpha}{e^{(n+1)^\mu x^\beta}} \right\} = \Sigma f_n; \qquad \alpha,\ \beta,\ \lambda \geq 0,\ \mu >$$

Here
$$F_n = -\frac{n^\lambda x^\alpha}{e^{n^\mu x^\beta}},$$

function whose uniform convergence was studied, 145, 3. We sa

$$F(x) = 0 \qquad \text{for any } x \geq 0.$$

Hence
$$F'(x) = 0 \qquad x \geq 0.$$

Let
$$G(x) = \Sigma f_n'(x).$$

Then
$$G_n(x) = F_n'(x) = -\frac{\alpha n^\lambda x^{\alpha-1}}{e^{n^\mu x^\beta}} + \frac{\beta n^{\lambda+\mu} x^{\alpha+\beta-1}}{e^{n^\mu x^\beta}}.$$

If $x > 0$,
$$G_n(x) \doteq 0,$$

ence
$$F'(x) = \Sigma f_n'(x),$$

nd we may differentiate the series termwise.

If $x = 0$, and $\alpha = 1$, $\lambda > 0$; $G_n(0) = -n^\lambda \doteq -\infty$ as $n \doteq \infty$.

In this case 2) does not hold, and we cannot differentiate t ries termwise.

For $x = 0$, and $\alpha > 1$, $G_n(0) = 0$, and now 2) holds; we m erefore differentiate the series termwise. But if we look at t niform convergence of the series 1), we see this takes place on hen

$$\frac{\alpha - 1}{\beta} > \frac{\lambda}{\mu}.$$

or simplicity let us take $s = 1$. Let the series on the right be denoted by $\phi(x)$. For each x in $\mathfrak{A}$ we show that

$$\epsilon > 0, \qquad \delta > 0, \qquad D = \left|\frac{\Delta F}{\Delta x} - \phi(x)\right| < \epsilon, \qquad |\Delta x| < \delta.$$

or

$$\frac{\Delta F}{\Delta x} = \Sigma \frac{f_n(x+\Delta x) - f_n(x)}{\Delta x} = \Sigma f'_n(\xi_n)$$

re ξ_n lies in $V_\delta(x)$.

hus

$$D = \sum_1^\infty \{ f'_n(\xi_n) - f'_n(x) \} = D_m + \bar{D}_m.$$

ut G being convergent, $\bar{G}_m < \epsilon/3$ if m is taken sufficiently larg nce

$$|\bar{D}_m| < \sum_{m+1}^\infty |f'_n(\xi_n)| + \sum_{m+1}^\infty |f'_n(x)| \leq 2\,\bar{G}_m < \tfrac{2}{3}\epsilon.$$

n the other hand, since $\frac{\Delta f_n}{\Delta x} \doteq f'_n(x)$ and since there are only ns in D_m, we may take δ so small that

$$|D_m| < \epsilon/3.$$

hus

$$|D| < \epsilon \qquad \text{for } |\Delta x| < \delta.$$

Example 1. Let

$$F(x) = \sum_0^\infty \frac{(-1)^n}{n!} \cdot \frac{1}{1 + a^n x} = \Sigma f_n(x) \qquad a > 1.$$

his series converges uniformly in $\mathfrak{A} = (0 < b)$, since

$$|f_n(x)| \leq \frac{1}{n!}.$$

lso

$$f'_n(x) = \frac{(-1)^{n+1}}{} \quad \frac{a^n}{}$$

3. *Example 2. The* ϑ *functions.*

These are defined by

$$\vartheta_1(x) = 2\sum_0^\infty(-1)^n q^{(n+\frac{1}{2})^2}\sin(2n+1)\pi x$$
$$= 2q^{\frac{1}{4}}\sin\pi x - 2q^{\frac{9}{4}}\text{si}$$

$$\vartheta_2(x) = 2\sum_0^\infty q^{(n+\frac{1}{2})^2}\cos(2n+1)\pi x$$
$$= 2q^{\frac{1}{4}}\cos\pi x + 2q^{\frac{9}{4}}\text{c}$$

$$\vartheta_3(x) = 1 + 2\sum_1^\infty q^{n^2}\cos 2n\pi x$$
$$= 1 + 2q\cos 2\pi x + 2$$

$$\vartheta_0(x) = 1 + 2\sum_1^\infty(-1)^n q^{n^2}\cos 2n\pi x$$
$$= 1 - 2q\cos 2\pi x + 2$$

Let us take
$$|q| < 1.$$

Then these series converge uniformly at every
let us consider as an example ϑ_1. The series

$$T = |q| + |q|^4 + |q|^9 + \cdots$$

is convergent since the ratio of two successive ter

$$\frac{q^{(n+1)^2}}{q^{n^2}} = q^{2n+1};$$

and this $\doteq 0$. Now each term in ϑ_1 is numerically

$$\leq |q|^{(n+\frac{1}{2})^2} < |q|^{n^2},$$

and hence $<$ the corresponding term in T.

Thus $\vartheta_1(x)$ is a continuous function of x for ev

$$\vartheta_2'(x) = -2\pi\sum_0^\infty (2n+1)q^{(n+\frac{1}{2})^2}\sin(2n+1)\pi x$$
$$= -2\pi(q^{\frac{1}{4}}\sin\pi x + 3q^{\frac{9}{4}}\sin 3\pi x$$

$$\vartheta_3'(x) = -4\pi\sum_1^\infty nq^{n^2}\sin 2n\pi x$$
$$= -4\pi(q\sin 2\pi x + 2q^4\sin 4\pi x$$

$$\vartheta_0'(x) = -4\pi\sum_1^\infty (-1)^n nq^{n^2}\sin 2n\pi x$$
$$= +4\pi(q\sin 2\pi x - 2q^4\sin 4\pi x$$

To show the uniform convergence of these series, let
sider the first and compare it with

$$S = 1 + 3|q| + 5|q|^4 + 7|q|^9 + \cdots$$

The ratio of two successive terms of this series is

$$\frac{2n+3}{2n+1}\frac{|q|^{(n+1)^2}}{|q|^{n^2}} = \frac{2n+3}{2n+1}|q|^{2n+1},$$

which $\doteq 0$. Thus S is convergent. The rest follows before.

156. 1. *Let*

$$\lim_{t=\tau}\frac{G(a+h, t_1 \cdots t_n) - G(a, t_1 \cdots t_n)}{h} = \frac{g(a+h) - g(}{h}$$

uniformly for $0 < |h| < \eta$, τ *finite or infinite.*

Let

$$G_a'(a, t) \textit{ exist}$$

for each t *near* τ. *Then* $g'(a)$ *exists and*

$$\lim_{t=\tau} G_a'(a, t) = g'(a).$$

converge for each x in $\mathfrak{A}$ which has $x = a$ as a p
Let $f'_\iota(a)$ exist for each $\iota = (\iota_1, \cdots \iota_n)$. Let

$$\sum \frac{f_\iota(a+h) - f_\iota(a)}{h}$$

converge uniformly with respect to h. Then

$$F'(a) = \sum f'_{\iota_1, \dots, \iota_n}(a).$$

CHAPTER VI

POWER SERIES

.57. On account of their importance in analysis we sl
vote a separate chapter to power series.

We have seen that without loss of generality we may emp
series

$$a_0 + a_1 x + a_2 x^2 + \cdots$$

tead of the formally more general one

$$a_0 + a_1(x - a) + a_2(x - a)^2 + \cdots$$

have seen that if 1) converges for $x = c$ it converges al
ely and uniformly in $(-\gamma, \gamma)$ where $0 < \gamma < |c|$. Fina
saw that if c is an end point of its interval of convergence
nilaterally continuous at this point. The series 1) is, of cou
continuous function of x at any point *within* its interval
ivergence.

.58. 1. *Let* $P(x) = a_0 + a_1 x + a_2 x^2 + \cdots$ *converge in the inte*
$= (-\alpha, \alpha)$ *which may not be complete. The series*

$$P_n = 1 \cdot 2 \cdot \cdots n a_n + 2 \cdot 3 \cdot \cdots (n+1) a_{n+1} x + \cdots$$

ained by differentiating each term of P n *times is absolutely*
formly convergent in $\mathfrak{B} = (-\beta, \beta)$, $\beta < \alpha$, *and*

$$\frac{d^n P}{dx^n} = P_n(x), \qquad \text{in } \mathfrak{B}.$$

But the series whose general term is the last term of the preceding inequality is convergent.

2. *Let*
$$P = a_0 + a_1 x + a_2 x^2 + \cdots$$
converge in the interval $\mathfrak{A}$. *Then*
$$Q = \int_a^x P dx = \int_a^x a_0 dx + \int_a^x a_1 x dx + \cdots$$
where α, x *lie in* $\mathfrak{A}$. *Moreover* Q *considered as a function of* x *converges uniformly in* $\mathfrak{A}$.

For by 137, P is uniformly convergent in (α, x). We may therefore integrate termwise by 150, 3. To show that Q is uniformly convergent in $\mathfrak{A}$ we observe that P being uniformly convergent in $\mathfrak{A}$ we may set
$$P = P_m + \overline{P}_m$$
where
$$|\overline{P}_m| < \sigma, \qquad m > m_0, \ \sigma \text{ small at pleasure.}$$
Then
$$Q = Q_m + \overline{Q}_m$$
where
$$|\overline{Q}_m| = \left|\int_a^x \overline{P}_m dx\right| \leq \sigma \widehat{\mathfrak{A}} < \epsilon$$
on taking σ sufficiently small.

159. 1. Let us show how the theorems in 2 may be used to obtain the developments of some of the elementary functions in power series.

The Logarithmic Series. We have
$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$
for any x in $\mathfrak{A} = (-1^*, 1^*)$. Thus
$$\int_0^x \frac{dx}{1-x} = -\log(1-x) = \int_0^x dx + \int_0^x x dx + \cdots$$
Hence
$$\log(1-x) = -\left\{x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right\} \quad ; \quad x \text{ in } \mathfrak{A}.$$
This gives also
$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \quad ; \quad x \text{ in } \mathfrak{A}. \qquad (1$$

The series 1) is also valid for $x=1$. For the series is convergent for $x=1$, and $\log(1+x)$ is continuous at $x=1$. We now apply 117, 6.

For $x=1$, we get

$$\log 2 = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots$$

2. *The Development of* $\arcsin x$. We have by the Binomial Series

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2}x^2 + \frac{1\cdot 3}{2\cdot 4}x^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6 + \cdots$$

for x in $\mathfrak{A} = (-1^*, 1^*)$. Thus

$$\int_0^x \frac{dx}{\sqrt{1-x^2}} = \arcsin x = x + \frac{1}{2\cdot 3}x^3 + \frac{1\cdot 3}{2\cdot 4\cdot 5}x^5 + \cdots \tag{2}$$

It is also valid for $x=1$. For the series on the right is convergent for $x=1$. We can thus reason as in 1.

For $x=1$ we get

$$\frac{\pi}{2} = 1 + \frac{1}{2\cdot 3} + \frac{1\cdot 3}{2\cdot 4\cdot 5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7} + \cdots$$

3. *The Arctan Series.* We have

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$$

for x in $\mathfrak{A} = (-1^*, 1^*)$. Thus

$$\int_0^x \frac{dx}{1+x^2} = \arctan x = \int_0^x dx - \int_0^x x^2 dx + \cdots$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \tag{3}$$

valid in $\mathfrak{A}$. The series 3) is valid for $x=1$ for the same reason as in 2.

For $x=1$ we get $\quad \dfrac{\pi}{4} = 1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \cdots$

4. *The Development of* e^x. We have seen that

$$E(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

converges for any x. Differentiating, we get

$$E'(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$$

Hence
$$E'(x) = E(x)$$
for any x. Let us consider now the function
$$f(x) = \frac{E(x)}{e^x}.$$
We have
$$f'(x) = \frac{e^x E' - E e^x}{e^{2x}} = \frac{E' - E}{e^x} = 0$$
by (a). Thus by I, 400, $f(x)$ is a constant. For
Hence
$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$
valid for any x.

5. *Development of* $\cos x$, $\sin x$.

The series
$$C = 1 - \frac{x^2}{2\,!} + \frac{x^4}{4\,!} - \frac{x^6}{6\,!} + \cdot$$
converges for every x. Hence, differentiating,
$$C' = -\frac{x}{1} + \frac{x^3}{3\,!} - \frac{x^5}{5\,!} + \cdots$$
$$C'' = -1 + \frac{x^2}{2\,!} - \frac{x^4}{4\,!} + \cdots$$

Hence adding,
$$C + C'' = 0.$$

Let us consider now the function
$$f(x) = C \sin x + C' \cos x.$$

Then
$$f'(x) = C \cos x + C' \sin x - C' \sin x + C'$$
$$= (C + C'') \cos x$$

If we multiply (c) by $\sin x$ and (d) by $\cos x$ and add, we g
$= \cos x$. Similarly we get $C' = -\sin x$. Thus finally

$$\cos x = 1 - \frac{x^2}{2\,!} + \frac{x^4}{4\,!} - \cdots$$

$$\sin x = \frac{x}{1\,!} - \frac{x^3}{3\,!} + \frac{x^5}{5\,!} - \cdots$$

lid for any x.

160. 1. *Let* $P = a_m x^m + a_{m+1}x^{m+1} + \cdots$, $a_m \neq 0$, *converge* *me interval* $\mathfrak{A}$ *about the origin. Then there exists an interv* $< \mathfrak{A}$ *in which* P *does not vanish except at* $x = 0$.

For

$$P = x^m(a_m + a_{m+1}x + \cdots)$$

$$= x^m Q.$$

Obviously Q converges in $\mathfrak{A}$. It is thus continuous at $x =$ nce $Q \neq 0$ at $x = 0$ it does not vanish in some interval $\mathfrak{B}$ abo $= 0$ by I, 351.

In analogy to polynomials, we say P has a *zero or root of ord* at the origin.

2. *Let* $P = a_0 + a_1 x + a_2 x^2 + \cdots$ *vanish at the points* $b_1, b_2, \cdots \doteq$ *hen all the coefficients* $a_n = 0$. *The points* b_n *are supposed to* *fferent from each other and from* 0.

For by hypothesis $P(b_n) = 0$. But P being continuous at $x =$

$$P(0) = \lim P(b_n).$$

Hence

$$P(0) = 0,$$

nd thus

$$a_0 = 0.$$

be equal for the points of an infinite sequence B whos
then $a_n = b_n$, $n = 0, 1, 2 \cdots$

For $P - Q$ vanishes at the points B.

Hence
$$a_n - b_n = 0 \quad , \quad n = 0, 1, 2 \cdots$$

4. Obviously if the two series P, Q are equa
little interval about the origin, the coefficients of
equal; that is
$$a_n = b_n \quad , \quad n = 0, 1, 2 \cdots$$

161. 1. Let
$$y = a_0 + a_1 x + a_2 x^2 + \cdots$$

converge in an interval $\mathfrak{A}$. As x ranges over $\mathfrak{A}$,
an interval $\mathfrak{B}$. Let
$$z = b_0 + b_1 y + b_2 y^2 + \cdots$$

converge in $\mathfrak{B}$. Then z may be considered as a f
fined in $\mathfrak{A}$. We seek to develop z in a power seri

To this end let us raise 1) to the 2°, 3°, 4° $\cdots$
series
$$y^2 = a_{20} + a_{21} x + a_{22} x^2 + \cdots$$
$$y^3 = a_{30} + a_{31} x + a_{32} x^2 + \cdots$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

which converge absolutely within $\mathfrak{A}$.

We note that a_{mn} is a polynomial.
$$a_{m,n} = F_{m,n}(a_0, a_1 \cdots a_n)$$

in $a_0, \cdots a_n$ with coefficients which are positive inte

If we put 3) in 2), we get a double series
$$D = (b_0 + b_1 a_0) + b_1 a_1 x + b_1 a_2 x^2 + \cdots$$

where
$$c_0 = b_0 + b_1 a_0 + b_2 a_{20} + b_3 a_{30} + \cdots$$
$$c_1 = b_1 a_1 + b_2 a_{21} + b_3 a_{31} + \cdots$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

We may now state the following theorem, which is a solut our problem.

Let the adjoint y-series,

$$\eta = \alpha_0 + \alpha_1 \xi + \alpha_2 \xi^2 + \cdots$$

converge for $\xi = \xi_0$ *to the value* $\eta = \eta_0$. *Let the adjoint z serie*

$$\zeta = \beta_0 + \beta_1 \eta + \beta_2 \eta^2 + \cdots$$

converge for $\eta = \eta_0$. *Then the z series* 2) *can be developed power series in x, viz. the series* 5), *which is valid for* $|x| \leq \xi_0$

For in the first place, the series 8) converges for $\eta \leq \eta_0$. show now that the positive term series

$$\begin{aligned} \mathfrak{D} = (\beta_0 + \beta_1 \alpha_0) &+ \beta_1 \alpha_1 \xi + \beta_1 \alpha_2 \xi^2 + \cdots \\ &+ \beta_2 \alpha_{20} + \beta_2 \alpha_{21} \xi + \beta_2 \alpha_{22} \xi^2 + \cdots \\ &+ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

converges for $\xi \leq \xi_0$. We observe that $\mathfrak{D}$ differs from A at most by its first term. To show the convergence of have, raising 7) to successive powers,

$$\eta^2 = A_{20} + A_{21} \xi + A_{22} \xi^2 + \cdots$$

$$\eta^3 = A_{30} + A_{31} \xi + A_{32} \xi^2 + \cdots$$

Putting these values of η, η^2, $\eta^3 \cdots$ in 8), we get

$$\begin{aligned}\Delta = (\beta_0 + \beta_1\alpha_0) + \beta_1\alpha_1\xi + \beta_1\alpha_2\xi^2 + \cdots \\ + \beta_2 A_{20} + \beta_2 A_{21}\xi + \beta_2 A_{22}\xi^2 + \cdots \\ + \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\end{aligned}$$

Summing by rows we get a convergent series whos or 8). But this series converges for $\xi \leq \xi_0$ since th and 8) converges by hypothesis for $\eta = \eta_0$. Now b term of $\mathfrak{D}$ is $\leq$ than the corresponding term in Δ. converges for $\xi \leq \xi_0$.

2. As a corollary of 1 we have:

Let

$$y = a_0 + a_1 x + a_2 x^2 + \cdots$$

converge in $\mathfrak{A}$, *and*

$$z = b_0 + b_1 y + b_2 y^2 + \cdots$$

converge for all $-\infty < y < +\infty$. *Then* z *can be deve* *power series in* x,

$$z = c_0 + c_1 x + c_2 x^2 + \cdots = C$$

for all x *within* $\mathfrak{A}$.

3. *Let the series*

$$y = a_m x^m + a_{m+1} x^{m+1} + \cdots, \qquad m \geq 1$$

converge for some $x > 0$. *If the series*

$$z = b_0 + b_1 y + b_2 y^2 + \cdots$$

converges for some $y > 0$, *it can be developed in a power s*

$$z = c_0 + c_1 x + c_2 x^2 + \cdots$$

convergent for some $x > 0$.

converge in $\mathfrak{A} = (-A, A)$. Then y can be developed i
series about any point c of $\mathfrak{A}$,

$$y = c_0 + c_1(x-c) + c_2(x-c)^2 + \cdots$$

which is valid in an interval $\mathfrak{B}$ whose center is c and lying

162. 1. As an *application* of the theorem 161, I let u

$$z = 1 + \frac{y}{1\,!} + \frac{y^2}{2\,!} + \frac{y^3}{3\,!} + \cdots$$

$$y = \frac{x}{1\,!} - \frac{x^3}{3\,!} + \frac{x^5}{5\,!} - \cdots$$

As the reader already knows,

$$z = e^y \quad , \quad y = \sin x,$$

hence z considered as a function of x is

$$z = e^{\sin x}.$$

We have

$$\begin{array}{rlllllll}
z = 1 + x & + \, 0 \cdot x^2 & - \frac{1}{6} x^3 & + \, 0 \cdot x^4 & + \frac{1}{120} x^5 & + \, 0 \cdot x^6 & + \cdot \\
 & + \frac{1}{2} x^2 & + \; 0 & - \frac{1}{6} x^4 & + \; 0 & + \frac{1}{45} x^6 & + \cdot \\
 & & + \frac{1}{6} x^3 & + \; 0 & - \frac{1}{12} x^5 & + \; 0 & + \cdot \\
 & & & + \frac{1}{24} x^4 & + \; 0 & - \frac{1}{36} x^6 & + \cdot \\
 & & & & + \frac{1}{120} x^5 & + \; 0 & + \cdot \\
 & & & & & + \frac{1}{720} x^6 & + \cdot
\end{array}$$

Summing by columns, we get

$$z = e^{\sin x} = 1 + x + \tfrac{1}{2} x^2 - \tfrac{1}{8} x^4 - \tfrac{1}{15} x^5 - \tfrac{1}{240} x^6 \cdots$$

2. As a *second application* let us consider the power

$$z = P(y) = a_0 + a_1 y + a_2 y^2 + \cdots$$

This may be regarded as a double series. E
summed by columns. Hence

$$P(x+h) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$
$$+ h(a_1 + 2\,a_2x + 3\,a_3x^2 +$$
$$+ \frac{h^2}{2!}(2\,a_2 + 2\cdot 3\,a_3x + 3\cdot 4$$
$$+ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$= P(x) + hP'(x) + \frac{h^2}{2!}P''(x) +$$

on using 158, 1.

This, as the reader will recognize, is Taylo
the series 1) about the point x. We thus have

A power series 1) *may be developed in Tayl*
any point x within its interval of convergence.
such that $x+h$ lies within the interval of converg

163. 1. The addition, subtraction, and mult
series may be effected at once by the principles
have if

$$P = a_0 + a_1x + a_2x^2 + \cdots$$
$$Q = b_0 + b_1x + b_2x^2 + \cdots$$

converge in a common interval $\mathfrak{A}$:

$$P + Q = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2$$
$$P - Q = (a_0 - b_0) + (a_1 - b_1)x + (a_2 - b_2)x^2$$
$$P \cdot Q = a_0b_0 + (a_1b_0 + a_0b_1)x + (a_2b_0 + a_1b_1$$

These are valid within $\mathfrak{A}$, and the first two i

2. Let us now consider the division of P by

Then $1/P$ can be developed in a power series

$$\frac{1}{P} = c_0 + c_1 x + c_2 x^2 + \cdots$$

valid in $\mathfrak{B}$. *The first coefficient* $c_0 = \dfrac{1}{a_0}$.

For

$$\frac{1}{P} = \frac{1}{a_0 + Q} = \frac{1}{a_0} \cdot \frac{1}{1 + \dfrac{Q}{a_0}}$$

$$= \frac{1}{a_0} \left\{ 1 - \frac{Q}{a_0} + \frac{Q^2}{a_0^2} - \frac{Q^3}{a_0^3} + \cdots \right\}$$

for all x in $\mathfrak{B}$. We have now only to apply 161, 1.

3. Suppose $\quad P = a_m x^m + a_{m+1} x^{m+1} + \cdots \qquad a_m \neq 0.$

To reduce this case to the former, we remark that

$$P = x^m Q$$

where

$$Q = a_m + a_{m+1} x + \cdots$$

Then

$$\frac{1}{P} = \frac{1}{x^m} \cdot \frac{1}{Q}.$$

But $1/Q$ has been treated in 2.

164. 1. Although the reasoning in 161 affords us a me
determining the coefficients in the development of the quo
two power series, there is a more expeditious method ap
also to many other problems, called the method of *undet
coefficients.* It rests on the hypothesis that $f(x)$ can be de
in a power series in a certain interval about some point, le

2. *If* $f(x) = a_0 + a_1x + a_2x^2 + \cdots$

is an even function, the right-hand side can
of x; if $f(x)$ is odd, only odd powers occur

For if f is even, $f(x) = f(-x)$.

But $f(-x) = a_0 - a_1x + a_2x^2$

Subtracting 3) from 1), we have by 2)

$$0 = 2(a_1x + a_3x^3 + a_5x^5$$

for all x near the origin. Hence by 160,

$$a_1 = a_3 = a_5 = \cdots$$

The second part of the theorem is simil

165. *Example 1.* $f(x) = \tan x$.

Since $$\tan x = \frac{\sin x}{\cos x},$$

and $$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

we have

$$\tan x = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots}$$

tan x is an odd function. From 1), 2) we have, clearin…
…ions,

$$\frac{x^3}{3!}+\frac{x^5}{5!}\cdots=(a_1x+a_3x^3+\cdots)\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\right)$$

$$=a_1x+\left(a_3-\frac{a_1}{2!}\right)x^3+\left(a_5-\frac{a_3}{2!}+\frac{a_1}{4!}\right)x^5$$

$$+\left(a_7-\frac{a_5}{2!}+\frac{a_3}{4!}-\frac{a_1}{6!}\right)x^7+\left(a_9-\frac{a_7}{2!}+\frac{a_5}{4!}-\frac{a_3}{6!}+\frac{a_1}{8!}\right)x^9+\cdot$$

…omparing coefficients on each side of this equation gives

$$a_1=1.$$

$$a_3-\frac{a_1}{2!}=-\frac{1}{3!}.\qquad\therefore a_3=\frac{1}{3}.$$

$$a_5-\frac{a_3}{2!}+\frac{a_1}{4!}=\frac{1}{5!}.\qquad\therefore a_5=\frac{2}{15}.$$

$$a_7-\frac{a_5}{2!}+\frac{a_3}{4!}-\frac{a_1}{6!}=-\frac{1}{7!}.\qquad\therefore a_7=\frac{17}{315}.$$

$$a_9-\frac{a_7}{2!}+\frac{a_5}{4!}-\frac{a_3}{6!}+\frac{a_1}{8!}=\frac{1}{9!}.\qquad\therefore a_9=\frac{62}{2835}.$$

…hus $\tan x=x+\frac{1}{3}x^3+\frac{2}{15}x^5+\frac{17}{315}x^7+\frac{62}{2835}x^9+\cdots$

…d in $\left(-\frac{\pi^*}{2},\frac{\pi^*}{2}\right)$.

…xample 2. $$f(x)=\operatorname{cosec}x=\frac{1}{\sin x}$$

$$=\frac{1}{x\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-\cdots\right)}=\frac{1}{xP}=\frac{1}{x(1-(\ldots}$$

…ince

$$\ldots\ 1\ \ \sin x$$

Hence

$$1 = (a_0 + a_2x^2 + \cdots)\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right)$$

$$= a_0 + \left(a_2 - \frac{a_0}{3!}\right)x^2 + \left(a_4 - \frac{a_2}{3!} + \frac{a_0}{5!}\right)x^4$$

$$+ \left(a_6 - \frac{a_4}{3!} + \frac{a_2}{5!} - \frac{a_0}{7!}\right)x^6 + \left(a_8 - \frac{a_6}{3!} + \frac{a_4}{5!}\right.$$

Comparing like coefficients gives

$$a_0 = 1.$$

$$a_2 - \frac{a_0}{3!} = 0. \qquad \therefore\ a_2 = \frac{1}{6}.$$

$$a_4 - \frac{a_2}{3!} + \frac{a_0}{5!} = 0. \qquad \therefore\ a_4 = \frac{7}{360}.$$

$$a_6 - \frac{a_4}{3!} + \frac{a_2}{5!} - \frac{a_0}{7!} = 0. \qquad \therefore\ a_6 =$$

Thus
$$\frac{1}{\sin x} = \frac{1}{x} + \frac{1}{6}x + \frac{7}{360}x^3 + \frac{31}{3\cdot 7!}x^5$$

valid in $(-\pi^*, \pi^*)$.

166. Let
$$F(x) = f_1(x) + f_2(x) + \cdots$$

where
$$f_n(x) = a_{n0} + a_{n1}x + a_{n2}x^2 + \cdots \qquad n =$$

Let the adjoint series

$$\alpha_{n0} + \alpha_{n1}\xi + \alpha_{n2}\xi^2 + \cdots$$

converge for $\xi = R$ and have ϕ_n as sums for th

Let
$$\Phi = \phi_1 + \phi_2 + \cdots$$

F converges uniformly in $\mathfrak{A}$. For as $|x| < \xi$,

$$|f_n(x)| \leq a_{n0} + a_{n1}|x| + a_{n2}|x|^2 + \cdots$$

$$\leq a_{n0} + a_{n1}\xi + a_{n2}\xi^2 + \cdots = \phi_n.$$

We now apply 136, 2 as $\Sigma\phi_n$ is convergent for $\xi = R$.

To prove the latter part of the theorem we observe that

$$\begin{aligned} &a_{10} + a_{11}R + a_{12}R^2 + \cdots \\ &+ a_{20} + a_{21}R + a_{22}R^2 + \cdots \\ &+ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

is convergent, since summing it by rows it has Φ as sum the double series 1) converges absolutely for $|x| \leq \xi$, b Thus the series 1) may be summed by columns by **130**, 1 $F(x)$ as sum, since 1) has F as sum on summing by rows

167. *Example.*

$$F(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{1}{1 + a^n x} = \Sigma f_n(x) \qquad a > 1.$$

This series we have seen converges in $\mathfrak{A} = (0, b)$, b posi arbitrarily large.

Since it is impossible to develop the $f_n(x)$ in a power ser the origin which will have a common interval of converg us develop F in a power series about $x_0 > 0$.

We have

$$\frac{1}{1 + a^n x} = \frac{1}{1 + a^n x_0} \cdot \frac{1}{1 + \frac{a^n(x - x_0)}{}}$$

Thus F give rise to the double series

$$\begin{aligned} D = A'_{00} + A'_{01}(x - x_0) + A'_{02}(x - x_0)^2 + \cdots \\ + A'_{10} + A'_{11}(x - x_0) + A'_{12}(x - x_0)^2 + \cdots \\ + \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

where
$$A'_{n\kappa} = \frac{(-1)^n}{n!} A_{n\kappa}.$$

The adjoint series to $f_n(x)$ is, setting $\xi = |x - x_0|$,

$$\phi_n(\xi) = \frac{1}{n!}\left(\frac{1}{1 + a^n x_0} + \frac{a^n\xi}{(1 + a^n x_0)^2} + \frac{a^{2n}\xi^2}{(1 + a^n x_0)^3}\right.$$

This is convergent if

$$\frac{a^n\xi}{1 + a^n x_0} < 1 \quad \text{or if} \quad \xi < x_0,$$

that is, if
$$0 < x < 2x_0,$$

For any x such that $x_0 \leq x < 2x_0$, $\xi = x - x_0$.

Then for such an x

$$\phi_n = \frac{1}{n!}\frac{1}{1 + a^n(2x_0 - x)}$$

and the corresponding series

$$\Phi = \Sigma\phi_n$$

is evidently convergent, since $\phi_n < \frac{1}{n!}$.

168. *Inversion of a Power Series.*

Let the series

$$v = b_0 + b_1 t + b_2 t^2 + \cdots$$

have $b_0 \neq 0$, and let it converge for $t = t_0$. If we set

$$t = x t_0, \qquad u = \frac{v - b_0}{b_1 t_0},$$

it goes over into a series of the form

$$u = x - a_2 x^2 - a_3 x^3 - \cdots$$

which converges for $x = 1$. Without loss of generality v
suppose that the original series 1) has the form 2) and co
for $x = 1$. We shall therefore take the given series to be 2
I, 437, 2 the equation 2) defines uniquely a function x of u
is continuous about the point $u = 0$, and takes on the valu
for $u = 0$.

We show that this function x may be developed in a
series in u, valid in some interval about $u = 0$.

To this end let us set

$$x = u + c_2 u^2 + c_3 u^3 + \cdots$$

and try to determine the coefficient c, so that 3) satis
formally. Raising 3) to successive powers, we get

$$x^2 = u^2 + 2c_2 u^3 + (c_2^2 + 2c_3)u^4 + (2c_4 + 2c_2 c_3)u^5 + \cdots$$

$$x^3 = u^3 + 3c_2 u^4 + (3c_2^2 + 3c_3)u^5 + \cdots$$

$$x^4 = u^4 + 4c_2 u^5 + \cdots$$

. . . .

Putting these in 2) it becomes

$$u = u + (c_2 - a_2)u^2 + (c_3 - 2a_2 c_2 - a_3)u^3$$
$$+ (c_4 - a_2(c_2^2 + 2c_3) - 3a_3 c_2 - a_4)u^4$$
$$+ (c_5 - 2a_2(c_4 + c_2 c_3) - 3a_3(c_2^2 + c_3) - 4a_4 c_2 - a_5)$$

This method enables us thus to deter
3) such that this series when put in 2
relation. We shall call the series 3) whe
the values given in 6), the *inverse series* be

Suppose now the inverse series 3) con
can we say it satisfies 2) for values of u
answer is, Yes. For by 161, 3, we may
double series which results by replacing in

$$x, \quad x^2, \quad x^3, \quad \cdots$$

by their values in 3), 4). But when we
2) goes over into the right side of 5), a
$= 0$ by 6) except the first.

We have therefore only to show that
verges for some $u \neq 0$. To show this we n
2) converges for $x = 1$. Then $a_n \doteq 0$, and

$$|a_n| < \text{some } \alpha \qquad n = 2,$$

On the other hand, the relations 6) show

$$c_n = f_n(a_2, a_3, \cdots a_n)$$

is a polynomial with integral positive co
replace $a_2, a_3 \cdots$ by α, getting

$$\gamma_n = f_n(\alpha, \alpha, \cdots \alpha).$$

Obviously $$|c_n| < \gamma_n.$$

Let us now replace all the a's in 2) by α
series

$$u = x - \alpha x^2 - \alpha x^3 -$$

$$= x - \frac{\alpha x^2}{1 - x}.$$

Let us set $1 - 2(2\alpha + 1)u + u^2 = 1 - v$. For u near $u =$ < 1. Then by the Binomial Theorem

$$\sqrt{1 - v} = 1 + d_1 v + d_2 v^2 + \cdots$$

Replacing v by its value in u, this becomes a power series i ich holds for u near the origin, by 161, 3. Thus 14) shows t an be developed in a power series about the origin. Thus nverges about $u = 0$. But then by 10) the inverse series nverges in some interval about $u = 0$.

We may, therefore, state the theorem:

Let

$$u = b + b_1 x + b_2 x^2 + b_3 x^3 + \cdots \quad , \quad b_1 \neq 0,$$

verge about the point $x = 0$. Then this relation defines x nction of u which admits the development

$$x = (u - b)\left\{\frac{1}{b_1} + a_1(u - b) + a_2(u - b)^2 + \cdots\right\}$$

ut the point $u = b$. The coefficients a may be obtained from the method of undetermined coefficients.

Example. We saw that

$$u = \log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$$

If we set

$$u = x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

have $\quad a_2 = -\frac{1}{2} \quad , \quad a_3 = \frac{1}{3} \quad , \quad a_4 = -\frac{1}{4} \quad , \quad a_5 = \frac{1}{5} \cdots$

If we invert 2), we get

$$x = u + c_2 u^2 + c_3 u^3 + \cdots$$

Thus we get

$$x = u + \frac{u^2}{2!} + \frac{u^3}{3!} + \frac{u^4}{4!} + \frac{u^5}{5!} + \cdots$$

But from 1) we have

$$1 + x = e^u = 1 + \frac{u}{1!} + \frac{u^2}{3!} + \cdots$$

which agrees with 3).

Taylor's Developme…

169. 1. We have seen, I, 409, that if … first n derivatives are continuous in $\mathfrak{A}$ …

$$f(a+h) = f(a) + \frac{h}{1!}f'(a) + \frac{h^2}{2!}f''(a) + \cdots$$

$$+ \frac{h^n}{n!}f^{(n)}(a + \theta h)$$

where $\quad a < a + h < b \quad , \quad 0 < \cdots$

Consider the infinite power series in h.

$$T = f(a) + \frac{h}{1!}f'(a) + \frac{h^2}{2!}f'' \cdots$$

We call it the *Taylor's series belonging* … terms of 1) and 2) are the same. Let …

$$R_n = \frac{h^n}{n!}f^{(n)}(a + \theta h)$$

We observe that R_n is a function of n, … variable θ lying between 0 and 1.

Then

$$f(a+h)=f(a)+\frac{h}{1!}f'(a)+\frac{h^2}{2!}f''(a)$$

The above theorem is called *Taylor's theorem;*
tion 5) is the development of $f(x)$ in the interval
series.

Another form of 5) *is*

$$f(x)=f(a)+\frac{(x-a)}{1!}f'(a)+\frac{(x-a)^2}{2!}f''(a$$

When the point a is the origin, that is, when a
gives

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+$$

This is called *Maclaurin's development* and the r
Maclaurin's series. It is of course only a special c
development.

2. Let us note the content of Taylor's Theorem.

If 1° $f(x)$ can be developed in this form i
$\mathfrak{A}=(a<b)$;

2° if $f(x)$ and all its derivatives are known
$x=a$;

then the value of f and all its derivatives are k
point x within $\mathfrak{A}$.

The remarkable feature about this result is tha
tion requires a knowledge of the values of $f(x)$
$(a, a+\delta)$ as small as we please. Since the valu
tion of a real variable takes on in a part of its inter
have no effect on the values that $f(x)$ may have in
interval $(c<b)$, the condition 1° must impose a co

For differentiating 1) n times, we get

$$f^{(n)}(x) = n!\,a_n + \frac{n+1!}{2!}\,a_{n+1}(x -$$

Setting here $x = a$, we get 2).

The above theorem says that if $f(x)$ ca
power series about $x = a$, this series can be n
series.

171. 1. *Let $f^{(n)}(x)$ exist and be numericall*
stant M for all $a \leq x \leq b$, and for every n.
developed in Taylor's series for all x in $(a,$

For then
$$|R_n| < M\frac{h^n}{n!}.$$

But obviously
$$\lim_{n=\infty} \frac{h^n}{n!} = 0.$$

2. The application of the preceding theore

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdot\cdot$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}$$

which are *valid for every x.*

Since
$$a^x = e^{x \log a}, \qquad a > 0,$$

we have

$$a^x = 1 + x\frac{\log a}{1!} + x^2\frac{\log^2 a}{2!} +$$

We introduce the auxiliary function defined over (a, b).

$$g(t) = f(t) + f'(t)(x-t) + \cdots + f^{(n-1)}(t)\frac{(x-t)^{n-1}}{n-1!}.$$

Then

$$g(x) = f(x)$$

d

$$g(a) = f(a) + f'(a)(x-a) + \cdots + f^{(n-1)}(a)\frac{(x-a)^{n-1}}{n-1!}.$$

Hence

$$R_n = g(x) - g(a).$$

If we differentiate 1), we find the terms cancel in pairs, leavi

$$g'(t) = \frac{(x-t)^{n-1}}{n-1!} f^{(n)}(t).$$

We apply now Cauchy's theorem, I, 448, introducing anoth bitrary auxiliary function $G(x)$ which satisfies the conditio that theorem.

Then

$$\frac{g(x)-g(a)}{G(x)-G(a)} = \frac{g'(c)}{G'(c)}, \qquad a < c < x.$$

Using 2) and 3), we get, since $x = a + h$,

$$R_n = \frac{G(a+h)-G(a)}{G'(a+\theta h)} \frac{h^{n-1}(1-\theta)^{n-1}}{n-1!} f^{(n)}(a+\theta h)$$

here $0 < \theta < 1$.

2. If we set

$$G(x) = (b-x)^{\mu}, \qquad \mu \neq 0,$$

e have a function which satisfies our conditions. Then 4) becom

$$R_n = \frac{h^n(1-\theta)^{n-\mu}}{\quad} f^{(n)}(a+\theta h)$$

For $\mu = n$, we get from 5)

$$R_n = \frac{h^n}{n!} \cdot f^{(n)}(a + \theta h),$$

which is Lagrange's formula already obtained.

173. 1. *We consider now the development of*

$$(1 + x)^\mu \qquad x > -1 \quad , \quad \mu \text{ arbitra}$$

The corresponding Taylor's series is

$$T = 1 + \frac{\mu}{1} x + \frac{\mu \cdot \mu - 1}{1 \cdot 2} x^2 + \frac{\mu \cdot \mu - 1 \cdot \mu}{1 \cdot 2 \cdot 3}$$

We considered this series in 99, where we saw

T converges for $|x| < 1$ and diverges for $|x|$

When $x = 1$, T converges only when μ

T converges only when $\mu \geq 0$.

We wish to know when

$$(1 + x)^\mu = 1 + \frac{\mu}{1} x + \frac{\mu \cdot \mu - 1}{1 \cdot 2} x^2$$

The cases when T diverges are to be thrown
sider in succession the cases that T converg
investigate when $\lim R_n = 0$.

Case 1°. $0 < |x| < 1$. It is convenient to
form of the remainder. This gives

$$R_n = (1 - \theta)^{n-1} \frac{\mu \cdot \mu - 1 \cdots \mu - n + 1}{1 \cdot 2 \cdots n} x^n$$

$$= \frac{1}{n} S_n U_n W_n,$$

In U_n,

$$|1+\theta x|<1+|x|,$$

which is finite. Hence U_n is < some constant M.

To show that $\lim S_n = 0$, we make use of the fact that
T converges for the values of x under consideration.
every μ

$$\lim \frac{\mu \cdot \mu-1 \cdot \cdots \mu-n+2}{1 \cdot 2 \cdot \cdots n-1} x^{n-1} = 0,$$

since the limit of the n^{th} term of a convergent series
this formula replace μ by $\mu - 1$, then

$$\lim \frac{\mu-1 \cdot \mu-2 \cdot \cdots \mu-n+1}{1 \cdot 2 \cdot \cdots n-1} x^{n-1} = \lim \frac{S_n}{\mu x} = 0.$$

Hence

$$\lim S_n = 0.$$

Thus

$$\lim R_n = 0.$$

Hence 1) is valid for $|x|<1$.

Case 2. $x=1$, $\mu>-1$. We employ here Lagrange's
the remainder, which gives

$$R_n = \frac{\mu \cdot \mu-1 \cdot \cdots \mu-n+1}{1 \cdot 2 \cdot \cdots n}(1+\theta)^{\mu-n}$$

$$= U_n W_n,$$

setting

$$U_n = \frac{\mu \cdot \mu-1 \cdot \cdots \mu-n+1}{1 \cdot 2 \cdot \cdots n},$$

$$W_n = (1+\theta)^{\mu-n}.$$

Consider W_n. Since n increases without limit, $\mu - n$

Case 3. $x = -1,\ \mu \gtreqless 0$. We use here for μ
Roche form of the remainder 172, 5). We set a

$$R_n = (-1)^n \frac{(1-\theta)^{n-\mu}}{n-1\,!\,\mu}\,\mu\cdot\mu-1\cdot\cdots\mu-n$$

$$= (-1)^n \frac{\mu-1\cdot\mu-2\cdot\cdots\mu-n+1}{1\cdot 2\cdot\cdots n-1}.$$

Applying I, 143, we see that $\lim R_n = 0$.

Hence 1) is valid here if $\mu > 0$.

When $\mu = 0$ equation 1) is evidently tru
reduce to 1.

Summing up, we have the theorem :

The development of $(1+x)^\mu$ *in Taylor's s*
$|x| < 1$ *for all* μ. *When* $x = +1$ *it is nece*
when $x = -1$ *it is necessary that* $\mu \gtreqless 0$.

2. We note the following formulas obtain
$x = 1$ and -1.

$$2^\mu = 1 + \frac{\mu}{1} + \frac{\mu\cdot\mu-1}{1\cdot 2} + \frac{\mu\cdot\mu-1\cdot\mu-2}{1\cdot 2\cdot 3} +$$

$$0 = 1 - \frac{\mu}{1} + \frac{\mu\cdot\mu-1}{1\cdot 2} - \frac{\mu\cdot\mu-1\cdot\mu-2}{1\cdot 2\cdot 3} +$$

174. 1. We develop now $\log(1+x)$.
Taylor's series is

$$T = 1 + \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

We saw, 89, Ex. 2, that T converges wh

Let $-1 < x < 0$. We use here Cauchy's rema
gives, setting $x = -\xi$, $0 < \xi < 1$,

$$|R_n| = \xi^n \cdot \frac{1}{1-\theta\xi} \cdot \left(\frac{1-\theta}{1-\theta\xi}\right)^{n-1}$$

$$= S_n U_n W_n,$$

if

$$S_n = \xi^n,$$

$$U_n = \frac{1}{1-\theta\xi},$$

$$W_n = \left(\frac{1-\theta}{1-\theta\xi}\right)^{n-1}.$$

Evidently
$$\lim S_n = 0.$$
Also
$$U_n < \frac{1}{1-\xi}.$$
Finally
$$\lim W_n = 0 \qquad \text{since } \frac{1-\theta}{1-\theta\xi} < 1.$$

We can thus sum up in the theorem:

Taylor's development of $\log(1+x)$ *is valid when a*
$|x| < 1$ *or* $x = 1$. That is, for such values of x

$$\log(1+x) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

2. We note the following special case:

$$\tfrac{1}{1} - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots = \log 2.$$

The series on the left we have already met with.

175. We add for completeness the development o
ing functions for which it can be shown that $\lim R_n =$

$$\arcsin x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{x^7}{7}$$

176. We wish now to call attention
which are prevalent regarding the devel
Taylor's series.

Criticism 1. It is commonly supposed,
belonging to a function $f(x)$ is convergen

$$f(x) = T.$$

That this is not always true we proceed
examples.

Example 1. For $f(x)$ take Cauchy's fu

$$C(x) = \lim_{n \to \infty} e^{-\frac{1}{x^{2}+\frac{1}{n}}}.$$

For $x \neq 0$ $\quad C(x) = e^{-\frac{1}{x^2}}$; for $x = 0$

1° *derivative.* For $x \neq 0$, $\quad C'(x) = \frac{2}{x^3}$

For $x = 0$, $\quad C'(0) = \lim_{h=0} \frac{C(h) - C(0)}{h}$

2° *derivative.* $x \neq 0$, $\quad C''(x) = C(x)\{$

$x = 0$, $\quad C''(0) = \lim \frac{C'(h) - C'(0}{h}$

3° *derivative.* $x \neq 0$, $\quad C'''(x) = C(x)$

$x = 0$, $\quad C'''(0) = \lim \frac{C''(}{h}$

That is, T is convergent for every x, but vanishes identical is thus obvious that $C(x)$ cannot be developed about the ori Taylor's series.

Example 2. Because the Taylor's series about the origin nging to $C(x)$ vanishes identically, the reader may be incli regard this example with suspicion, yet without reason.

Let us consider therefore the following function,

$$f(x) = C(x) + e^x = C(x) + g(x).$$

Then $f(x)$ and its derivatives of every order are continuous

Since

$$f^{(n)}(x) = C^{(n)}(x) + g^{(n)}(x) \qquad n = 1, 2 \dots$$

d

$$C^{(n)}(0) = 0$$

have

$$f^{(n)}(0) = 1.$$

Hence Taylor's development for $f(x)$ about the origin is

$$T = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

This series is convergent, but it does not converge to the ri lue since

$$T = e^x.$$

177. 1. *Example 3.* The two preceding examples leave no g to be desired from the standpoint of rigor and simplici ey involve, however, a function, namely, $C(x)$, which is fined in the usual way; it is therefore interesting to have ples of functions defined in one of the ordinary everyd ays, *e.g.* as infinite series. Such examples have been given *ringsheim.*

The Taylor's series about the origin for

$$T(x) = \sum_{\lambda=0}^{\infty} \frac{x^\lambda}{\lambda!} F^{(\lambda)}(0) \quad ; \quad \lambda ! =$$

and by 2)

$$\frac{F^{(\lambda)}(0)}{\lambda!} = (-1)^\lambda \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$$

Hence

$$T(x) = \sum_{\lambda=0}^{\infty} \frac{(-1)^\lambda}{e^{a^\lambda}} x^\lambda = \Sigma$$

As $t_\lambda \geqq 0$ and $\lim t_\lambda = 0$, $t_{\lambda+1} < t_\lambda$, this se for any x in $\mathfrak{A}$. Hence T converges in $\mathfrak{A}$

2. Readers familiar with the element tions of a complex variable will know wi ing that our Taylor's series T given in 3 function F in any interval $\mathfrak{A}$, however sm $F(x)$ is an analytic function for which th singular point, since F has the poles $-\frac{1}{a}$ limiting point is 0.

3. To show by elementary means tha oped about the origin in a Taylor's serie prove now, however, with *Pringsheim*:

If we take $a \geqq \left(\frac{e+1}{e-1}\right)^2 = 4.68 \cdots$, $T($ *throughout any interval* $\mathfrak{A} = (0, b)$, *howeve*

We show 1° that if $F(x) = T(x)$ throu true in $\mathfrak{B} = (0, 2b^*)$.

In fact let $\qquad 0 < x_0 < b.$

By 161, 4 we can develop T about

equal for $0 \leq x < 2x_0$. As we can take x_0 as near b as w
$F = T$ in $\mathfrak{B}$.

By repeating the operation often enough, we can show
T in any interval $(0, B)$ where $B > 0$ is arbitrarily large
To prove our theorem we have now only to show
some one $x > 0$.

Since

$$F(x) = \left(\frac{1}{1+x} - \frac{1}{1+ax}\right) + \left(\frac{1}{2!}\frac{1}{1+a^2x} - \frac{1}{3!}\frac{1}{1+a^3x}\right)$$

we have

$$F(x) > \frac{1}{1+x} - \frac{1}{1+ax} = G(x).$$

On the other hand

$$T(x) = \frac{1}{e} - \left(\frac{x}{e^a} - \frac{x^2}{e^{a^2}}\right) - \left(\quad\right) - \cdots$$

Hence

$$T(x) < \frac{1}{e}.$$

To find a value of x for which $G \geq \frac{1}{e}$ take $x = a^{-\frac{1}{2}}$.
value of x

$$G = \frac{a^{\frac{1}{2}} - 1}{a^{\frac{1}{2}} + 1}.$$

Observe that G considered as a function of a is an i
function. For

$$a = \left(\frac{e+1}{e-1}\right)^2, \quad G = \frac{1}{e}.$$

Hence $F > T$ for $x > a^{-\frac{1}{2}}$.

178. *Criticism 2.* It is commonly thought if $f(x)$
derivatives of every order are continuous in an interv
then the corresponding Taylor's series is convergent in $\mathfrak{B}$
That this is not always so is shown by the following

Taylor's series about the origin is

$$T = \frac{1}{2}\sum_{\lambda=0}^{\infty}(-1)^{\lambda}(e^{a^{\lambda}} + e^{-a^{\lambda}} \ldots$$

since

$$F^{(\lambda)}(0) = (-1)^{\lambda}\lambda!\sum_{n=0}^{\infty}\frac{a^{2\lambda n}}{(2n)!} \ldots$$

The series T is divergent for $x > 0$, as is

179. *Criticism 3.* It is commonly thou derivatives vanish for a certain value of then $f(x)$ vanishes identically. One reas

The development of $f(x)$ about $x = a$ is

$$f(x) = f(a) + \frac{x-a}{1!}f'(a) + \frac{(x-a}{2!} \ldots$$

As f and all its derivatives vanish at a, t

$$f(x) = 0 + 0 \cdot (x-a) + 0 \cdot (x- \ldots$$

$$= 0 \text{ whatever } x \text{ is.}$$

There are two tacit assumptions which in

First, one assumes because f and all it are finite at $x = a$, that therefore $f(x)$ Taylor's series. An example to the contra $C(x)$. We have seen that $C(x)$ and all i $x = 0$, yet $C(x)$ is not identically 0; in fac viz. at $x = 0$.

Secondly, suppose $f(x)$ were developable certain interval $\mathfrak{A} = (a-h, a+h)$. Then out $\mathfrak{A}$, but we cannot infer that it is the fact* from Dirichlet's definition of a functi

Since Taylor's series T is a power series, it converges no
in $\mathfrak{A}$, but also within $\mathfrak{B} = (2a - b, a)$. It is commonly su
that $f(x) = T$ also in $\mathfrak{B}$. A moment's reflection shows s
assumption is unjustified without further conditions on f

2. *Example.* We construct a function by the method con
in I, 333, viz.

$$f(x) = \lim_{n=\infty} \frac{(1+x)^n \cos x + 1 + \sin x}{1 + (1+x)^n}.$$

Then $f(x) = \cos x$, in $\mathfrak{A} = (0, 1)$

$= 1 + \sin x$, within $\mathfrak{B} = (0, -1)$.

We have therefore as a development in Taylor's serie
in $\mathfrak{A}$,

$$f(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = T.$$

It is obviously not valid within $\mathfrak{B}$, although T converges i

3. We have given in 1) an arithmetical expression fo
Our example would have been just as conclusive if we had

Let $f(x) = \cos x$ in $\mathfrak{A}$,

and $= 1 + \sin x$ within $\mathfrak{B}$.

181. 1. *Criticism 5.* The following error is sometimes
Suppose Taylor's development

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots$$

valid in $\mathfrak{A} = (a < b)$.

It may happen that T is convergent in a larger
$\mathfrak{B} = (a < B)$.

One must not therefore suppose that 1) is also valid in

2. *Example.*

182. Let $f(x)$ have finite derivativ
$\mathfrak{A} = (a < b)$. In order that $f(x)$ can be de
series

$$f(x) = f(a+h) = f(a) + hf'(a) +$$

valid in the interval $\mathfrak{A}$ we saw that it is
that

$$\lim_{n=\infty} R_n = 0.$$

But R_n is not only a function of the inde
of the unknown variable θ which lies wit
and is a function of n and h.

Pringsheim has shown how the above con
by the following one in which θ is an indep

For the relation 1) *to be valid for all h s*
necessary and sufficient that Cauchy's form o

$$R_n(h, \theta) = \frac{(1-\theta)^{n-1}h^n}{n-1!} f^{(n)}($$

the h and θ being independent variables, con
for the rectangle D whose points (h, θ) satisf

$$0 \leq h \leq H$$

$$0 \leq \theta \leq 1.$$

1° *It is sufficient.* For then there exis
such that

$$| R_n(h, \theta) | < \epsilon$$

for every point (h, θ) of D.

The demonstration depends upon the fact that $R_n(h,$ times the n^{th} term $f_n(\alpha, k)$ of the development of $f'(x)$ ab point $a+\alpha$. In fact let $h=\alpha+k$. Then by 158

$$f'(a+h)=f'(a+\alpha+k)=f'(a+\alpha)+\cdots+\frac{k^{n-1}}{n-1!}f^{(n)}(a+$$

whose n^{th} term is

$$f_n(\alpha, k)=\frac{k^{n-1}}{n-1!}f^{(n)}(a+\alpha).$$

Let $\alpha=\theta h$, then

$$R_n(h, \alpha)=hf_n(\alpha, k)$$

as stated.

The image Δ_0, of D_0 is the half of a square of side h_0, be diagonal.

To show that R_n converges uniformly to 0 in D_0 we ha to show that

$$f_n(\alpha, k)\doteq 0 \qquad \text{uniformly in } \Delta_0$$

To this end we have from 1) for all t in $\mathfrak{A}$

$$f'(a+t)=f'(a)+tf''(a)+\frac{t^2}{2!}f'''(a)+\cdots$$

Its adjoint

$$G(t)=|f'(a)|+t|f''(a)|+\cdots$$

also converges in $\mathfrak{A}$.

By 161, 4 we can develop 4) about $t=\alpha$, which gives

$$G(\alpha, k)=G(\alpha)+kG'(\alpha)+\cdots+\frac{k^{n-1}}{n-1!}G^{(n-1)}(\alpha)+\cdot$$

But obviously $G(\alpha, k)$ is continuous in Δ_0, and evidently

To prove 6) we have from 1)

$$f^{(n)}(a+\alpha)=f^{(n)}(a)+\alpha f^{(n+1)}(a)+\frac{\alpha}{2}$$

and from 4)

$$G^{(n-1)}(\alpha)=|f^{(n)}(a)|+\alpha|f^{(n+1)}(a)|+$$

The comparison of 7), 8) proves 6).

Circular and Hyperbolic F

183. 1. We have defined the circular f
of certain lines; from this definition thei
may be deduced as is shown in trigonomet
From this geometric definition we have
cal expression for these functions. In par

$$\sin x=\frac{x}{1!}-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}$$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}$$

valid for every x.

As an interesting and instructive exerc
we propose now to develop some of the p
tions purely from their definition as infini
these series respectively S and C.

Let us also define $\tan x=\dfrac{\sin x}{\cos x}$, $\sec x=$

2. To begin, we observe that both S and

. Since S involves only odd powers of x, and C only ev
ers,

$$\sin x \text{ is an } \textit{odd}, \qquad \cos x \text{ is an } \textit{even} \text{ function.}$$

. Since S and C are power series which converge for every
have derivatives of every order. In particular

$$\frac{dS}{dx} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = C.$$

$$\frac{dC}{dx} = -\frac{x}{1} + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \cdots = -S.$$

Hence

$$\frac{d \sin x}{dx} = \cos x \quad , \quad \frac{d \cos x}{dx} = -\sin x.$$

. To get the *addition theorem*, let an index as x, y attached
C indicate the variable which occurs in the series. Then

$$S_x C_y = x - \left(\frac{x^3}{3!} + \frac{xy^2}{2!}\right) + \left(\frac{x^5}{5!} + \frac{x^3y^2}{3!2!} + \frac{xy^4}{1!4!}\right)$$
$$- \left(\frac{x^7}{7!} + \frac{x^5}{5!}\frac{y^2}{2!} + \frac{x^3}{3!}\frac{x^4}{4!} + \frac{xy^6}{6!}\right) + \cdots$$

$$C_x S_y = y - \left(\frac{y^3}{3!} + \frac{x^2y}{2!}\right) + \left(\frac{y^5}{5!} + \frac{y^3}{3!}\frac{x^2}{2!} + \frac{x^4y}{4!1!}\right)$$
$$- \left(\frac{y^7}{7!} + \frac{y^5}{5!}\frac{x^2}{2!} + \frac{y^3}{3!}\frac{x^4}{4!} + \frac{y}{1!}\frac{x^6}{6!}\right) + \cdots$$

Adding,

$$C_y + C_x S_y = x + y - \frac{1}{3!}\left\{x^3 + \binom{3}{1}x^2y + \binom{3}{1}xy^2 + y^3\right\}$$

$$+ \frac{1}{\cdot}\left\{x^5 + \binom{5}{1}x^4y + \binom{5}{2}x^3y^2 + \binom{5}{2}x^2y^3 + \binom{5}{1}xy^4 + y^5\right\} + \cdots$$

7. We can get now the important relati

$$\sin^2 x + \cos^2 x = 1$$

directly from the addition theorem. Let
aid of the series. We have

$$S'^2 = \frac{x^2}{1} - x^4\left(\frac{1}{3!} + \frac{1}{3!}\right) + x^6\left(\frac{1}{5!} + \frac{1}{3!}\right.$$

$$- x^8\left(\frac{1}{7!} + \frac{1}{3!\,5!} + \frac{1}{5!\,3!} + \frac{1}{7}\right.$$

$$C'^2 = 1 - x^2\left(\frac{1}{2!} + \frac{1}{2!}\right) + x^4\left(\frac{1}{4!} + \frac{1}{2!}\right.$$

$$- x^6\left(\frac{1}{6!} + \frac{1}{4!\,2!}\frac{1}{2!} + \frac{1}{4!} + \frac{1}{6}\right.$$

$$+ x^8\left(\frac{1}{8!} + \frac{1}{6!\,2!} + \frac{1}{4!\,4!} + \frac{1}{6}\right.$$

Hence

$$S'^2 + C'^2 = 1 - \frac{x^2}{2!}\left(1 - \binom{2}{1} + 1\right) + \frac{x^4}{4!}\left(1 - \left(\right.\right.$$

$$- \frac{x^6}{6!}\left(1 - \binom{6}{1} + \binom{6}{2} - \binom{6}{3} + \binom{6}{4}\right.$$

Now by I, 96,

$$1 - \binom{m}{1} + \binom{m}{2} - \binom{m}{3} + \cdots$$

Thus

$$S'^2 + C'^2 = \sin^2 x + \cos^2 x$$

Similarly we see $y = 1$ for $x = 0$. Γ crosses the y-axi and is parallel to the x-axis.

Since
$$S = x\left(1 - \frac{x^2}{2 \cdot 3}\right) + \frac{x^5}{5!}\left(1 - \frac{x^2}{6 \cdot 7}\right) + \cdots$$

and each parenthesis is positive for $0 < x^2 < 6$,

$$\sin x > 0 \qquad \text{for } 0 < x < \sqrt{6} = 2.449 \cdots$$

Since
$$C = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}\left(1 - \frac{x^2}{5 \cdot 6}\right) + \frac{x^8}{8!}\left(1 - \frac{x^2}{9 \cdot 10}\right) + \cdots$$

we see

$$\cos x > 0 \qquad \text{for } 0 < x < \sqrt{2} = 1.414 \cdots$$

Since
$$C = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}\left(1 - \frac{x^2}{7 \cdot 8}\right) - \frac{x^{10}}{10!}\left(1 - \frac{x^2}{11 \cdot 1}\right.$$

$$\cos x < 0 \qquad \text{for } x = 2.$$

Since $D_x \cos x = -\sin x$ and $\sin x > 0$ for $0 < x < \sqrt{6}$, $\cos x$ is a decreasing function for these values of x. As it tinuous and > 0 for $x = \sqrt{2}$, but < 0 for $x = 2$, $\cos x$ vanisl and only once in $(\sqrt{2}, 2)$.

This root, uniquely determined, of $\cos x$ we denote by $\frac{\pi}{2}$ first approximation, we have

$$\sqrt{2} < \frac{\pi}{2} < 2.$$

From 4) we have $\sin^2 \frac{\pi}{2} = 1$. As we saw $\sin x > 0$ for we have

$$\sin \frac{\pi}{2} = +1.$$

Thus $\sin x$ increases constantly from 0 to 1 while $\cos x$ d

Knowing how $\sin x$, $\cos x$ march in I_1
how they march in $I_2 = \left(\frac{\pi}{2}, \pi\right)$.

From the addition theorem,

$$\sin(\pi + x) = -\sin x, \qquad \cos(\pi +$$

Knowing how $\sin x$, $\cos x$ march in $(0, \pi)$
us about their march in $(0, 2\pi)$.

The addition theorem now gives

$$\sin(x + 2\pi) = \sin x, \qquad \cos(x +$$

Thus the functions $\sin x$, $\cos x$ are periodic

The graph of $\sin x \cos x$ for negative
recalling that $\sin x$ is odd and $\cos x$ is even

10. As a first approximation of π we fo

$$\sqrt{2} < \frac{\pi}{2} < 2.$$

By the aid of the development given 15

$$\operatorname{arctg} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}$$

we can compute π as accurately as we plea

In fact, from the addition theorem we d

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}, \quad \cos\frac{\pi}{4} =$$

Hence

$$\tan\frac{\pi}{4} = 1.$$

This in 5) gives *Leibnitz's formula*,

11. To get a more rapid means of computation, we make u the addition theorem.

To start with, let

$$\alpha = \operatorname{arctg} \tfrac{1}{5}.$$

Then 5) gives

$$\alpha = \frac{1}{5} - \frac{1}{3}\frac{1}{5^3} + \frac{1}{5}\frac{1}{5^5} - \frac{1}{7}\frac{1}{5^7} + \cdots$$

rapidly converging series.

The error E_α committed in breaking off the summation at t term is

$$E_\alpha < \frac{1}{2n-1}\frac{1}{5^{2n-1}}.$$

By virtue of the formula for duplicating the argument

$$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha},$$

e have

$$\tan 2\alpha = \tfrac{5}{12}.$$

Similarly

$$\tan 4\alpha = \tfrac{120}{119}.$$

Let

$$\beta = 4\alpha - \frac{\pi}{4}.$$

The addition theorem gives

$$\tan \beta = \frac{\tan 4\alpha - 1}{1 + \tan 4\alpha} = \frac{1}{239}.$$

Then 5) gives

$$\beta = \frac{1}{239} - \frac{1}{3}\frac{1}{239^3} + \frac{1}{5}\frac{1}{239^5} - \cdots$$

so a very rapidly converging series.

184. *The Hyperbolic Functions.* Clos
cular functions are the hyperbolic func
by the equations

$$\sinh x = \frac{e^x - e^{-x}}{2}.$$

$$\cosh x = \frac{e^x + e^{-x}}{2}.$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

$$\operatorname{sech} x = \frac{1}{\cosh x}, \quad \operatorname{cosech} x$$

Since

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} +$$

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} +$$

we have

$$\sinh x = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

valid for every x. From these equation

$$\sinh(-x) = -\sinh x \quad ; \quad \cosh($$

$$\sinh 0 = 0. \qquad \text{c}$$

$$\frac{d}{dx} \sinh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

Evidently from 3), 4)

$$\lim_{x=+\infty} \sinh x = +\infty \quad , \quad \lim_{x=+\infty} \cosh x = +\infty .$$

At $x=0$, $\cosh x$ has a minimum, and $\sinh x$ cuts the
at 45°.

For $x > 0$, $\cosh x > \sinh x$ since

$$e^x + e^{-x} > e^x - e^{-x}.$$

The two curves approach each other asymptotically as $x \doteq$
For the difference of their ordinates is e^{-x} which $\doteq 0$ as $x \doteq$

The addition theorem is easily obtained from that of e^x.

$$\sinh x \cosh y = \frac{e^x - e^{-x}}{2} \cdot \frac{e^y + e^{-y}}{2}$$

$$= \tfrac{1}{4}(e^{x+y} + e^{x-y} - e^{-x+y} - e^{-x-y}).$$

Similarly $\quad \cosh x \sinh y = \tfrac{1}{4}(e^{x+y} - e^{x-y} + e^{-x+y} - e^{-x-y}).$

Hence

$$\sinh x \cosh y + \cosh x \sinh y = \tfrac{1}{2}(e^{x+y} - e^{-(x+y)}) = \sinh (x +$$

Similarly we find

$$\cosh (x+y) = \cosh x \cosh y + \sinh x \sinh y.$$

In the same way we may show that

$$\cosh^2 x - \sinh^2 x = 1.$$

The Hypergeometric Function

185. This function, although known to Wallis, Euler,
earlier mathematicians, was first studied in detail by Ga
may be defined by the following power series in x:

$$F(\alpha, \beta, \gamma;\ x) = 1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} x + \frac{\alpha \cdot \alpha+1 \cdot \beta \cdot \beta+1}{1 \cdot 2 \cdot \gamma \cdot \gamma+1} x^2$$

The convergence of the series F was
main result obtained there is that F conv
$|x| < 1$, whatever values the parameters l
γ a negative integer or zero.

186. For special values of the param
mentary functions in the following cases :

1. If α or β is a negative integer
degree n.

2. $$F(1, 1, 2;\ -x) = \frac{1}{x} \log(1 +$$

For $$F(1, 1, 2, -x) = 1 - \frac{x}{2} + \frac{x^2}{3}$$

Also
$$\log(1+x) = x\left(1 - \frac{x}{2} +\right.$$

The relation 1) is now obvious.

Similarly we have
$$F(1, 1, 2;\ x) = -\frac{1}{x}\log(1 -$$

$$F(\tfrac{1}{2}, 1, \tfrac{3}{2}, x^2) = \frac{1}{2x}\log\frac{1+x}{1-x}.$$

3. $$F(-\alpha, \beta, \beta;\ x) = 1 - \frac{\alpha}{1}x + \frac{\alpha\cdot\alpha}{1\cdot}$$
$$= (1-x)^\alpha.$$

4. $$xF(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}, x^2) = \arcsin x.$$

5. $$xF(\tfrac{1}{2}, 1, \tfrac{3}{2}, -x^2) = \arctan x.$$

Let $0 < G < \beta$. Then

$$F\left(\beta, 1, 1;\ \frac{G}{\beta}\right) = 1 + \frac{G}{1!} + \left(1 + \frac{1}{\beta}\right)\frac{G^2}{2!} + \left(1 + \frac{1}{\beta}\right)\left(1 \right.$$

is convergent since its argument is numerically <
3), 4) we see each term of 3) is numerically $\leq$ the
term of 4) for any $|x| < G$ and any $\alpha \geq \beta$. Th
considered as a function of α is uniformly con
interval $(\beta + \infty)$ by 136, 2; and hereby x may l
in $(-G, G)$. Applying now 146, 4 to 3) and l
we see 3) goes over into 2).

7. $$\lim_{\alpha = +\infty} xF\left(\alpha, \alpha, \frac{3}{2};\ -\frac{x^2}{4\alpha^2}\right) = \sin x.$$

For

$$xF\left(\alpha, \alpha, \frac{3}{2};\ -\frac{x^2}{4\alpha^2}\right) = x - \frac{x^3}{3!} + \left(1 + \frac{1}{\alpha}\right)^2 \frac{x^5}{5!} - \left(1 + \frac{1}{\alpha}\right)^2$$

Let $x = G > 0$ and $\alpha = G$. Then

$$GF\left(G, G, \frac{3}{2};\ \frac{1}{4}\right) = G + \frac{G^3}{3!} + \left(1 + \frac{1}{G}\right)^2 \frac{G^5}{5!}$$

is convergent by 185. We may now reason as in (

8. Similarly we may show:

$$\lim_{\alpha = +\infty} F\left(\alpha, \alpha, \frac{1}{2};\ -\frac{x^2}{4\alpha^2}\right) = \cos x.$$

$$\lim_{\alpha = +\infty} F\left(\alpha, \alpha, \frac{3}{2},\ \frac{x^2}{4\alpha^2}\right) = \sinh x.$$

$$\lim F\left(\alpha, \alpha, \frac{1}{2},\ \frac{x^2}{4\alpha^2}\right) = \cosh x.$$

Between F and two of its contiguous
relation. As the number of such pairs of

$$\frac{6 \cdot 5}{1 \cdot 2} = 15,$$

there are 15 such linear relations. Le
We set

$$Q_n = \frac{\alpha + 1 \cdot \alpha + 2 \cdots \alpha + n - 1 \cdot \beta}{1 \cdot 2 \cdots n \cdot \gamma \cdot \gamma + 1}$$

Then the coefficient of x^n in $F(\alpha\beta\gamma x)$ is

$$\alpha(\beta + n - 1)Q_n;$$

in $F(\alpha + 1, \beta, \gamma, x)$ it is

$$(\alpha + n)(\beta + n - 1)Q_n$$

in $F(\alpha, \beta, \gamma - 1, x)$ it is

$$\frac{\alpha(\beta + n - 1)(\gamma + n - 1}{\gamma - 1}$$

Thus the coefficient of x^n in

$$(\gamma - \alpha - 1)F(\alpha, \beta, \gamma, x) + \alpha F(\alpha$$
$$+ (1$$

is 0. This being true for each n, we have

$$(\gamma - \alpha - 1)F(\alpha, \beta, \gamma, x) + \alpha F(\alpha +$$
$$+ (1 - \gamma)F($$

Again, the coefficient of x^n in $F(\alpha, \beta$
in $xF(\alpha + 1, \beta, \gamma, x)$ it is $n(\gamma + n - 1)Q_n$.
Hence using the above coefficients, we g

Eliminating $F(\alpha, \beta, \gamma - 1, x)$ from 1), 3) gives

$$(\beta - \alpha)F(\alpha, \beta, \gamma, x) + \alpha F(\alpha + 1, \beta, \gamma, x) - \beta F(\alpha, \beta + 1, \gamma, x) =$$

Permuting α, β in 2) gives

$$(\gamma - \alpha - \beta)F(\alpha, \beta, \gamma, x) + \beta(1 - x)F(\alpha, \beta + 1, \gamma, x) + (\alpha - \gamma)F(\alpha - 1, \beta, \gamma, x) =$$

From 3), 5) let us eliminate $F(\alpha, \beta + 1, \gamma, x)$, getting

$$(\alpha - 1 - (\gamma - \beta - 1)x)F(\alpha, \beta, \gamma, x) + (\gamma - \alpha)F(\alpha - 1, \beta,$$
$$+ (1 - \gamma)(1 - x)F(\alpha, \beta, \gamma - 1, x) =$$

In 1) let us replace α by $\alpha - 1$ and γ by $\gamma + 1$; we get

$$(\gamma - \alpha + 1)F(\alpha - 1, \beta, \gamma + 1, x) + (\alpha - 1)F(\alpha, \beta, \gamma + 1, x) - \gamma F(\alpha - 1, \beta, \gamma, x) = 0.$$

In 6) let us replace γ by $\gamma + 1$; we get

$$(\alpha - 1 - (\gamma - \beta)x)F(\alpha, \beta, \gamma + 1, x) + (\gamma + 1 - \alpha)F(\alpha - 1, \beta, \gamma$$
$$- \gamma(1 - x)F(\alpha, \beta, \gamma, x) =$$

Subtracting (b) from (a), eliminates $F(\alpha - 1, \beta, \gamma + 1,$ gives

$$\gamma(1 - x)F(\alpha\beta\gamma x) - \gamma F(\alpha - 1, \beta, \gamma, x) + (\gamma - \beta)xF(\alpha, \beta, \gamma + 1, x) =$$

From 6), 7) we can eliminate $F(\alpha - 1, \beta, \gamma, x)$, getting

$$\gamma\{\gamma - 1 + (\alpha + \beta + 1 - 2\gamma)x\}F(\alpha, \beta, \gamma, x) + (\gamma - \alpha)(\gamma - \beta)xF(\alpha, \beta, \gamma + 1, x) + \gamma(1 - \gamma)(1 - x)F(\alpha, \beta, \gamma - 1, x) =$$

In this manner we may proceed, getting the remaining

For let p, q, r be any three integers.

$$F(\alpha\beta\gamma x), \qquad F(\alpha+1, \beta, \gamma, x)$$

$$F(\alpha+p, \beta+1, \gamma, x), \quad F(\alpha+p, \beta+2, \gamma$$

$$F(\alpha+p, \beta+q, \gamma+1, x), F(\alpha+p, \beta+q, \gamma+2$$

We have $p+q+r+1$ functions, a
are contiguous. There are thus $p+$
between them. We can thus by elimin
between any three of these functions.

189. *Derivatives.* We have

$$F'(\alpha, \beta, \gamma, x) = \sum_{n=1}^{\infty} n \frac{\alpha \cdot \alpha+1 \cdots \alpha+n-1}{1 \cdot 2 \cdots n \cdot \gamma \cdot}$$

$$= \sum_{n=0}^{\infty} \frac{\alpha \cdot \alpha+1 \cdots \alpha+n \cdot}{1 \cdot 2 \cdots n+1 \cdot \gamma}$$

$$= \frac{\alpha\beta}{\gamma} \sum_{0}^{\infty} \frac{\alpha+1 \cdots \alpha+n \cdot}{1 \cdot 2 \cdots n+1 \cdot \gamma}$$

$$= \frac{\alpha\beta}{\gamma} F(\alpha+1, \beta+1, \gamma+$$

Hence

$$F''(\alpha, \beta, \gamma, x) = \frac{\alpha \cdot \beta}{\gamma} F'(\alpha+1, \beta+$$

$$= \frac{\alpha \cdot \alpha+1 \cdot \beta \cdot \beta+1}{\gamma \cdot \gamma+1}$$

and so on for the higher derivatives.
functions.

The coefficient of x^n in x^2F'' is

$$n(n-1)P_n,$$

in $-xF''$ it is

$$-\frac{n(\alpha+n)(\beta+n)}{\gamma+n}P_n,$$

in $(\alpha+\beta+1)xF'$ it is

$$n(\alpha+\beta+1)P_n,$$

in $-\gamma F'$ it is

$$-\gamma\frac{(\alpha+n)(\beta+n)}{\gamma+n}P_n,$$

in $\alpha\beta F$ it is

$$\alpha\beta P_n.$$

Adding all these gives the coefficient of x^n in the left sid
We find it is 0.

191. *Expression of $F(\alpha\beta\gamma x)$ as an Integral.*

We show that for $|x|<1$,

$$B(\beta,\gamma-\beta)\cdot F(\alpha\beta\gamma x)=\int_0^1 u^{\beta-1}(1-u)^{\gamma-\beta-1}(1-xu)^-$$

where $B(p,q)$ is the Beta function of I, 692,

$$B(p,q)=\int_0^1 u^{p-1}(1-u)^{q-1}du.$$

For by the Binomial Theorem

$$(1-xu)^{-\alpha}=1+\frac{\alpha}{1}xu+\frac{\alpha\cdot\alpha+1}{1\cdot 2}x^2u^2+\frac{\alpha\cdot\alpha+1\cdot\alpha+2}{1\cdot 2\cdot 3}x^3u$$

for $|xu|<1$. Hence

$$J=\int_0^1 u^{\beta-1}(1-u)^{\gamma-\beta-1}(1-xu)^{-\alpha}du$$

Now from I, 692, 10)

$$B(\beta+1, \gamma-\beta) = \frac{\beta}{\gamma} B($$

Hence

$$B(\beta+2, \gamma-\beta) = \frac{\beta+1}{\gamma+1} B(\beta+1, \gamma \quad \beta$$

etc. Putting these values in 2) we get

192. *Value of* $F(\alpha, \beta, \gamma, x)$ *for* $x = 1.$

We saw that the F series converge $\alpha+\beta-\gamma<0$. The value of F when teresting. As it is now a function of α, it by $F(\alpha, \beta, \gamma)$. The relation betwee function may be established, as Gauss sh viz.:

$$\gamma\{\gamma-1+(\alpha+\beta+1-2\gamma)$$

$$+(\gamma-\alpha)(\gamma-\beta)xF$$

$$+\gamma(1-\gamma)(1-x)F$$

Assuming that

$$\alpha+\beta-\gamma<0$$

we see that the first and second terms but we cannot say this in general for t for this that $\alpha+\beta-(\gamma-1)<0$. W

$$L \lim_{x=1} (1-x) F(\alpha, \beta, \gamma$$

supposing 2) to hold. For if $|x|<1$,

$$F(\alpha, \beta, \gamma-1, x) = a_0 + a_1$$

Now by 100, this series also converge

and this establishes 3). Thus passing to the limit $x = 1$ gives

$$\gamma(\alpha + \beta - \gamma) F(\alpha, \beta, \gamma) + (\gamma - \alpha)(\gamma - \beta) F(\alpha, \beta, \gamma + 1)$$

or
$$F(\alpha, \beta, \gamma) = \frac{(\gamma - \alpha)(\gamma - \beta)}{\gamma(\gamma - \alpha - \beta)} F(\alpha, \beta, \gamma + 1).$$

Replacing γ by $\gamma + 1$, this gives

$$F(\alpha, \beta, \gamma + 1) = \frac{(\gamma + 1 - \alpha)(\gamma + 1 - \beta)}{(\gamma + 1)(\gamma + 1 - \alpha - \beta)} F(\alpha, \beta, \gamma$$

etc. Thus in general

$$F(\alpha, \beta, \gamma) =$$
$$\frac{(\gamma - \alpha)(\gamma + 1 - \alpha)\cdots(\gamma + n - 1 - \alpha)\cdot(\gamma - \beta)(\gamma + 1 - \beta)\cdots(\gamma + n -}{\gamma(\gamma + 1)\cdots(\gamma + n - 1)(\gamma - \alpha - \beta)(\gamma - \alpha - \beta + 1)\cdots(\gamma - \alpha - \beta +}$$
$$\cdot F(\alpha, \beta, \gamma$$

Gauss sets now

$$\Pi(n, x) = \frac{n!\, n^x}{(x + 1)(x + 2)\cdots(x + n)}.$$

Hence the above relation becomes

$$F(\alpha\beta\gamma) = \frac{\Pi(n, \gamma - 1)\,\Pi(n, \gamma - \alpha - \beta - 1)}{\Pi(n, \gamma - \alpha - 1)\,\Pi(n, \gamma - \beta - 1)} F(\alpha, \beta, \gamma +$$

Now
$$\lim_{n \to \infty} F(\alpha, \beta, \gamma + n) = 1.$$

For the series

$$F(\alpha, \beta, \gamma) = 1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} + \frac{\alpha \cdot \alpha + 1 \cdot \beta \cdot \beta + 1}{1 \cdot 2 \cdot \gamma \cdot \gamma + 1} + \cdots$$

converges absolutely when 2) holds. Hence

We shall show in the next chapter th

$$\lim_{n=\infty} \Pi(n, x)$$

exists for all x different from a negativ
it by $\Pi(x)$; as we shall see,

$$\Gamma(x) = \Pi(x-1) \quad ,$$

Letting $n \doteq \infty$, 6) gives

$$F(\alpha, \beta, \gamma) = \frac{\Pi(\gamma - 1)\Pi(\gamma}{\Pi(\gamma - \alpha - 1)}$$

We must of course suppose that

$$\gamma, \qquad \gamma - \alpha, \qquad \gamma - \beta,$$

are not negative integers or zero, as ot
Π or F function are not defined.

Bessel Functio

193. 1. The infinite series

$$J_n(x) = x^n \sum_{s=0}^{\infty} (-1)^s \frac{x^{2s}}{2^{n+2s} s!(n+s)}$$

converges for every x. For the ratio o
the adjoint series is

$$\frac{|x|^2}{2^2(s+1)(n+s+}$$

which $\doteq 0$ as $s \doteq \infty$ for any given x.

The series 1) thus define functions of
continuous. They are called Bessel fun

$$n = 0, 1, 2 \ldots$$

2. The following linear relation exists between three cons
Bessel functions:

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x) \qquad n > 0.$$

For $$J_{n-1} = \frac{x^{n-1}}{2^{n-1}(n-1)!} + \sum_{s=1}^{\infty}(-1)^s \frac{x^{2s+n-1}}{2^{n+2s-1}s!(n-1+s)!}$$

$$J_{n+1} = -\sum_{s=1}^{\infty}(-1)^s \frac{x^{2s+n-1}}{2^{n+2s-1}(s-1)!(n+s)!}.$$

Hence

$$J_{n-1} + J_{n+1}$$

$$= \frac{x^{n-1}}{2^{n-1}(n-1)!} + \sum_{1}^{\infty}(-1)^s \frac{x^{2s+n-1}}{2^{n+2s-1}} \left\{ \frac{1}{s!(n-1+s)!} - \frac{1}{(s-1)!(n+s)!} \right.$$

$$= \frac{x^{n-1}}{2^{n-1}(n-1)!} + n\sum_{1}^{\infty}(-1)^s \frac{x^{2s+n-1}}{2^{n+2s-1}s!(n+s)!}$$

$$= \frac{n}{x}\sum_{0}^{\infty}(-1)^s \frac{x^{2s+n}}{2^{n+2s-1}s!(n+s)!}$$

$$= \frac{2n}{x} J_n(x).$$

3. We show next that

$$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x) \qquad n > 0.$$

For subtracting 7) from 6) gives

$$J_{n-1} - J_{n+1} = \frac{x^{n-1}}{2^{n-1}(n-1)!} + \sum_{1}^{\infty}(-1)^s \frac{x^{2s+n-1}}{2^{n+2s-1}} \cdot \frac{n+2s}{s!(n+s)!}$$

$$= \sum_{0}^{\infty}(-1)^s \frac{(n+2s)x^{2s+n-1}}{2^{n+2s-1}s!(n+s)!}$$

$$= 2J_n'.$$

From 8) we get, on replacing J_{n+1} by its value as given

$$J_n'(x) = -\frac{n}{x}J_n(x) + J_{n-1}(x) \qquad n > 0.$$

This may be shown by direct differen
ply thus: Differentiating 9) gives

$$J_n'' = \frac{n}{x^2} J_n \quad \frac{n}{x} J_n' +$$

Equation 10) gives

$$J_{n-1}' = \frac{n \quad 1}{x} J_{n-1}$$

Replacing here J_{n-1} by its value as g

$$J_{n-1}' = \frac{n-1}{x} J_n' + \left(\frac{n(n}{x^2} \right.$$

Putting this in 12) gives 11).

5. $$e^{\frac{x}{2}\left(u - \frac{1}{u}\right)} = \sum u^n J_n$$

for any x, and for $u \neq 0$.

For

$$e^{\frac{x}{2}\left(u - \frac{1}{u}\right)} = e^{\frac{1}{2}xu} e^{-\frac{1}{2}\frac{x}{u}}$$

$$= \left\{ 1 + \frac{xu}{2} + \frac{x^2u^2}{2^2 \cdot 2!} + \frac{x^3u^3}{2^3 \cdot 3!} \right.$$

$\cdot$ $\cdot$

Now for any x and for any $u \neq 0$, th
absolutely convergent. Their product
in the form

$$\left(1 - \frac{x^2}{2^2} + \left(\frac{x}{2}\right)^4 \frac{1}{2!\,2!} - \cdots\right.$$

$$+ u\left(\frac{x}{2} - \frac{1}{2!}\left(\frac{x}{2}\right)^3 + \frac{1}{3!\,2}\right.$$

$$- u^{-1}\left(\frac{x}{2} - \frac{1}{2!}\left(\frac{x}{2}\right)^3 + \frac{1}{3!}\right.$$

194. 1. *Expression of $J_n(x)$ as an Integral.*

$$J_n(x) = \frac{x^n}{2^n\sqrt{\pi}} \cdot \frac{1}{\Gamma\left(\frac{2n+1}{2}\right)} \int_0^\pi \cos(x\cos\phi)\sin^{2n}\phi\, d$$

For

$$\cos u = \sum_0^\infty (-1)^s \frac{u^{2s}}{(2s)!}.$$

Hence

$$\cos(x\cos\phi) = \sum_0^\infty \frac{(-1)^s}{(2s)!} x^{2s}\cos^{2s}\phi$$

and thus

$$\cos(x\cos\phi)\sin^{2n}\phi = \sum_0^\infty \frac{(-1)^s}{(2s)!} x^{2s}\cos^{2s}\phi\sin^{2n}\phi.$$

As this series converges uniformly in $(0, \pi)$ for any val
we may integrate termwise, getting

$$\int_0^\pi \cos(x\cos\phi)\sin^{2n}\phi\, d\phi = \sum_0^\infty \frac{(-1)^s}{(2s)!} x^{2s} \int_0^\pi \cos^{2s}\phi\sin^{2n}\phi\, d$$

$$= \sum_0^\infty \frac{(-1)^s}{(2s)!} x^{2s} B\left(\frac{2s+1}{2}, \frac{2n+1}{2}\right) \text{ b}$$

$$= \sum_0^\infty \frac{(-1)^s}{(2s)!} x^{2s} \frac{\Gamma\left(\frac{2s+1}{2}\right)\Gamma\left(\frac{2n+1}{2}\right)}{\Gamma(s+n)} \text{ by}$$

We shall show in 225, 6, that

$$\Gamma\left(\frac{2s+1}{2}\right) = \frac{1\cdot 3\cdot 5 \cdots 2s-1}{2^s}\sqrt{\pi}.$$

Thus the last series above

CHAPTER VI

INFINITE PRODUC

195. 1. Let $\{a_{\iota_1 \cdots \iota_s}\}$ be an infinite s

indices $\iota = (\iota_1 \cdots \iota_s)$ ranging over a la

space. The symbol

$$P = \Pi_{\nu} a_{\iota_1 \cdots \iota_s} = \Pi_{\nu}$$

is called an *infinite product*. The numbe

P_μ denote the product of all the factor

R_μ. If

$$\lim_{\mu = \infty} P_\mu$$

is finite or definitely infinite, we call i

customary to represent a product and its

when no ambiguity will arise.

When the limit 2) is finite and $\neq 0$ or

$= 0$, we say P is *convergent*, otherwise P

We shall denote by P_μ the product ob

factors $a_\iota = 1$, whose indices ι lie in the c

co-product of P_μ.

The products most often occurring in

$$P = a_1 \cdot a_2 \cdot a_3 \cdots =$$

The factor P_μ is here replaced by

$$P_m = a_1 \cdot a_2 \cdots a_m$$

2. Let
$$P = \frac{1}{1} \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} \cdots$$
Obviously the product $P = 0$, as
$$P_n = \frac{1}{n} \doteq 0.$$

Hence $P = 0$, although no factor is zero. Such produc called *zero products*. Now we saw in I, 77 that the produ finite number of factors cannot vanish unless one of its vanishes. For this reason zero products hold an exception tion and will not be considered in this work. We therefor classed them among the divergent products. In the fol theorems relative to convergence, we shall suppose, for sim that *there are no zero factors*.

196. 1. *For $P = \Pi a_{\iota_1 \cdots \iota_s}$ to converge it is necessary that is convergent. If one of these P_μ converges, P is convergent*
$$P = P_\mu \cdot \bar{P}_\mu.$$
The proof is obvious.

2. *If the simple product $P = a_1 \cdot a_2 \cdot a_3 \cdots$ is convergent, tors finally remain positive.*

For, when P is convergent, $|P_n| >$ some positive num $n >$ some m. If now the factors after a_m were not all posi and P_ν could have opposite signs $\nu > n$, however large n i Thus P_n has no limit.

197. 1. To investigate the convergence or divergence infinite product $P = \Pi a_{\iota_1 \cdots \iota_s}$ when $a_\iota > 0$, it is often conve consider the series
$$L = \sum_{\mathfrak{A}} \log a_{\iota_1 \cdots \iota_s} = \Sigma l_{\iota_1 \cdots \iota_s},$$

If P is convergent, P_μ converges to a
L_μ is convergent by 1). If L_μ is conve
finite limit $\neq 0$ by 2).

2. *Example 1.*

$$P = \Pi\left(1+\frac{x}{n}\right)e^{-\frac{x}{n}} = \Pi a_n$$

is convergent for every x.

For, however large $|x|$ is taken and t
so large that

$$1+\frac{x}{n} > 0 \qquad n >$$

Instead of P we may therefore consid

Then $$L_m = \sum_{m+1}^{\infty}\left\{-\frac{x}{n}+\log\left(1\right.\right.$$

But by I, 413

$$\log\left(1+\frac{x}{n}\right) = \frac{x}{n} + M_n\frac{x^2}{n^2},$$

Hence $$L_m = \sum_{m+1} M_n x^2 \cdot \frac{1}{n^2}$$

which is convergent.

The product P occurs in the express
product.

Let us now consider the product

$$Q = \Pi\left(1+\frac{x}{n}\right)e^{-\frac{x}{n}} \qquad n = \pm$$

The associate logarithmic series L is
We may break it into two parts L', L''
positive n, the second over negative n.
these as we did on the series 3), and c

For let p be taken so large that $|x| < p$. We show t
co-product

$$G_p = \prod_{p+1}^{\infty} \frac{\left(1+\frac{1}{n}\right)^x}{1+\frac{x}{n}}$$

converges for this x. The corresponding logarithmic seri

$$L = \sum_{p+1}^{\infty}\left\{x\log\left(1+\frac{1}{n}\right) - \log\left(1+\frac{x}{n}\right)\right\}$$

$$= \sum_{p+1}^{\infty}\left\{\frac{x}{n} - \log\left(1+\frac{x}{n}\right)\right\} - x\sum_{p+1}^{\infty}\left\{\frac{1}{n} - \log\left(1\right.\right.$$

As each of the series on the right converges, so does L.
G converges for this value of x.

198. 1. When the associate logarithmic series

$$L = \Sigma \log a_{\iota_1 \dots \iota_s} \quad , \quad a_\iota > 0$$

is convergent,

$$\lim_{|\iota| = \infty} \log a_{\iota_1 \dots \iota_s} = 0, \qquad \text{by } 121, 1,$$

and therefore

$$\lim_{|\iota| = \infty} a_{\iota_1 \dots \iota_s} = 1.$$

For this reason it is often convenient to write the
$a_{\iota_1 \dots \iota_s}$ of an infinite product P in the form $1 + b_{\iota_1 \dots \iota_s}$. W
written in the form

$$P = \Pi(1 + b_{\iota_1 \dots \iota_s}),$$

we shall say it is written in its *normal form*. The series

$$\Sigma b_{\iota_1 \dots \iota_s} = \Sigma b_\iota$$

For P and

$$L = \Sigma \log (1 + a$$

converge or diverge simultaneously by 19
verge or diverge simultaneously by 123, 4.

3. *If the simple product $P = a_1 \cdot a_2 \cdot a_3 \cdots$*

For by 196, 2 the factors a_n finally becom
Hence by 197, 1 the series

$$\sum_{n=m}^{\infty} \log a_n \qquad a_n > 0$$

is convergent. Hence $\log a_n \doteq 0$. $\therefore a_n \doteq 1$

199. *Let $R_{\lambda_1} < R_{\lambda_2} < \cdots$ $\lambda \doteq \infty$ be a
cells. Then if P is convergent,*

$$P = P_{\lambda_1} + \sum_1^{\infty} (P_{\lambda_{n+1}} - P_{\lambda_n}).$$

For P is a telescopic series and

$$P_{\lambda_{\nu+1}} = P_{\lambda_1} + \sum_1^{\nu} (P_{\lambda_{n+1}} - P_{\lambda_n}).$$

200. 1. Let $P = \Pi(1 + a_{\iota_1 \cdots \iota_s})$.

We call $\mathfrak{P} = \Pi(1 + \alpha_{\iota_1 \cdots \iota_s})$, $\alpha_\iota =$
the *adjoint of P*, and write

$$\mathfrak{P} = \text{Adj } P.$$

2. *P converges, if its adjoint is convergent.*

$$\epsilon > 0, \quad \lambda, \qquad | P_\mu - P_\nu | < \epsilon$$

3. When the adjoint of P converges, we say P is *absolut*
nvergent.

The reader will note that absolute convergence of infin
oducts is defined quite differently from that of infin
ries. At first sight one would incline to define the adjoint o

$$P = \Pi a_{\iota_1 \dots \iota_s}$$

be

$$\mathfrak{P} = \Pi \mid a_{\iota_1 \dots \iota_s} \mid.$$

With this definition the fundamental theorem 2 would be fal
or let

$$P = \Pi (-1)^n ;$$

adjoint would be, by this definition,

$$\mathfrak{P} = 1 \cdot 1 \cdot 1 \cdot \dots$$

Now $\mathfrak{P}_n = 1$. $\therefore \mathfrak{P}$ is convergent. On the other ha
$_n = (-1)^n$ and this has no limit, as $n \doteq \infty$. Hence P
vergent.

4. *In order that* $P = \Pi(1 + a_{\iota_1 \dots \iota_s})$ *converge absolutely, it*
cessary and sufficient that $\Sigma\, a_{\iota_1 \dots \iota_s}$
nverges absolutely.

Follows at once from 198, 2.

Example.

$$\prod_1^\infty \left(1 - \frac{x^2}{n^2}\right)$$

nverges absolutely for every x.

For

$$\sum \frac{x^2}{n^2} = x^2 \sum \frac{1}{n^2}$$

We have now the following theorems :

2. *If an associate simple product Q is*
$P = Q$.

For since Q is convergent, we may assu
> 0 by 196, 2. Then

$$Q = e^{\Sigma \log a_n}$$

$$= e^{\Sigma \log a_{\iota}}$$

$$= P$$

3. *If the associate simple product Q is*
is P.

For let

$$P = \Pi(1 + a_{\iota_1 \ldots \iota_s})$$

$$Q = \Pi(1 + a_n).$$

Since Q is absolutely convergent,

$$\Pi(1 + \alpha_n) \quad , \quad \alpha_n = |$$

is convergent. Hence $\Pi(1 + \alpha_{\iota_1 \ldots \iota_s})$ is co

4. *Let $P = \Pi(1 + a_{\iota_1 \ldots \iota_m})$ be absolutely*
associate simple product $Q = \Pi(1 + a_n)$ is a
$P = Q$.

For since P is absolutely convergent,

$$\Sigma \alpha_{\iota_1 \ldots \iota_s}$$

converges by 200, 4. But then by 124, 5

$$\Sigma \alpha_n$$

is convergent. Hence Q is absolutely conv

For $$P = P_\mu \cdot P'_\mu;$$

l all the factors of P'_μ are >0, if μ is sufficiently large. T
shall feel at liberty to do, without further remark.

7. $$A = \Pi(1 + a_{\iota_1 \cdots \iota_s}) \qquad a_\iota > 0$$

l $$L = \Sigma \log(1 + a_{\iota_1 \cdots \iota_s})$$

verge or diverge simultaneously.

For if A is convergent,

$$\Sigma a_{\iota_1 \cdots \iota_s}$$

convergent by 200, 4. But then L is convergent by 123.
e converse follows similarly.

202. 1. As in 124, 10 we may form from a given m-tu
oduct
$$A = \Pi a_{\iota_1 \cdots \iota_m}$$
infinite number of conjugate n-tuple products
$$B = \Pi b_{j_1 \cdots j_n}$$
ere $a_\iota = b_j$ if ι and j are corresponding lattice points in the t
stems.

We have now:

2. *If A is absolutely convergent, so is B, and $A = B$.*

For by 201, 6, without loss of generality, we may take all
ctors > 0.

Then
$$A = e^{\Sigma \log a_{\iota_1 \cdots \iota_m}}$$
$$= e^{\Sigma \log a_\mu}$$
$$= e^{\Sigma \log a_{j_1 \cdots j_n}}$$

For $\Sigma \log \alpha_{\iota}$ is conv

Hence $\Sigma \log \beta_j$ is.

Arithmetical Oper

203. *Absolutely convergent products a* *versely.*

For let

$$A = \Pi a_{\iota_1 \cdots \iota_m}$$

be absolutely convergent. Then its asso

$$\mathfrak{A} = \Pi a_n$$

is absolutely convergent and $A = \mathfrak{A}$, by arrange the factors of A, getting the p sponds a simple associate series $\mathfrak{B}$ and B $\mathfrak{A}$ is absolutely convergent. Hence A

Conversely, let A be commutative. T finally become > 0. For if not, let

$$R_1 < R_2 < \cdots \doteq x$$

be a sequence of rectangular cells such t in some cell. We may arrange the factor products corresponding to 1),

$$A_1 \quad , \quad A_2 \quad , \quad A_3$$

have opposite signs alternately. Then A is a contradiction. We may therefore Then

$$A = e^{\Sigma \log a_{\iota_1 \cdots \iota_m}}$$

remains unaltered however the factors or Hence

$$\Sigma \log a$$

204. 1. *Let*

$$A = \prod a_{\iota_1 \cdots \iota_s}$$

be absolutely convergent. Then the s-tuple iterated product

$$B = \prod_{\iota_1'} \prod_{\iota_2'} \cdots \prod_{\iota_s'} a_{\iota_1 \cdots \iota_s}$$

is absolutely convergent and $A = B$ where $\iota_1' \cdots \iota_s'$ is a permu
$\iota_1, \iota_2 \cdots \iota_s$.

For by 202, 3 all the products of the type

$$\prod_{\iota_{s-1}\iota_s} a_{\iota_1 \cdots \iota_s} \qquad \prod_{\iota_s} a_{\iota_1 \cdots \iota_s}$$

are absolutely convergent, and by I, 324

$$\prod_{\iota_{s-1}\iota_s} = \prod_{\iota_{s-1}} \prod_{\iota_s}.$$

Similarly the products of the type

$$\prod_{\iota_{s-2}\iota_{s-1}\iota_s}$$

are absolutely convergent and hence

$$\prod_{\iota_{s-2}\iota_{s-1}\iota_s} = \prod_{\iota_{s-2}} \prod_{\iota_{s-1}} \prod_{\iota_s}.$$

In this way we continue till we reach A and B.

2. We may obviously generalize 1 as follows:

Let

$$A = \prod a_{\iota_1 \cdots \iota_s}$$

be absolutely convergent. Let us establish a 1 to 1 corres
between the lattice system $\mathfrak{L}$ over which $\iota = (\iota_1 \cdots \iota_s)$ ranges
lattice system $\mathfrak{M}$ over which

3. An important special case of 2 is th

Let
$$A = \Pi a_n \quad , \quad n = 1, 2, \cdot$$

converge absolutely. Let us throw the a_n *i*

$$a_{11} \quad , \quad a_{12} \cdots$$
$$a_{21} \quad , \quad a_{22} \cdots$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$a_{r1} \quad , \quad a_{r2} \cdots$$

Then
$$B_1 = \Pi a_{1n} \quad , \quad B_2 =$$

converge absolutely, and
$$A = B_1 B_2 \cdots B_r.$$

4. *The convergent infinite product*
$$P = (1 + a_1)(1 + a_2$$
is associative.

For let
$$m_1 < m_2 < \cdots \doteq \infty.$$

Let
$$1 + b_1 = (1 + a_1) \cdots (1 +$$
$$1 + b_2 = (1 + a_{m_1+1}) \cdots ($$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

We have to show that
$$Q = (1 + b_1)(1 + b_2$$
is convergent and $P = Q$.

205. *Let* $A = \Pi a_{\iota_1 \dots \iota_n}$, $B = \Pi b_{\iota_1 \dots \iota_n}$

convergent. Then

$$C = \Pi a_\iota \cdot b_\iota \quad , \quad D = \Pi \frac{a_\iota}{b_\iota}$$

re convergent and

$$C = A \cdot B \quad , \quad D = \frac{A}{B}.$$

Moreover if A, B are absolutely convergent, so are C, D.

Let us prove the theorem regarding C; the rest follows si rly. We have $C_\mu = A_\mu \cdot B_\mu$.

Now by hypothesis $A_\mu \doteq A$, $B_\mu \doteq B$ as $\mu \doteq \infty$.

Hence $C_\mu \doteq A \cdot B$.

To show that C is absolutely convergent when A, B are, let rite $a_\iota = 1 + \mathfrak{a}_\iota$, $b_\iota = 1 + \mathfrak{b}_\iota$ and set $|\mathfrak{a}_\iota| = \alpha_\iota$, $|\mathfrak{b}_\iota| =$

Since A, B converge absolutely,

$$\Sigma \log(1 + \alpha_\iota) \quad , \quad \Sigma \log(1 + \beta_\iota)$$

re convergent. Hence

$$\Sigma \{\log(1 + \alpha_\iota) + \log(1 + \beta_\iota)\} = \Sigma \log(1 + \alpha_\iota)(1 + \beta_\iota)$$

s absolutely convergent. Hence C is absolutely converg y 201, 7.

206. *Example.* The following infinite products occur in heory of elliptic functions:

$$Q_1 = \Pi(1 + q^{2n})$$

$$Q_2 = \Pi(1 + q^{2n-1}) \qquad n = 1, 2, \cdots$$

For by 205,
$$P = \Pi(1 + q^{2n})(1 +$$
$$= \Pi(1 + q^{2n})(1$$

Now all integers of the type $2n$, are of th
Hence by 204, 3,
$$\Pi(1 - q^{2n}) = \Pi(1 - q^{4n})\Pi(1$$
or
$$\Pi(1 - q^{4n-2}) = \frac{\Pi(1 - q^{2n})}{\Pi(1 - q^{4n})}.$$

Thus
$$P = \Pi\frac{(1 + q^{2n})(1}{1 - q^{4n}}$$
$$= 1.$$

Uniform Convergence

207. *In the limited or unlimited domain* $\mathfrak{A}$,
$$L = \Sigma \log f_{\iota_1 \ldots \iota_s}(x_1 \cdots x_m)$$
be uniformly convergent and limited. Then
$$F = \Pi f_{\iota_1 \ldots \iota_s}$$
is uniformly convergent in $\mathfrak{A}$.

For
$$F_\lambda = e^{L_\lambda}.$$

Now $L_\lambda \doteq L$ uniformly. Hence by 144, 1,
vergent.

208. *If the adjoint of*
$$F = \Pi(1 + f_{\iota_1 \ldots \iota_s}(x_1 \cdots x_m)$$

But as already noticed in 200, 2, 1)

$$| P_\mu - P_\nu | \leq | \mathfrak{P}_\mu - \mathfrak{P}_\nu |.$$

Hence F is uniformly convergent.

209. *The product*

$$F = \Pi(1 + f_{\iota_1 \cdots \iota_s}(x_1 \cdots x_m))$$

is uniformly convergent in the limited or unlimited domain $\mathfrak{A}$,

$$\Phi = \Sigma\phi_{\iota_1 \cdots \iota_s}(x_1 \cdots x_m) \quad , \quad \phi_\iota = |f_\iota|$$

is limited and uniformly convergent in $\mathfrak{A}$.

For by 138, 2 the series

$$L = \Sigma \log (1 + \phi_\iota)$$

is uniformly convergent and limited in $\mathfrak{A}$. Then by 2(
adjoint of F is uniformly convergent, and hence by 208, F

210. *Let*

$$F(x_1 \cdots x_m) = \Pi f_{\iota_1 \cdots \iota_s}(x_1 \cdots x_m)$$

be uniformly convergent at $x = a$. *If each* f_ι *is continuous* (
is also continuous at a.

This is a corollary of 147, 1.

211. 1. *Let* $G = \Sigma | f_{\iota_1 \cdots \iota_s}(x_1 \cdots x_m) |$ *converge in the complete domain* $\mathfrak{A}$ *having* a *as a limiting point. Let* G *an* f_ι *be continuous at* a. *Then*

$$F(x_1 \cdots x_m) = \Pi(1 + f_{\iota_1 \cdots \iota_s}(x_1 \cdots x_m))$$

is continuous at a.

For by 149, 5, G is uniformly converg
limited near $x = a$. Thus by 209,

$$\Pi(1 + f_\iota)$$

is uniformly convergent at a. To establi
to apply 146, 1.

212. 1. Let $F = \Pi f_{\iota_1 \dots \iota_s}(x)$,

converge in $\mathfrak{A} = (a, a + \delta)$. Then

$$\log F = L = \Sigma \log$$

If we can differentiate this series termw

$$\frac{d}{dx} \log F = \sum \frac{f'_\iota}{f_\iota}$$

Thus to each infinite product 1) of this
nite series 3). Conditions for termwise di
2) are given in 153, 155, 156. Other con
Chapter XVI.

2. *Example*. Let us consider the infin

$$\theta(x) = 2 q^{\frac{1}{4}} Q \sin \pi x \prod_1^\infty (1 - 2 q^{2n}$$

which occurs in the elliptic functions.

Let us set

$$1 - u_n = 1 - 2 q^{2n} \cos 2 \pi x$$

Then $|u_n| < 2 |q|^{2n} + |q$

Thus if $|q| < 1$, the product 1) is absolute
It is uniformly convergent for any x and

Now the series Σa_n converges if $|q| < 1$. For settin
the series Σb_n is convergent in this case. Moreover,

$$\lim_{n=\infty} \frac{a_n}{b_n} = 1.$$

Thus we may differentiate termwise.

The Circular Functions

213. 1. *Sin x and cos x as Infinite Products.*

From the addition theorem

$$\sin(mx + x) = \sin(m+1)x = \sin mx \cos x + \cos$$

$m = 1, 2, 3 \cdots$ we see that for an odd n

$$\sin nx = a_0 \sin^n x + a_1 \sin^{n-1} x + \cdots + a_{n-1} \sin$$

where the coefficients a are integers. If we set $t = \sin$

$$\sin nx = F_n(t) = a_0 t^n + a_1 t^{n-1} + \cdots + a_{n-1} t$$

Now F_n being a polynomial of degree n, it has n root

$$0, \quad \pm \sin\frac{\pi}{n}, \quad \pm \sin\frac{2\pi}{n}, \quad \cdots \pm \sin\frac{n-1}{2}\frac{\pi}{n}$$

corresponding to the values of x which make $\sin nx =$

$$F_n(t) = a_0 t\left(t - \sin\frac{\pi}{n}\right)\left(t + \sin\frac{\pi}{n}\right)\cdots$$

$$= a_0 t\left(t^2 - \sin^2\frac{\pi}{n}\right)\cdots\left(t^2 - \sin^2\frac{n-1}{n} \cdot\right.$$

To find α we observe that this equation g

$$\frac{\sin nx}{\sin x} = \alpha\left[1 - \frac{\sin^2 x}{\sin^2\frac{\pi}{n}}\right]\cdot$$

Letting $x \doteq 0$ we now get $\alpha = n$. Thus

in 3), and replacing x by $\frac{x}{n}$, we have finally

$$\sin x = n \sin\frac{x}{n}\, P(x, n)$$

where

$$P(x, n) = \Pi\left[1 - \frac{\sin^2\frac{x}{n}}{\sin^2\frac{r\pi}{n}}\right] \qquad r = 1,$$

We note now that as $n \doteq \infty$,

$$n \sin\frac{x}{n} = x\,\frac{\sin\frac{x}{n}}{\frac{x}{n}} \doteq x.$$

Similarly

$$\frac{\sin^2\frac{x}{n}}{\sin^2\frac{r\pi}{n}} \doteq \frac{x^2}{r^2\pi^2}.$$

It seems likely therefore that if we pass
4), we shall get

$$\sin x = x P(x)$$

where

$$P(x) = \prod_1^{\infty}\left(1 - \frac{x^2}{r^2\pi^2}\right).$$

We observe that

$$\lim_{n=\infty} P(x, n) = \lim_{n=\infty} e^{L(x, n)} = e^{L(x)} = P(x)$$

ovided

$$\lim_{n=\infty} L(x, n) = L(x).$$

We have thus only to prove 7). Let us denote the sum of t
st m terms in 6) by $L_m(x, n)$ and the sum of the remaini
$\bar{L}_m(x, n)$. Then

$$(x, n) - L(x) | < | L_m(x, n) - L_m(x) | + | \bar{L}_m(x, n) | + | \bar{L}_m(x) |.$$

Since for $0 < x < \frac{\pi}{2}$,

$$\frac{x}{2} < \sin x < x,$$

have

$$\frac{\sin^2 \frac{x}{n}}{\sin^2 \frac{r\pi}{n}} > \frac{\frac{x^2}{4 n^2}}{\frac{\pi^2 r^2}{n^2}} = \frac{x^2}{4 \pi^2 r^2},$$

d hence for an m_1 so large that $\frac{|x|}{m_1} < 1$, we have,

$$\log \left[1 - \frac{\sin^2 \frac{x}{n}}{\sin^2 \frac{r\pi}{n}} \right] < \log \left(1 - \frac{x^2}{4 \pi^2 r^2} \right), \qquad r > m.$$

But the series

2. In algebra we learn that every polynomial

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

can be written as a product

$$a_n(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n),$$

where $\alpha_1, \alpha_2 \cdots$ are its roots. Now

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

is the limit of a polynomial, viz. the first n te natural to ask, Can we not express $\sin x$ as the which vanishes at the zeros of $\sin x$? That thi have just shown in 1.

3. If we set $x = \pi/2$ in 5), it gives,

$$1 = \frac{\pi}{2}\Pi\left(1 - \frac{1}{4r^2}\right) = \frac{\pi}{2}\Pi\frac{(2r-1)(2r}{2r \cdot 2r}$$

Hence
$$\frac{\pi}{2} = \Pi\frac{2r \cdot 2r}{(2r-1)(2r+1)} = \frac{2 \cdot 2 \cdot 4 \cdot 4}{1 \cdot 3 \cdot 3 \cdot 5}$$

a formula due to *Wallis*.

4. From 5) we can get another expression fo

$$\sin x = x\Pi\left(1 - \frac{x}{r\pi}\right)e^{\frac{x}{r\pi}} \qquad r = \pm 1, \pm$$

For the right side is convergent by 197, 2. the factors in pairs, we have

14. *We now show that*

$$\cos x = \prod_1^\infty \left(1 - \frac{4x^2}{(2n-1)^2\pi^2}\right).$$

To this end we use the relation

$$\sin 2x = 2 \sin x \cos x.$$

Hence

$$x = \frac{1}{2} \cdot \frac{2x}{x} \frac{\Pi\left(1 - \frac{4x^2}{n^2\pi^2}\right)}{\Pi\left(1 - \frac{x^2}{n^2\pi^2}\right)} = \frac{\Pi\left(1 - \frac{4x^2}{(2m)^2\pi^2}\right)\left(1 - \frac{4x^2}{(2m-1)^2\pi^2}\right)}{\Pi\left(1 - \frac{x^2}{n^2\pi^2}\right)}$$

$$= \frac{\Pi\left(1 - \frac{x^2}{m^2\pi^2}\right)}{\Pi\left(1 - \frac{x^2}{n^2\pi^2}\right)} \cdot \Pi\left(1 - \frac{4x^2}{(2m-1)^2\pi^2}\right)$$

n which 1) is immediate.

From 1) we have, as in 213, 4,

$$\cos x = \Pi\left(1 - \frac{2x}{(2n-1)\pi}\right)e^{\frac{2x}{(2n-1)\pi}} \qquad n = 0, \pm 1, \pm 2, \cdots$$

15. From the expression of $\sin x$, $\cos x$ as infinite product ir *periodicity* is readily shown. Thus from 213, 12)

$$\sin x = \lim_{n=\infty} P_n(x).$$

But

$$\frac{P_n(x+\pi)}{P_n(x)} = \frac{x+(n+1)\pi}{x-n\pi} \doteq -1 \quad , \quad \text{as } n \doteq \infty.$$

Hence

$$\lim P_n(x+\pi) = -\lim P_n(x),$$

$$\sin(x+\pi) = -\sin x.$$

Similarly 214, 1) gives

$$\log \cos x = \sum_1^\infty \log\left(1 - \frac{4x^2}{(2s-1)^2\pi^2}\right),$$

To get formulae having a wider range we have the products 213, 5) and 214, 1). We then get

$$\log \sin^2 x = \log x^2 + \sum_1^\infty \log\left(1 - \frac{x^2}{s^2\pi^2}\right)$$

valid for any x *such that* $\sin x \neq 0$; and

$$\log \cos^2 x = \sum_1^\infty \log\left(1 - \frac{4x^2}{(2s-1)^2\pi^2}\right)$$

valid for any x *such that* $\cos x \neq 0$.

If we differentiate 3), 4) we get

$$\cot x = \frac{1}{x} + 2\sum_1^\infty \frac{x}{x^2 - s^2\pi^2},$$

$$\tan x = 2\sum_1^\infty \frac{x}{\left(\frac{2s-1}{2}\right)^2\pi^2 - x^2}$$

valid as in 3), 4).

Remark. The relations 5), 6) exhibit $\cot x$, tan rational functions whose poles are precisely the p functions. They are analogous to the representa of a fraction as the sum of partial fractions.

2. *To get developments of* $\sec x$, $\operatorname{cosec} x$, *we obse*

$$\operatorname{cosec} x = \tan \tfrac{1}{2} x + \cot x.$$

Hence

3. *To get* sec x, we observe that

$$\operatorname{cosec}\left(\frac{\pi}{2}-x\right)=\sec x.$$

Now
$$\operatorname{cosec} x = \frac{1}{x} + \sum_1^{\infty}(-1)^{s-1}\left\{\frac{1}{s\pi - x} - \frac{1}{s\pi + x}\right\}.$$

Hence

$$\operatorname{cosec}\left(\frac{\pi}{2}-x\right) = \frac{1}{\frac{\pi}{2}-x} + \sum_1^{\infty}(-1)^{s-1}\left\{\frac{1}{s\pi - \frac{\pi}{2} + x} - \frac{1}{s\pi + \frac{\pi}{2} -}\right.$$

Let us regroup the terms of S, forming the series

$$T = \left\{\frac{1}{\frac{\pi}{2}-x} + \frac{1}{\frac{\pi}{2}+x}\right\} - \left\{\frac{1}{\frac{3\pi}{2}-x} + \frac{1}{\frac{3\pi}{2}+x}\right\} + \cdots$$

As
$$|S_n - T_n| = \frac{1}{\left|\frac{2n-1}{2}\pi - x\right|} \doteq 0,$$

we see that T is convergent and $= S$. Thus

$$\sec x = \sum_1^{\infty}(-1)^{s-1}\frac{(2s-1)\pi}{\left(\frac{2s-1}{2}\right)^2\pi^2 - x^2},$$

valid for all x such that $\cos x \neq 0$.

217. As an exercise let us show the periodicity of 216, 5). We have

$$\cot x = \lim F_n(x) = \lim \sum^{n} \frac{1}{x + s\pi} \qquad x \neq s\pi.$$

218. *Development of* log sin x, tan x, *etc., in power serie*

From 216, 1)

$$\log \frac{\sin x}{x} = \sum_1^\infty \log\left(1 - \frac{x^2}{s^2\pi^2}\right).$$

If we give to $\frac{\sin x}{x}$ its limiting value 1 as $x \doteq 0$, the re

holds for $|x| < \pi$.

Now for $|x| < \pi$

$$-\log\left(1 - \frac{x^2}{s^2\pi^2}\right) = \frac{x^2}{s^2\pi^2} + \frac{1}{2}\frac{x^4}{s^4\pi^4} + \cdots$$

Thus

$$-\log \frac{\sin x}{x} = \frac{x^2}{\pi^2} + \frac{1}{2}\frac{x^4}{\pi^4} + \frac{1}{3}\frac{x^6}{\pi^6} + \cdots$$

$$+ \frac{x^2}{2^2\pi^2} + \frac{1}{2}\frac{x^4}{2^4\pi^4} + \frac{1}{3}\frac{x^6}{2^6\pi^6} + \cdots$$

$$+ \frac{x^2}{3^2\pi^2} + \frac{1}{2}\frac{x^4}{3^4\pi^4} + \frac{1}{3}\frac{x^6}{3^6\pi^6} + \cdots$$

$$+ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

provided we sum this double series by rows. But since
is a positive term series, we may sum by columns, b
Doing this we get

$$-\log \frac{\sin x}{x} = H_2 \frac{x^2}{\pi^2} + \frac{1}{2} H_4 \frac{x^4}{\pi^4} + \frac{1}{3} H_6 \frac{x^6}{\pi^6} + \cdots$$

where

$$H_n = \frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \cdots$$

The relation 2) *is valid for* $|x| < \pi$.

The terms of G_n are a part of H_n. Obviously

$$G_n = \frac{2^n - 1}{2^n} H_n.$$

These coefficients put in 3) give

$$-\log \cos x = (2^2 - 1) H_2 \frac{x^2}{\pi^2} + \tfrac{1}{2}(2^4 - 1) H_4 \frac{x^4}{\pi^4} + \tfrac{1}{3}(2^6 - 1) H_6 \frac{x^6}{\pi^6}$$

valid for $|x| < \frac{\pi}{2}$. If we differentiate 4) and 2), we get

$$\tan x = 2(2^2 - 1) H_2 \frac{x}{\pi^2} + 2(2^4 - 1) H_4 \frac{x^3}{\pi^4} + 2(2^6 - 1) H_6 \frac{x^5}{\pi^6}$$

valid for $|x| < \frac{\pi}{2}$;

$$\cot x = \frac{1}{x} - 2 H_2 \frac{x}{\pi^2} - 2 H_4 \frac{x^3}{\pi^4} - 2 H_6 \frac{x^5}{\pi^6} - \cdots$$

valid for $0 < |x| < \pi$.

Comparing 5) with the development of $\tan x$ given

gives

$$H_2 = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6} = \frac{1}{6} \cdot \frac{2\,\pi^2}{2\,!} = B_1 \cdot \frac{2\,\pi^2}{2\,!}.$$

$$H_4 = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \frac{\pi^4}{90} = \frac{1}{30} \cdot \frac{2^3\,\pi^4}{4\,!} = B_3 \cdot \frac{2^3\,\pi^4}{4\,!}$$

$$H_6 = \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \cdots = \frac{\pi^6}{945} = \frac{1}{42} \cdot \frac{2^5\,\pi^6}{6\,!} = B_5 \cdot \frac{2^5\,\pi^6}{6\,!}$$

$$H_8 = \frac{1}{1^8} + \frac{1}{2^8} + \frac{1}{3^8} + \cdots = \frac{\pi^8}{9450} = \frac{1}{30} \cdot \frac{2^7\,\pi^8}{8\,!} = B_7 \cdot \frac{2^7\,\pi^8}{8\,!}$$

. . . .

Let us set

$$H_{2n} = \frac{2^{2n-1}\pi^{2n}}{(2\,n)\,!} B_{2n-1}.$$

From 6), 8) we get

$$\operatorname{cotan} x - \frac{1}{x} = -\sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} B_{2n-1} x^{2n-1}$$

valid for $0 < |x| < \pi$.

219. *Recursion formula for the Bernouillian Num*

If we set $$f(x) = \tan x,$$

we have by Taylor's development

$$f(x) = x f'(0) + x^3 \frac{f'''(0)}{3!} + x^5 \frac{f^{v}(}{5}$$

where $$\frac{f^{(2n-1)}(0)}{(2n-1)!} = \frac{2(2^{2n}-1)H_{2n}}{\pi^{2n}} = \frac{2^{2n}(2^{2n} \quad 1}{(2n)!}$$

Now by I, 408,

$$f^{(2n-1)}(0) - \binom{2n-1}{2} f^{(2n-3)}(0) - \binom{2n-1}{4} f^{(2n-5)}(0) -$$

From 1), 2) we get

$$\frac{2^{2n-1}(2^{2n}-1)}{n} B_{2n-1} - \binom{2n-1}{2} \frac{2^{2n-3}(2^{2n-2}-1)}{n-1} B$$
$$+ \binom{2n-1}{4} \frac{2^{2n-5}(2^{2n-4}-1)}{n-2} B_{2n-5} - \cdots$$

We have already found B_1, B_3, B_5, B_7; it is n successively:

$$B_9 = \tfrac{5}{66} \qquad B_{11} = \tfrac{691}{2730} \qquad B_{13} = \tfrac{7}{6} \qquad B_1$$
$$B_{17} = \tfrac{43867}{798}, \quad B_{19} = \tfrac{174611}{330}.$$

Thus to calculate B_9, we have from 3)

$$2^9(2^{10}-1) \; B_9 - 9 \cdot 8 \; 2^7(2^8-1) \cdot 1 + 9 \cdot 8 \cdot 7 \cdot$$

The B *and* Γ *Functions*

220. In Volume I we defined the B and Γ functions by mea integrals:

$$B(u, v) = \int_0^{\infty} \frac{x^{u-1}dx}{(1+x)^{u+v}}$$

$$\Gamma(u) = \int_0^{\infty} e^{-x}x^{u-1}dx$$

ich converge only when $u, v > 0$. Under this condition we s at

$$B(u, v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}.$$

We propose to show that $\Gamma(u)$ can be developed in the infin oduct

$$G = \frac{1}{u}\prod_1^{\infty}\frac{\left(1+\frac{1}{n}\right)^u}{1+\frac{u}{n}}.$$

This product converges, as we saw, 197, 3, for any $u \neq 0, -$ 2, ⋯ From 201, 7 and 207 it is obvious that G converges ab ely and uniformly at any point u different from these singu ints. Thus the expression 4) has a wider domain of definit an that of 2). Since $G = \Gamma$, as we said, for $u > 0$, we shall nd the definition of the Γ function in accordance with 4), gative u.

It frequently happens that a function $f(x)$ can be represen different analytic expressions whose domains of converge different. For example, we saw 218, 9), that $\tan x$ can be loped in a power series

and

$$\tan x = 2\sum_1^\infty \frac{x}{\left(\frac{2s-1}{2}\right)^2\pi^2 - x^2} \text{ by}$$

are analytic expressions valid for every x for w
$\tan x$ is defined.

221. 1. Before showing that G and Γ have th
$u > 0$, let us develop some of the properties of the
in 220, 4). In the first place, we have, by 210:

The function $G(u)$ is continuous, except at the
$-2, \cdots$

Since the factors of 4) are all positive for $u >$

$G(u)$ is positive for $u > 0$.

2. *In the vicinity of the point $x = -m$,* $m = ($

$$G(u) = \frac{H(u)}{x+m}$$

where $H(u)$ is continuous near this point, and
this point.

For

$$G(u) = \frac{\left(1+\frac{1}{m}\right)^u}{1+\frac{u}{m}} H(u)$$

where H is the infinite product G with one factor
may reason on H as we did on G, we see H conv
$x = -m$. Hence $H \neq 0$ at this point. But H al
formly about this point; hence H is continuous

Also

$$(u+1)(u+2)\cdots(u+n-1)=(n-1)!\left(1+\frac{u}{1}\right)\left(1+\frac{u}{2}\right)\cdots$$

Thus $P_n = G_n$. But $G_n \doteq G$, hence P_n, is conver lim P_n.

223. *Euler's Constant.* This is defined by the con

$$C=\sum_1^\infty\left\{\frac{1}{n}-\log\left(1+\frac{1}{n}\right)\right\}.$$

It is easy to see at once that

$$C<\frac{1}{2}\sum\frac{1}{n^2}=\frac{1}{2}H_2=\frac{\pi^2}{12}=.82\cdots$$

by 218, 7). By calculation it is found that

$$C=.577215\cdots$$

224. *Another expression of G is*

$$G=\frac{e^{-Cu}}{u\Pi\left(1+\frac{u}{n}\right)e^{-\frac{u}{n}}}\ ,\quad n=1,2,\cdots$$

where C is the Eulerian constant.

For when $a>0$, $\quad a^u=e^{u\log a}$.

Hence

$$G=\frac{1}{u}\Pi\frac{e^{u\log\left(1+\frac{1}{n}\right)}}{1+\frac{u}{n}}$$

are convergent. Hence

$$G = \frac{\Pi e^{u\left[\log\left(1+\frac{1}{n}\right) - \frac{1}{n}\right]}}{u\Pi\left(1+\frac{u}{n}\right)e^{-\frac{u}{n}}}$$

from which 1) follows at once, using 223.

225. *Further Properties of G.*

1. $$G(u+1) = uG(u).$$

Let us use the product

$$P_n(u) = \frac{1}{u} \cdot \frac{(n-1)!}{(u+1)\cdots(u+n-1}$$

employed in 222. Then

$$P_n(u+1) = \frac{nuP_n(u)}{u+n}.$$

As

$$\frac{nu}{u+n} \doteq u \qquad \text{as } n \doteq$$

we get 1) from 2) at once on passing to the lin

2. $$G(u+n) = u(u+1)\cdots(u+n$$

This follows from 1) by repeated application

3. $$G(n) = 1 \cdot 2 \cdot \cdots n-1 = (n$$

where n is a positive integer.

4. $$G(u)\,G(1-u) = \frac{\pi}{\sin \pi u}.$$

For

$$G(1-u) = -uG(-u) \qquad \text{by}$$

$$= \frac{e^{Cu}}{\quad} \qquad \text{by}$$

We now use 213, 5).

Let us note that by virtue of 1, 2 the value of G is know all $u > 0$, when it is known in the interval (0, 1). By virt 5) G is known for $u > 0$ when its value is known for Moreover the relation 5) shows the value of G is known in (when its value is known in $(0, \frac{1}{2})$.

As a result of this we see G is known when its values i interval $(0, \frac{1}{2})$ are known; or indeed in any interval of leng

Gauss has given a table of $\log G(u)$ for $1 < u < 1.5$ calcu to 20 decimal places. A four-place table is given in "A Table of Integrals" by *B. O. Peirce*, for $1 < u < 2$.

5. $$G(\tfrac{1}{2}) = \sqrt{\pi}.$$

For in 5) set $u = \frac{1}{2}$. Then

$$G^2(\tfrac{1}{2}) = \pi.$$

Hence $$G(\tfrac{1}{2}) = \pm\sqrt{\pi}.$$

We must take the plus sign here, since $G > 0$ when $u > 0$, b

6. $$G\left(\frac{2n+1}{2}\right) = \frac{1 \cdot 3 \cdot 5 \ldots 2n-1}{2^n} \cdot \sqrt{\pi}$$

where n is a positive integer.

For $$G\left(\frac{2n+1}{2}\right) = G\left(1 + \frac{2n-1}{2}\right) = \frac{2n-1}{2} G\left(\frac{2n-1}{2}\right),$$

Similarly $$G\left(\frac{2n-1}{2}\right) = \frac{2n-3}{2} G\left(\frac{2n-3}{2}\right), \text{ etc.}$$

Thus $$G\left(\frac{2n+1}{2}\right) = \frac{2n-1}{2} \cdot \frac{2n-3}{2} \ldots \frac{3}{2} \cdot \frac{1}{2} G\Big($$

We may write 2)

$$L' = -C + \sum_1^\infty \left\{ \frac{1}{n} - \frac{1}{u+n-1} \right\}.$$

That the relations 2), 3) hold for any $u \neq 0,$ $-$
by reasoning similar to that employed in 216. In

$$L^{(r)} = (-1)^r (r-1)! \sum_1^\infty \frac{1}{(u+n-1)^r},$$

In particular,

$$L'(1) = -C.$$

$$L^{(r)}(1) = (-1)^r (r-1)! \sum \frac{1}{n^r} = (-1)^r (r$$

227. *Development of* log $G(u)$ *in a Power Seri*
development is valid about the point $u = 1$, we ha

$$\log G(u) = L(u) = L(1) + \frac{u-1}{1!} L'(1) + \frac{(u-1}{2!}$$

or using 226, 5), and setting $u = 1 + x$,

$$\log G(1+x) = -Cx + \sum_{n=2}^\infty \frac{(-1)^n}{n} H$$

We show now *this relation is valid for* $-\frac{1}{2} < x$
that

$$R_s = \frac{x^s}{s!} L^{(s)}(1 + \theta x), \qquad 0 < \theta < 1$$

converges to 0, as $s \doteq \infty$.

For, if $0 \leq x \leq 1$, then

$$| R_s | \leq \frac{1}{s} \sum^\infty \frac{1}{n^s} \doteq 0.$$

s shown how the series 1) may be made to converge m
pidly. We have for any x in $\mathfrak{A}$

$$\log(1+x) = x - \sum_2^\infty (-1)^n \frac{x^n}{n}.$$

This on adding and subtracting from 1) gives

$$\log G(1+x) = -\log(1+x) + (1-C)x + \sum_2^\infty (-1)^n (H_n - 1)$$

Changing here x into $-x$ gives

$$\log G(1-x) = -\log(1-x) - (1-C)x + \Sigma(H_n - 1)\frac{x^n}{n}.$$

Subtracting this from the foregoing gives

$$\log G(1+x) - \log G(1-x)$$
$$= -\log\frac{1+x}{1-x} + 2(1-C)x - \sum_1^\infty \frac{x^{2m+1}}{2m+1}(H_{2m+1} -$$

From 225, 4

$$\log G(1+x) + \log G(1-x) = \log\frac{\pi x}{\sin \pi x}.$$

This with the preceding relation gives

$$\log G(1+x)$$
$$(1-C)x - \tfrac{1}{2}\log\frac{1+x}{1-x} + \log\frac{\pi x}{\sin\pi x} - \tfrac{1}{2}\sum_1^\infty (H_{2m+1} - 1)\frac{x^{2m+1}}{2m+1}$$

lid in $\mathfrak{A}$.

This series converges rapidly for $0 \leq x \leq \frac{1}{2}$, and enables us
mpute $G(u)$ in the interval $1 \leq u \leq \frac{3}{2}$. The other values of
y be readily obtained as already observed.

Now for any x in the interval $(0, n)$,

$$x^u \leq n^u \quad , \quad x^u > x n^{u-1}$$

since $u \geq 0$ and $u - 1 \leq 0$.

Also for any x in the interval (n, ∞)

$$x^u < x n^{u-1} \quad , \quad x^u > n^u.$$

Hence

$$n^{u-1}\int_0^n e^{-x} x^n dx + n^u \int_n^\infty e^{-x} x^{n-1} dx < \Gamma(u+n)$$

$$< n^u \int_0^n e^{-x} x^{n-1}$$

Thus

$$\frac{\Gamma(u+n)}{n^u} < \int_0^n e^{-x} x^{n-1} dx + \frac{1}{n}\int_n^\infty e^{-x} x^n dx$$

$$< \int_0^n e^{-x} x^{n-1} dx + \frac{1}{n}\int_0^\infty e^{-x}$$

Let us call these integrals A, B, C respective
We see at once that

$$B = \frac{\Gamma(n+1)}{n} = \frac{n!}{n} = (n-1)!$$

Also, integrating by parts,

$$A = \left[\frac{e^{-x} x^n}{n}\right]_0^n + \frac{1}{n}\int_0^n e^{-x} x^n dx = \frac{n^n}{ne^n}$$

Thus $$\frac{\Gamma(u+n)}{n^u} < (n-1)! + \frac{n^{n-1}}{e^n}.$$

Similarly $$\frac{\Gamma(u+n)}{n^u} > (n-1)! - \frac{n^{n-1}}{}.$$

But

$$\nu_n > 1 + \frac{n}{n+1} + \cdots + \frac{n^m}{(n+1)\cdots(n+m)} \quad , \quad \text{for any } m$$

$$> \frac{mn^m}{(n+1)\cdots(n+m)} = \frac{m}{\left(1+\frac{1}{n}\right)\cdots\left(1+\frac{m}{n}\right)} > \frac{m}{\left(1+\frac{m}{n}\right)^m}.$$

Let us take

$$n > m^2 \quad \text{or} \quad \frac{m}{n} < \frac{1}{m}.$$

Then

$$\nu_n > \frac{m}{\left(1+\frac{1}{m}\right)^m} > \frac{m}{e}.$$

Since m may be taken large at pleasure,

$$\lim \nu_n = \infty$$

nd hence

$$\lim q_n = 0.$$

Thus

$$\lim_{n=\infty} \frac{\Gamma(u+n)}{(n-1)!\, n^u} = 1 \quad , \quad 0 \leq u \leq 1.$$

But from $\Gamma(u+1) = u\Gamma(u)$ we have

$$\frac{\Gamma(u+1+n)}{n^{u+1}(n-1)!} = \frac{u+n}{n}\,\frac{\Gamma(u+n)}{n^u(n-1)!} = 1$$

so, as $n \doteq \infty$. Thus the relation 1) holds for $1 \leq u \leq 2$, and
ct for any $u > 0$.

As

$$\Gamma(u+n) = u(u+1)\cdots(u+n-1)\Gamma(u),$$

e have

$$\Gamma(u) = \frac{\Gamma(u+n)}{u(u+1)\cdots(u+n-1)}.$$

Hence using 1),

$$\Gamma(u) = \frac{(n-1)!\, n^u}{u(u+1)\cdots(u+n-1)} \cdot \frac{\Gamma(u+n)}{(n-1)!\, n^u}$$

Letting $n \doteq \infty$, we get $\Gamma(u) = G(u)$ for any $u > 0$, making u
1) and 222, 1).

CHAPTER VIII

AGGREGATES

Equivalence

229. 1. Up to the present the aggrega
have been point aggregates. We now c
general. Any collection of well-determin
able one from another, and thought of as
an *aggregate* or *set*.

Thus the class of prime numbers, the cl
tions, the inhabitants of the United States,

Some of the definitions given for point
ously to aggregates in general, and we sha
them here, as it is only necessary to repl
object or element.

As in point sets, $\mathfrak{A} = 0$ shall mean that $\mathfrak{A}$

Let $\mathfrak{A}$, $\mathfrak{B}$ be two aggregates such that
associated with some one element b of $\mathfrak{B}$, an
that $\mathfrak{A}$ is *equivalent* to $\mathfrak{B}$ and write

$$\mathfrak{A} \sim \mathfrak{B}.$$

We also say $\mathfrak{A}$ and $\mathfrak{B}$ are in *one to one c*
uniform correspondence. To indicate that
in this correspondence we write

$$a \sim b.$$

For we can associate the elements of $\mathfrak{A}$ with t
keeping precisely the correspondence which exist
elements of $\mathfrak{B}$ and B, of $\mathfrak{C}$ and C, etc.

Example 1. $\mathfrak{A} = 1, 2, 3, \cdots$

$$\mathfrak{B} = a_1, a_2, a_3, \cdots$$

If we set $a_n \sim n$, $\mathfrak{A}$ and $\mathfrak{B}$ will stand in 1, 1 corre

Example 2. $\mathfrak{A} = 1, 2, 3, 4, \cdots$

$$\mathfrak{B} = 2, 4, 6, 8, \cdots$$

If we set n of $\mathfrak{A}$ in correspondence with $2n$ of $\mathfrak{B}$
be in uniform correspondence.

We note that $\mathfrak{B}$ is a part of $\mathfrak{A}$; we have thus t
infinite aggregate may be put in uniform corresp
partial aggregate of itself.

This is obviously impossible if $\mathfrak{A}$ is finite.

Example 3. $\mathfrak{A} = 1, 2, 3, 4, \cdots$

$$\mathfrak{B} = 10^1, 10^2, 10^3, 10^4, \cdots$$

If we set $n \sim 10^n$, we establish a uniform corr
tween $\mathfrak{A}$ and $\mathfrak{B}$. We note again that $\mathfrak{A} \sim \mathfrak{B}$ althou

Example 4. Let $\mathfrak{C} = \{\xi\}$, where, using the triad

$$\xi = \cdot\, \xi_1 \xi_2 \xi_3 \cdots \qquad \xi_n = 0, 2$$

denote the Cantor set of I, 272. Let us associate

$$x = \cdot\, x_1 x_2 x_3 \cdots$$

where $x_n = 0$ when $\xi_n = 0$, and $= 1$ when $\xi_n = 2$
the dyadic system

viously we can select enough of these del
their lower content is as near 1 as we choose

$$\underline{\mathrm{Cont}}\, C' = 1.$$

As $\overline{\mathrm{Cont}}\, C \leq 1$, C' is metric and its cont
discrete.

230. 1. *Let* $\mathfrak{A} = \alpha + A$, $\mathfrak{B} = \beta + B$, *wh*
of $\mathfrak{A}$, $\mathfrak{B}$ *respectively.* *If* $\mathfrak{A} \sim \mathfrak{B}$, *then* $A \sim B$

For, since $\mathfrak{A} \sim \mathfrak{B}$, each element a of $\mathfrak{A}$ is
one element b of $\mathfrak{B}$, and the same holds for
that $\alpha \sim \beta$, the uniform correspondence of
on the contrary $\alpha \sim b'$ and $\beta \sim a'$, the unifor
tween A, B can be established by setting a
other elements in A, B correspond as in $\mathfrak{A} \sim$

2. We state as obvious the theorems:

No part $\mathfrak{B}$ *of a finite set* $\mathfrak{A}$ *can be* $\sim \mathfrak{A}$.

No finite part $\mathfrak{B}$ *of an infinite set* $\mathfrak{A}$ *can be*

Cardinal Numbers

231. 1. We attach now to each aggre
called its *cardinal number*, which is defined a

1° Equivalent aggregates have the same c

2° If $\mathfrak{A}$ is $\sim$ to a part of $\mathfrak{B}$, but $\mathfrak{B}$ is no
of $\mathfrak{A}$, the cardinal number of $\mathfrak{A}$ is less tha
cardinal number of $\mathfrak{B}$ is greater than that

2. It is a property of any two *finite* cardinal nu
either

$$\mathfrak{a} = \mathfrak{b} \quad , \quad \text{or } \mathfrak{a} > \mathfrak{b} \quad , \quad \text{or } \mathfrak{a} < \mathfrak{b}.$$

This property has not yet been established for
dinal numbers. There is in fact a fourth alterna
$\mathfrak{A}$, $\mathfrak{B}$, besides the three involved in 1). For unt
has been shown, there is the possibility that:

No part of $\mathfrak{A}$ is $\sim \mathfrak{B}$, and no part of $\mathfrak{B}$ is $\sim \mathfrak{A}$.

The reader should thus guard against expre
assuming that one of the three relations 1) mus
two cardinal numbers.

3. We note here another difference. If $\mathfrak{A}$, $\mathfrak{B}$
out common element,

$$\text{Card}\,(\mathfrak{A} + \mathfrak{B}) > \text{Card}\,\mathfrak{A}.$$

Let now $\mathfrak{A}$ denote the positive even and $\mathfrak{B}$ t
numbers. Obviously

$$\text{Card}\,(\mathfrak{A} + \mathfrak{B}) = \text{Card}\,\mathfrak{A} = \text{Card}\,\mathfrak{B}$$

and the relation 2) does not hold for these transfin

4. We have, however, the following:

Let $\mathfrak{A} > \mathfrak{B}$, *then*

$$\text{Card}\,\mathfrak{A} \geq \text{Card}\,\mathfrak{B}.$$

For obviously $\mathfrak{B}$ is $\sim$ to a part of $\mathfrak{A}$, viz. $\mathfrak{B}$ itse

5. This may be generalized as follows:

Let

$$\mathfrak{A} = \mathfrak{B} + \mathfrak{C} + \mathfrak{D} + \cdots$$

$$A = B + C + D + \cdots$$

Enumerable Sets

232. 1. An aggregate which is equiva
positive integers $\mathfrak{J}$ or to a part of $\mathfrak{J}$ is

Thus all finite aggregates are enumerabl
ber attached to an infinite enumerable set i

At times we shall also denote this cardin

$$\mathfrak{e} = \aleph_0.$$

2. *Every infinite aggregate $\mathfrak{A}$ contains an i*

For let a_1 be an element of $\mathfrak{A}$ and

$$\mathfrak{A} = a_1 + \mathfrak{A}_1.$$

Then $\mathfrak{A}_1$ is infinite; let a_2 be one of its e

$$\mathfrak{A}_1 = a_2 + \mathfrak{A}_2.$$

Then $\mathfrak{A}_2$ is infinite, etc.

Then

$$\mathfrak{B} = a_1, a_2, \cdots$$

is a part of $\mathfrak{A}$ and forms an infinite enumer

3. From this follows that

$\aleph_0$ *is the least transfinite cardin*

233. *The rational numbers are enumerab*

For any rational number may be written

$$r = \frac{m}{n}$$

where, as usual, m is relatively prime to n.

The equation

$$|m| + |n| = p$$

admits but a finite number of solutions for

Let us now arrange these solutions in a sequence, putting tho
rresponding to $p = q$ before those corresponding to $p = q +$
We get

$$r_1 \;,\; r_2 \;,\; r_3 \cdots$$

hich is obviously enumerable.

234. *Let the indices $\iota_1, \iota_2, \cdots \iota_p$ range over enumerable sets. Th*

$$\mathfrak{A} = \{a_{\iota_1 \cdots \iota_p}\}$$

enumerable.

For the equation

$$\nu_1 + \nu_2 + \cdots + \nu_p = n,$$

here the ν's are positive integers, admits but a finite numb
solutions for each $n = p,\ p+1,\ p+2,\ p+3 \cdots$ Thus t
ements of

$$\mathfrak{B} = \{b_{\nu_1 \cdots \nu_p}\}$$

ay be arranged in a sequence

$$b_1 \;,\; b_2 \;,\; b_3 \cdots$$

giving to n successively the values $p,\ p+1, \cdots$ and putting t
ements $b_{\nu_1 \cdots \nu_p}$ corresponding to $n = q + 1$ after those correspon
g to $n = q$.

Thus the set $\mathfrak{B}$ is enumerable. Consider now $\mathfrak{A}$. Since ea
dex ι_m ranges over an enumerable set, each value of ι_m as ι_m'
sociated with some positive integer as m' and conversely.
ay now establish a 1, 1 correspondence between $\mathfrak{A}$ and $\mathfrak{B}$
tting

$$b_{m_1' m_2' \cdots m_p'} \sim a_{\iota_1' \iota_2' \cdots \iota_p'}.$$

Hence $\mathfrak{A}$ is enumerable.

235. 1. *An enumerable set of enumerable aggregates form*

Thus the a-elements in 1) form a set

$$\{a_{mn}\} \qquad m, n, = 1, 2,$$

which is enumerable by 234.

2. *The real algebraic numbers form an en*

For each algebraic number is a root of irreducible equation of the form

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0$$

the a's being rational numbers. Thus the t numbers may be represented by

$$\{\rho_{n,\, a_1 a_2 \cdots a_n}\}$$

where the index n runs over the positive int over the rational numbers.

3. *Let $\mathfrak{A}$, $\mathfrak{B}$ be two enumerable sets. The*

$$\text{Card } \mathfrak{A} = \text{Card } \mathfrak{B} = \aleph_0$$

$$\text{Card } (\mathfrak{A} + \mathfrak{B}) = \aleph_0$$

And in general if $\mathfrak{A}_1$, $\mathfrak{A}_2 \cdots$ are an enume aggregates, $\quad \text{Card } (\mathfrak{A}_1, \mathfrak{A}_2, \cdots) = \aleph_0.$

This follows from 1.

236. *Every isolated aggregate $\mathfrak{A}$, limited o able set.*

For let us divide $\mathfrak{R}_m$ into cubes of side 1. an enumerable set C_1, $C_2 \cdots$. About each as center we describe a cube of side σ, so sm other point of $\mathfrak{A}$. This is possible since $\mathfrak{A}$ is is

where $\mathfrak{A}_\iota$ denotes the isolated points of $\mathfrak{A}$ and
ing points of $\mathfrak{A}$.

Similarly,

$$\mathfrak{A}'_p = \mathfrak{A}'_{p,\iota} + \mathfrak{A}''_p$$

$$\mathfrak{A}''_p = \mathfrak{A}''_{p,\iota} + \mathfrak{A}'''_p$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Thus,

$$\mathfrak{A} = \mathfrak{A}_\iota + \mathfrak{A}'_{p,\iota} + \mathfrak{A}''_{p,\iota} + \cdots +$$

But $\mathfrak{A}^{(n)}$ is finite and $\mathfrak{A}^{(n)}_p \leq \mathfrak{A}^{(n)}$.

Thus $\mathfrak{A}$ being the sum of $n+1$ enumerable

2. *If $\mathfrak{A}'$ is enumerable, so is $\mathfrak{A}$.*

For as in 1,

$$\mathfrak{A} = \mathfrak{A}_\iota + \mathfrak{A}'_p$$

and

$$\mathfrak{A}'_p \leq \mathfrak{A}'.$$

238. 1. *Every infinite aggregate $\mathfrak{A}$ contains*
$\mathfrak{B} \sim \mathfrak{A}$.

For let $\mathfrak{E} = (a_1, a_2, a_3 \cdots)$ be an infinite e
so that

$$\mathfrak{A} = \mathfrak{E} + \mathfrak{F}.$$

Let

$$\mathfrak{E} = a_1 + E.$$

To establish a uniform correspondence b
associate a_n in $\mathfrak{E}$ with a_{n+1} in E. Thus $\mathfrak{E} \sim E$.

We now set

$$\mathfrak{B} = E + \mathfrak{F}.$$

239. 1. A theorem of great impor
whether two aggregates are equivalent i
the converse of 238, 2.

Let $\mathfrak{A}_1 < \mathfrak{A}$, $\mathfrak{B}_1 < \mathfrak{B}$. *If* $\mathfrak{A}_1 \sim \mathfrak{B}$ *and* $\mathfrak{B}$
then
$$\mathfrak{A} \sim \mathfrak{B}.$$

In the correspondence $\mathfrak{A}_1 \sim \mathfrak{B}$, let $\mathfrak{A}_2$
associated with $\mathfrak{B}_1$. Then
$$\mathfrak{A}_2 \sim \mathfrak{B}_1 \sim \mathfrak{A}$$
and hence
$$\mathfrak{A} \sim \mathfrak{A}_2.$$

But as $\mathfrak{A}_1 > \mathfrak{A}_2$, we would infer from 1)
$$\mathfrak{A} \sim \mathfrak{A}_1.$$

As $\mathfrak{A}_1 \sim \mathfrak{B}$ by hypothesis, the truth of
once from 2).

To establish 2) we proceed thus. In the
$\mathfrak{A}_3$ be that part of $\mathfrak{A}_2$ which $\sim \mathfrak{A}_1$ in $\mathfrak{A}$.
$\mathfrak{A}_1 \sim \mathfrak{A}_3$, let $\mathfrak{A}_4$ be that part of $\mathfrak{A}_3$ which $\sim$
Continuing in this way, we get the indef
$$\mathfrak{A} > \mathfrak{A}_1 > \mathfrak{A}_2 > \mathfrak{A}_3 > \cdot$$
such that
$$\mathfrak{A} \sim \mathfrak{A}_2 \sim \mathfrak{A}_4 \sim \cdots$$
$$\mathfrak{A}_1 \sim \mathfrak{A}_3 \sim \mathfrak{A}_5 \sim \cdots$$

Let now
$$\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{C}_1 \quad , \quad \mathfrak{A}_1 = \mathfrak{A}$$
$$\mathfrak{D} = Dv(\mathfrak{A}, \mathfrak{A}_1, \mathfrak{A}_2 \cdots)$$

Then
$$\mathfrak{A} = \mathfrak{D} + \mathfrak{C}_1 + \mathfrak{C}_2 + \mathfrak{C}_3$$
and similarly
$$\mathfrak{A}_1 = \mathfrak{D} + \mathfrak{C}_2 + \mathfrak{C}_3 + \mathfrak{C}_4$$

2. In connection with the foregoing proof, *Bernstein*, the reader must guard against the foll does not *in general* follow from

$$\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{C}_1 \quad , \quad \mathfrak{A}_2 = \mathfrak{A}_3 + \mathfrak{C}_3 \quad , \quad \mathfrak{A} \sim \mathfrak{A}_2$$

that

$$\mathfrak{C}_1 \sim \mathfrak{C}_3$$

which is the first relation in 5).

Example. Let $\mathfrak{A} = (1, 2, 3, 4, \cdots)$.

$$\mathfrak{A}_1 = (2, 3, 4, 5 \cdots) \quad , \quad \mathfrak{A}_2 = (3, 4, 5, 6$$

$$\mathfrak{A}_3 = (5, 6, 7, 8 \cdots).$$

Then

$$\mathfrak{C}_1 = 1 \qquad \mathfrak{C}_3 = (3, 4).$$

Now $\mathfrak{A}$, $\mathfrak{A}_1$, $\mathfrak{A}_2$, $\mathfrak{A}_3$ are all enumerable sets; her

$$\mathfrak{A} \sim \mathfrak{A}_2 \quad , \quad \mathfrak{A}_1 \sim \mathfrak{A}_3.$$

But obviously $\mathfrak{C}_1$ is not equivalent to $\mathfrak{C}_3$, since only one element cannot be put in 1 to 1 corres set consisting of two elements.

240. 1. *If* $\mathfrak{A} > \mathfrak{B} > \mathfrak{C}$, *and* $\mathfrak{A} \sim \mathfrak{C}$, *then* $\mathfrak{A} \sim \mathfrak{B}$.

For by hypothesis a part of $\mathfrak{B}$, viz. $\mathfrak{C}$, is $\sim \mathfrak{A}$. is $\sim \mathfrak{B}$, viz. $\mathfrak{B}$ itself. We apply now 239.

2. *Let* α *be any cardinal number. If*

$$\alpha \leq \text{Card}\, \mathfrak{B} \leq \alpha,$$

then

$$\alpha = \text{Card}\, \mathfrak{B}.$$

For let Card $\mathfrak{A} = \alpha$. Then from

$$\alpha \leq \text{Card}\, \mathfrak{B}$$

it follows that $\mathfrak{A} \sim$ a part or the whole of $\mathfrak{B}$; whi

4. *Two infinite enumerable sets are equivalent.*

For both are equivalent to $\mathfrak{Z}$, the set of positive i

241. 1. *Let $\mathfrak{E}$ be any enumerable set in $\mathfrak{A}$; set $\mathfrak{A}$ $\mathfrak{B}$ is infinite, $\mathfrak{A} \sim \mathfrak{B}$.*

For $\mathfrak{B}$ being infinite, contains an infinite enum Let $\mathfrak{B} = \mathfrak{F} + \mathfrak{G}$. Then

$$\mathfrak{A} = \mathfrak{E} + \mathfrak{F} + \mathfrak{G},$$

$$\mathfrak{B} = \mathfrak{F} + \mathfrak{G}.$$

But $\mathfrak{E} + \mathfrak{F} \sim \mathfrak{F}$. Hence $\mathfrak{A} \sim \mathfrak{B}$.

2. We may state 1 thus:

$$\text{Card}\,(\mathfrak{A} - \mathfrak{E}) = \text{Card}\,\mathfrak{A}$$

provided $\mathfrak{A} - \mathfrak{E}$ is infinite.

3. From 1 follows at once the theorem:

Let $\mathfrak{A}$ be any infinite set and $\mathfrak{E}$ an enumerable set.

$$\text{Card}\,(\mathfrak{A} + \mathfrak{E}) = \text{Card}\,\mathfrak{A}.$$

Some Space Transformations

242. 1. Let T be a transformation of space such point x corresponds a single point x_T, and conversel Moreover, let x, y be *any* two points of space. A formation they go over into x_T, y_T. If

$$\text{Dist}\,(x, y) = \text{Dist}\,(x_T, y_T)$$

If $\mathfrak{R}$ denotes the original space, and $\mathfrak{R}_T$ th
after displacement, we have, obviously,

$$\mathfrak{R} \sim \mathfrak{R}_T.$$

2. Let $\quad y_1 = tx_1 \quad , \quad \cdots \quad y_m = tx_m \quad , \quad t$

Then when x ranges over the m-way space
m-way space $\mathfrak{Y}$. If we set $x \sim y$ as defined b

$$\mathfrak{X} \sim \mathfrak{Y}.$$

Also $$\text{Dist }(0, y) = t \text{ Dist }(0, x)$$

We call 1) a transformation of *similitude*.
space is *dilated*; if $t < 1$, it is *contracted*.

3. Let Q be any point in space. About it
scribe a sphere S of radius R. Let P be any
join of P, Q let us take a point P' such that

$$\text{Dist }(P', Q) = \frac{R^2}{\text{Dist }(P, Q)}.$$

Then P' is called the *inverse of P with resp*
formation of space is called *inversion*. Q is th
Obviously points without S go over into po
versely. As $P \doteq \infty$, $P' \doteq Q$.
The correspondence between the old and ne
except there is no point corresponding to Q.

The Cardinal c

243. 1. *All or any part of space* $\mathfrak{S}$ *may b*

correspondence between $\mathfrak{S}$ and the resulting aggreg
uniform since all the transformations employed ar

As a result of this and 240, 1 we see that the
real numbers is $\sim$ to those lying in the interval (0,
the aggregate of all points of $\Re_m$ is $\sim$ to the point
or a unit sphere, etc.

244. 1. *The points lying in the unit interval*
not enumerable.

For if they were, they could be arranged in a se

$$a_1, a_2, a_3 \cdots$$

Let us express the a's as decimals in the norm

$$a_n = \cdot a_{n1} a_{n2} a_{n3} \cdots$$

Consider the decimal

$$b = \cdot b_1 b_2 b_3 \cdots$$

also written in the normal form, where

$$b_1 \neq a_{1,1} \quad , \quad b_2 \neq a_{2,2} \quad , \quad b_3 \neq a_{3,3} \quad ,$$

Then b lies in $\mathfrak{A}$ and is yet different from any nu

2. We have $\quad (0^*, 1^*) \sim (0, 1) \quad$, by 241, 3,

$$\sim (a, b) \quad , \text{ by } 243,$$

where a, b are finite or infinite.

Thus the cardinal number of any interval, fin
with or without its end points is the same.

We denote it by $\mathfrak{c}$ and call it *the cardinal numb*
linear continuum, or *of the real number system* $\Re$.

4. *The cardinal number of the Cantor set* $\mathfrak{C}$ *of* I, 272 *is*

For each point a of $\mathfrak{C}$ has the representation in th
system

$$a = \cdot a_1 a_2 a_3 \cdots \quad , \quad a = 0, 2.$$

But if we read these numbers in the dyadic system,
each $a_n = 2$ by the value 1, we get all the points in the
(0, 1). As there is a uniform correspondence between
sets of points, the theorem is established.

245. *An enumerable set* $\mathfrak{A}$ *is not perfect, and conversely*
set is not enumerable.

For suppose the enumerable set

$$\mathfrak{A} = a_1, a_2 \cdots$$

were perfect. In $D_{r_1}^*(a_1)$ lies an infinite partial set
since by hypothesis $\mathfrak{A}$ is perfect. Let a_{m_2} be the point
index in $\mathfrak{A}_1$. Let us take $r_2 < r_1$ such that $D_{r_2}(a_{m_2}$
$D_{r_1}^*(a_1)$. In $D_{r_2}^*(a_{m_2})$ lies an infinite partial set $\mathfrak{A}_2$ of
a_{m_3} be the point of lowest index in $\mathfrak{A}_2$, etc.

Consider now the sequence

$$a_1 \quad , \quad a_{m^2} \quad , \quad a_{m_3} \quad \cdots$$

It converges to a point α by I, 127, 2. But α lies in $\mathfrak{A}$,
is perfect. Thus α is some point of 1), say $\alpha = a_s$.
leads to a contradiction. For a_s lies in every $D_{r_{m_n}}^*(a_{m_n}$
other hand, no point in this domain has an index as l
which $\doteq \infty$, as $n \doteq \infty$. Thus $\mathfrak{A}$ cannot be perfect.

Conversely, suppose the perfect set $\mathfrak{A}$ were enumerab
is impossible, for we have just seen that when $\mathfrak{A}$ is enun
cannot be perfect.

But $\quad \text{Card } \mathfrak{B}_n \leq \text{Card } \mathfrak{C}_n$.

Hence $\quad \text{Card } \mathfrak{A} \leq \mathfrak{c} \quad , \quad$ by 2

On the other hand, $\quad \text{Card } \mathfrak{A} \geq \text{Card } \mathfrak{A}_1 =$

From 1), 2) we have the theorem, by 24

247. 1. As already stated, the complex
notes a point in n-way space. Let $x_1, x_2,$
enumerable set. We may also say that the

$$x = (x_1, x_2, \cdots \text{ in inf.})$$

denotes *a point in* ∞*-way space* $\mathfrak{R}_x$.

2. *Let* $\mathfrak{A}$ *denote a point set in* $\mathfrak{R}_n$, *n finite*

$$\text{Card } \mathfrak{A} \leq \mathfrak{c}.$$

For let us first consider the unit cube $\mathfrak{C}$
range over $\mathfrak{B} = (0^*, 1^*)$. Let $\mathfrak{D}$ denote the

$$\mathfrak{c} = \text{Card } \mathfrak{D} \leq \text{Card } \mathfrak{C}.$$

On the other hand we show Card $\mathfrak{C} \leq \mathfrak{c}$.

For let us express each coördinate x_m as
form. Then

$$x_1 = \cdot a_{11}a_{12}a_{13}a_{14} \cdots$$

$$x_2 = \cdot a_{21}a_{22}a_{23}a_{24} \cdots$$

$$x_3 = \cdot a_{31}a_{32}a_{33}a_{34} \cdots$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Let us now form the number

$$y = \cdot a_{11}a_{12}a_{21}a_{13}a_{22}a_{23} \cdots$$

obtained by reading the above table diagon

Let us now complete $\mathfrak{C}$ by adding its faces, ob
By a transformation of similitude T we can b
Hence

$$\text{Card } \mathfrak{C} \geq \text{Card } C.$$

On the other hand, $\mathfrak{C}$ is a part of C, hence

$$\text{Card } \mathfrak{C} \leq \text{Card } C.$$

Thus Card $C = \mathfrak{c}$. The rest of the theorem fo

248. *Let $\mathfrak{F} = \{f\}$ denote the aggregate of one-
functions over a unit cube $\mathfrak{C}$ in $\mathfrak{R}_n$.*

Then

$$\text{Card } \mathfrak{F} = \mathfrak{c}.$$

Let C denote the rational points of $\mathfrak{C}$, *i.e.*
whose coördinates are rational. Then any f is
values over C are known. For if α is an irra
we can approach it over a sequence of rational p
But f being continuous, $f(\alpha) = \lim f(a_n)$, and
On the other hand, C being enumerable, we can
in a sequence

$$C = c_1, c_2, \cdots$$

Let now $\mathfrak{R}_\infty$ be a space of an infinite enume
dimensions, and let $y = (y_1, y_2 \cdots)$ denote any
Let f have the value η_1 at c_1, the value η_2 at
the points of C. Then the complex $\eta_1, \eta_2, \cdots$
mines f in $\mathfrak{C}$. But this complex also dete
$\eta = (\eta_1, \eta_2 \cdots)$ in $\mathfrak{R}_\infty$. We now associate f w

$$\text{Card } \mathfrak{F} \leq \text{Card } \mathfrak{R} \;= \mathfrak{c}.$$

But obviously Card $\mathfrak{F} \geq \mathfrak{c}$, for among the ele

aggregate formed of all possible a's of thi
cardinal number.

Let β be an arbitrary element of $\mathfrak{B}$. L
that a which has the value 1 for $b=\beta$ an
other b's. This establishes a correspond
part of $\mathfrak{A}$. Hence

$$\mathfrak{a} > \mathfrak{b}.$$

Suppose $\mathfrak{a}=\mathfrak{b}$. Then there exists a
associates with each b some one a and
impossible.

For call a_b that element of $\mathfrak{A}$ which is ass
a_b has the value 1 or 2 for each β of $\mathfrak{B}$.
in $\mathfrak{A}$ an element a' which for each β of
determination than the one a_b has. But
associated with some element of $\mathfrak{B}$, say th

$$a' = a_{b'}.$$

Then for $b=b'$, a' must have that one
which $a_{b'}$ has. But it has not, hence the co

250. *The aggregate of limited integrable f*
$\mathfrak{A}=(0, 1)$ *has a cardinal number* $\mathfrak{f} > \mathfrak{c}$.

For let $f(x)=0$ in $\mathfrak{A}$ except at the poi
Cantor set of I, 272, and 229, Ex. 4. At
have the value 1 or 2 at pleasure. The ag
all possible such functions has a cardina
reasoning of 249 shows. But each f is co
which is discrete. Hence f is integrable.

We have now the following obvious relations:

$$\aleph_0 + n = \aleph_0 \quad , \quad n \text{ a positive integer}$$

$$\aleph_0 + \aleph_0 + \cdots + \aleph_0 = \aleph_0 \quad , \quad n \text{ terms.}$$

$$\aleph_0 + \aleph_0 + \cdots = \aleph_0 \quad , \quad \text{an infinite enumera}$$

If the cardinal numbers of $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ *are* $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$.

$$\mathfrak{a} + (\mathfrak{b} + \mathfrak{c}) = (\mathfrak{a} + \mathfrak{b}) + \mathfrak{c},$$

$$\mathfrak{a} + \mathfrak{b} = \mathfrak{b} + \mathfrak{a}.$$

The first relation states that addition is *as* that it is *commutative.*

252. *Multiplication.*

1. Let $\mathfrak{A} = \{a\}$, $\mathfrak{B} = \{b\}$ have the cardinal union of all the pairs (a, b) forms a set called $\mathfrak{B}$. It is denoted by $\mathfrak{A} \cdot \mathfrak{B}$. We agree that same as (b, a). Then

$$\mathfrak{A} \cdot \mathfrak{B} = \mathfrak{B} \cdot \mathfrak{A}.$$

We define the product of $\mathfrak{a}$ and $\mathfrak{b}$ to be

$$\text{Card } \mathfrak{A} \cdot \mathfrak{B} = \text{Card } \mathfrak{B} \cdot \mathfrak{A} = \mathfrak{a} \cdot \mathfrak{b} =$$

2. *We have obviously the following formal cardinal numbers:*

$$\mathfrak{a}(\mathfrak{b} \cdot \mathfrak{c}) = (\mathfrak{a} \cdot \mathfrak{b})\mathfrak{c},$$

$$\mathfrak{a} \cdot \mathfrak{b} = \mathfrak{b} \cdot \mathfrak{a},$$

$$\mathfrak{a}(\mathfrak{b} + \mathfrak{c}) = \mathfrak{a}\mathfrak{b} + \mathfrak{a}\mathfrak{c},$$

which express respectively the associative, co tributive properties of cardinal numbers.

Example 2. Let $\mathfrak{A} = \{a\}$ denote the famil

$$x^2 + y^2 = a^2.$$

Let $\mathfrak{B} = \{b\}$ denote a set of segments o
interpret (a, b) to be the points on a cylin
and whose height is b. Then $\mathfrak{A} \cdot \mathfrak{B}$ is th
cylinders.

253. 1. $$\aleph_0 = n \cdot \aleph_0 \quad , \quad \textit{or } n\mathfrak{c} = \mathfrak{c}.$$

For let

$$\mathfrak{N} = (a_1, a_2, \cdots a_n),$$

$$\mathfrak{E} = (e_1, e_2 \cdots \text{in inf.})$$

Then

$$\mathfrak{N} \cdot \mathfrak{E} = (a_1, e_1) \quad , \quad (a_1, e_2) \quad ,$$

$$(a_2, e_1) \quad , \quad (a_2, e_2) \quad ,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$= \mathfrak{E}_1 + \mathfrak{E}_2 + \cdots + \mathfrak{E}_n.$$

The cardinal number of the set on the l
cardinal number of the set on the right is $\aleph_0$

2. $$e\mathfrak{c} = \mathfrak{c}.$$

For let $\mathfrak{C} = \{c\}$ denote the points on a righ
3, ...).

Then $$\mathfrak{E}\mathfrak{C} = \{(n, c)\}$$

may be regarded as the points on a right

$$\text{Card } \{l_n\} = \mathfrak{c}.$$

Hence

$$e\mathfrak{c} = \text{Card } \mathfrak{E}\mathfrak{C} = \mathfrak{c}.$$

object from any one of these classes in $C_2 \cdots$ a
compartment. The result is a certain *distrib*
from these k classes K, among the γ compartm

The number of distributions of objects from
compartments is k^γ.

For in C_1 we may put an object from any
Thus C_1 may be filled in k ways. Similarly
k ways. Thus the compartments C_1, C_2 may
Similarly C_1, C_2, C_3 may be filled in k^3 ways,

255. 1. The totality of distributions of ob
K among the γ compartments C form an aggr
denoted by

$$K^C.$$

We call it the *distribution of K over C.* Th
bution of this kind may be called the cardinal
and we have then

$$\text{Card } K^C = k^\gamma.$$

2. What we have here set forth for finite
tended to any aggregates, $\mathfrak{A} = \{a\}$, $\mathfrak{B} = \{b\}$ w
bers we call $\mathfrak{a}$, $\mathfrak{b}$. Thus the totality of distr
among the b's, or the *distribution of* $\mathfrak{A}$ *over* $\mathfrak{B}$,

$$\mathfrak{A}^{\mathfrak{B}},$$

and its cardinal number is taken to be the defi
$\mathfrak{a}^{\mathfrak{b}}$. Thus,

$$\text{Card} \cdot \mathfrak{A}^{\mathfrak{B}} = \mathfrak{a}^{\mathfrak{b}}.$$

256. *Example 1.* Let

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0$$

have rational number coefficients. Each coe
over the enumerable set of elements in t
system $R = \{r\}$, whose cardinal number is $\aleph_0$.
form a set $\mathfrak{A} = (a_\iota, \cdots a_n) = \{a\}$. To the tota
corresponds a distribution of the r's among th

As
$$\text{Card } R^{\mathfrak{A}} = \aleph_0 = \mathfrak{c}$$
we have the relation:
$$\aleph_0^n = \aleph_0 \quad , \quad \text{or } \mathfrak{c}^n = \mathfrak{c}$$
for any integer n.

On the other hand, the equations 1) ma
the complex
$$(a_1, \cdots a_n),$$

and the totality of equations 1) is associated
$$\mathfrak{C} = \{(a_1, \cdots a_n)\}.$$
But
$$\{(a_1, a_2)\} = \{a_1\} \cdot \{a_2\},$$
$$\{(a_1, a_2, a_3)\} = \{(a_1, a_2)\} \cdot \{a_3\}$$
Hence
$$\mathfrak{C} = \{a_1\} \cdot \{a_2\} \cdots \{a_{}$$
Thus
$$\text{Card } \mathfrak{C} = \mathfrak{c} \cdot \mathfrak{c} \cdot \cdots \mathfrak{c} \quad , \quad n$$
But
$$\text{Card } \mathfrak{C} = \text{Card } R^{\mathfrak{A}},$$

since each of these sets is associated uniforml
1). Thus
$$\mathfrak{c}^n = \mathfrak{c} \cdot \mathfrak{c} \cdot \cdots \mathfrak{c} \quad , \quad n$$

257. *Example 2.* Any point x in m-way s
m coördinates $x_1, x_2, \cdots x_m$, each of which ma
of real numbers $\mathfrak{R}$, whose cardinal number i
nates $x_1 \cdots x_m$ form a finite set
$$\mathfrak{X} = (x_1, \cdots x_m).$$

Thus to $\mathfrak{R}_m = \{x\}$ corresponds the distributio
$\mathfrak{R}$, among the m elements of $\mathfrak{X}$, or the set

258. $$\mathfrak{a}^{\mathfrak{b}+\mathfrak{c}} = \mathfrak{a}^{\mathfrak{b}} \cdot \mathfrak{a}^{\mathfrak{c}}.$$

To prove this we have only to show that

$$\mathfrak{A}^{\mathfrak{B}+\mathfrak{C}} \text{ and } \mathfrak{A}^{\mathfrak{B}} \cdot \mathfrak{A}^{\mathfrak{C}}$$

can be put in 1–1 correspondence. But this is o
the set on the left is the totality of all the distrib
elements of $\mathfrak{A}$ among the sets formed of $\mathfrak{B}$ and $\mathfrak{C}$.
hand, the set on the right is formed of a combinatio
bution of the elements of $\mathfrak{A}$ among the $\mathfrak{B}$, and among
such a distribution may be regarded as the distribut
sidered.

259. $$(\mathfrak{a}^{\mathfrak{b}})^{\mathfrak{c}} = \mathfrak{a}^{\mathfrak{b}\cdot\mathfrak{c}}.$$

We have only to show that we can put in 1–1 co
the elements of

$$(\mathfrak{A}^{\mathfrak{B}})^{\mathfrak{C}} \text{ and } \mathfrak{A}^{\mathfrak{B}\cdot\mathfrak{C}}.$$

Let $\mathfrak{A} = \{a\}$, $\mathfrak{B} = \{b\}$, $\mathfrak{C} = \{c\}$. We note that $\mathfrak{A}^{\mathfrak{B}}$
distributions of the a's among the b's, and that the l
is formed of the distributions of these sets among th
are obviously associated uniformly with the distrib
a's among the elements of $\mathfrak{B} \cdot \mathfrak{C}$.

260. 1. $$\mathfrak{c}^n = (m^{\mathfrak{c}})^n = m^{n\mathfrak{c}} = m^{\mathfrak{c}} = \mathfrak{c}$$

where m, n are positive integers.

For each number in the interval $\mathfrak{C} = (0, 1^*)$ can b
in normal form once and once only by

$$\cdot\, a_1 a_2 a_3 \cdots \text{ in the } m\text{-adic system},$$

Hence, $$m^{\mathfrak{e}} = \mathfrak{c}.$$

As $n^{\mathfrak{e}} = \mathfrak{e}$, we have 1), using 1) in 257.

2. *The result obtained in* 247, 2 *may be stated:*

$$\mathfrak{c}^{\mathfrak{e}} = \mathfrak{c}.$$

3. $$\mathfrak{e}^{\mathfrak{c}} = \mathfrak{c}.$$

For obviously $$n^{\mathfrak{e}} \leq \mathfrak{e}^{\mathfrak{c}} \leq \mathfrak{c}^{\mathfrak{e}}.$$

But by 3), $\mathfrak{c}^{\mathfrak{e}} = \mathfrak{c}$ and by 1) $n^{\mathfrak{e}} = \mathfrak{c}$.

261. 1. *The cardinal number* $\mathfrak{t}$ *of all functions* f (*take on but two values in the domain of definition* $\mathfrak{A}$, *o ber* $\mathfrak{a}$, *is* $2^{\mathfrak{A}}$.

Moreover, $$2^{\mathfrak{A}} > \mathfrak{a}.$$

This follows at once from the reasoning of 249.

2. *Let* $\mathfrak{f}$ *be the cardinal number of the class of a fined over a domain* $\mathfrak{A}$ *whose cardinal number is* $\mathfrak{c}$.

$$\mathfrak{f} = \mathfrak{c}^{\mathfrak{c}} = 2^{\mathfrak{c}} > \mathfrak{c}.$$

For the class of functions which have but two va 1, $2^{\mathfrak{c}}$.

On the other hand, obviously

$$\mathfrak{f} = \mathfrak{c}^{\mathfrak{c}}.$$

But

$$\begin{aligned} \mathfrak{c}^{\mathfrak{c}} &= (2^{\mathfrak{e}})^{\mathfrak{c}}, && \text{by 260, 1)} \\ &= 2^{\mathfrak{e}\mathfrak{c}}, && \text{by 259, 1)} \\ &= 2^{\mathfrak{c}}, && \text{by 253, 2).} \end{aligned}$$

Thus, $$\mathfrak{c}^{\mathfrak{c}} = 2^{\mathfrak{c}}.$$

For let $\mathfrak{D}$ be a Cantor set in $\mathfrak{A}$ [I, 272]. Bein
limited function defined over $\mathfrak{D}$ is integrable. Bu
the points of $\mathfrak{A}$ may be set in uniform correspon
points of $\mathfrak{D}$.

4. *The set of all functions*

$$f(x) = f_1(x) + f_2(x) + \cdots$$

which are the sum of convergent series, and whose te
ous in $\mathfrak{A}$, has the cardinal number $\mathfrak{c}$.

For the set $\mathfrak{F}$ of continuous functions in $\mathfrak{A}$ ha
number $\mathfrak{c}$ by 248. These functions are to be dist
the enumerable set $\mathfrak{E}$ of terms in 2). Hence th
functions is

$$\mathfrak{F}^{\mathfrak{E}},$$

whose cardinal number is

$$\mathfrak{c}^{\mathfrak{e}} = \mathfrak{c}.$$

Remark. Not every integrable function can be
the series 2).

For the class of integrable functions has a cardi
by 250.

5. *The cardinal number of all enumerable sets in*
$\mathfrak{R}_m$ *is* $\mathfrak{c}$.

For it is obviously the cardinal number of the
$\mathfrak{R}_m$ over an enumerable set $\mathfrak{E}$, or

$$\text{Card } \mathfrak{R}_m^{\mathfrak{E}} = \mathfrak{c}^{\mathfrak{e}} = \mathfrak{c}.$$

Numbers of Liouville

262. In I, 200 we have defined algebraic numb

The first to actually show the existence of
bers was *Liouville*. He showed how to form
numbers. At present we have practical
whether a given number is algebraic or not.
signal achievements of *Hermite* to have shown
is transcendental.

Shortly after *Lindemann*, adapting Hermit
that $\pi = 3.14159 \cdots$ is also transcendental.
problem the *Quadrature of the Circle* was answ
The researches of Hermite and Lindemann e
an infinity of transcendental numbers. It is, h
pose to give an account of these famous resu
our considerations to certain numbers which
of Liouville.

In passing let us note that the existence of
bers follows at once from 235, 2 and 244, 2.

For the cardinal number of the set of real
c, and that of the set of all real numbers is c,

263. In algebra it is shown that any alge
root of an *irreducible* equation,

$$f(x) = a_0x^m + a_1x^{m-1} + \cdots + a_m$$

whose coefficients are integers without comm
the *order of α is m*.

We prove now the theorem

Let

$$r_n = \frac{p_n}{q_n} \quad , \quad p_n, q_n \textit{ relatively } p$$

$\doteq \alpha$, *an algebraic number of order m, as $n \doteq \infty$.*

for $n > s$, since the numerator of the middle member is an i
and hence ≥ 1.

On the other hand, by the Law of the Mean [I, 397],

$$f(r_n) - f(\alpha) = (r_n - \alpha) f'(\beta)$$

where β lies in $D_\delta(\alpha)$. Now $f(\alpha) = 0$ and $f'(\beta) <$ so
Hence

$$|r_n - \alpha| > \frac{f(r_n)}{M} \geq \frac{1}{Mq_n^m},$$

on using 3). But however large M is, there exists a ν, su
$q_n > M$, for any $n > \nu$. This in 4) gives 2).

264. **1.** *The numbers*

$$L = \frac{a_1}{10^{1!}} + \frac{a_2}{10^{2!}} + \frac{a_3}{10^{3!}} + \cdots$$

where $a_n < 10^n$, and not all of them vanish after a certain in
transcendental.

For if L is algebraic, let its order be m. Then if L_n
the sum of the first n terms of 1), there exists a ν such tha

$$\eta = |L - L_n| > \frac{1}{10^{(m+1)n!}}, \quad \text{for } n > \nu.$$

But

$$\eta = \frac{a_{n+1}}{10^{(n+1)!}} + \cdots < \frac{1}{10^{(m+1)n!}}, \quad n > \nu',$$

ν' being taken sufficiently large. But 3) contradicts 2).

The numbers 1) we call the *numbers of Liouville.*

2. *The set of Liouville numbers has the cardinal number* c

For all real numbers in the interval (0*, 1) can be repr
by

$$\beta = \frac{b_1}{10^1} + \frac{b_2}{10^2} + \frac{b_3}{10^3} + \cdots, \quad 0 \leq b_n \leq 9,$$

CHAPTER IX

ORDINAL NUMBERS

Ordered Sets

265. An aggregate $\mathfrak{A}$ is *ordered*, when $a,$
its elements, either a precedes b, or a succeeds
law; such that if a precedes b, and b precede
cede c. The fact that a precedes b may be i

$$a < b.$$

Then

$$a > b$$

states that a succeeds b.

Example 1. The aggregates

$$1, 2, 3, \cdots$$

$$2, 4, 6, \cdots$$

$$a_1, a_2, a_3, \cdots$$

$$\cdots -3, -2, -1, 0, 1, 2, 3,$$

$$\cdots a_{-3}, a_{-2}, a_{-1}, a_0, a_1, a_2, a$$

are ordered.

Example 2. The rational number system
an infinite variety of ways. For, being enu
arranged in a sequence

Example 4. The positive integers $\mathfrak{J}$ may be
nite variety of ways besides their natural order.
may write them in the order

$$1, 3, 5, \cdots 2, 4, 6, \cdots$$

so that the odd numbers precede the even. Or

$$1, 4, 7, 10, \cdots 2, 5, 8, 11, \cdots 3, 6, 9$$

and so on. We may go farther and arrange th
of sets. Thus in the first set put all primes;
the products of two primes; in the third se
three primes; etc., allowing repetitions of the
number in set m precede all the numbers in set
bers in each set may be arranged in order of ma

Example 5. The points of the plane $\mathfrak{R}_2$ may
infinite variety of ways. Let L_y denote the rig
the x-axis at a distance y from this axis, taking
of y. We order now the points of $\mathfrak{R}_2$ by sti
point on $L_{y'}$ precedes the points on any $L_{y''}$ w
points on any L_y shall have the order they alrea
line due to their position.

266. *Similar Sets.* Let $\mathfrak{A}$, $\mathfrak{B}$ be ordered and
$a \sim b$, $\alpha \sim \beta$. If when $a < \alpha$ in $\mathfrak{A}$, $b < \beta$ in $\mathfrak{B}$,
to $\mathfrak{B}$, and write

$$\mathfrak{A} \simeq \mathfrak{B}.$$

Thus the two ordered and equivalent aggr
when corresponding elements in the two sets
relative order.

In the correspondence $\mathfrak{A} \sim \mathfrak{B}$, let $a_r \sim r$ for $r =$
let $b_n \sim m + n$, $n = 1, 2 \cdots$ Then $\mathfrak{A} \simeq \mathfrak{B}$.

Example 3. Let

$$\mathfrak{A} = 1, 2, 3, \cdots$$

$$\mathfrak{B} = b_1, b_2 \cdots a_1, a_2 \cdots$$

Let the correspondence between $\mathfrak{A}$ and $\mathfrak{B}$ be
Ex. 2. Then $\mathfrak{A}$ is not similar to $\mathfrak{B}$. For 1 is the
$\mathfrak{A}$ while its associated element a_1 is not first in $\mathfrak{B}$.

Example 4. Let

$$\mathfrak{A} = 1, 2, 3, \cdots$$

$$\mathfrak{B} = a_1, a_2 \cdots b_1, b_2 \cdots$$

Let $a_n \sim 2n$, $b_n \sim 2n - 1$. Then $\mathfrak{A} \sim \mathfrak{B}$ but $\mathfrak{A}$

267. *Let* $\mathfrak{A} \simeq \mathfrak{B}$, $\mathfrak{B} \simeq \mathfrak{C}$. *Then* $\mathfrak{A} \simeq \mathfrak{C}$.

For let $a \sim b$, $a' \sim b'$ in $\mathfrak{A} \sim \mathfrak{B}$. Let $b \sim c$, $b' \sim c'$
us establish a correspondence $\mathfrak{A} \sim \mathfrak{C}$ by setting $a \sim$
if $a < a'$ in $\mathfrak{A}$, $c < c'$ in $\mathfrak{C}$. Hence $\mathfrak{A} \simeq \mathfrak{C}$.

Eutactic Sets

268. Let $\mathfrak{A}$ be any ordered aggregate, and $\mathfrak{B}$ a
elements of $\mathfrak{B}$ being kept in the same relative ord
and each $\mathfrak{B}$ both have a first element, we say that $\mathfrak{A}$
or *eutactic*.

Example 1. $\mathfrak{A} = 2, 3, \cdots 500$ is well ordered. F
element 2. Moreover any part of $\mathfrak{A}$ as 6, 15, 25,
first element.

Example 3. Let $\mathfrak{A}$ = rational n
arranged in their order of magnit
also has a first element, viz. 0. It
For the partial set $\mathfrak{B}$ consisting of
$\mathfrak{A}$ has no first element.

Example 4. An ordered set whic
times be made so by ordering its
law.

Thus in Ex. 3, let us arrange
Obviously $\mathfrak{A}$ is now well ordered.

Example 5. $\mathfrak{A} = a_1, a_2 \cdots b_1, b_2 \cdots$
first element of $\mathfrak{A}$; and any part of

$$a_1, \ a_2 \cdots$$

$$b_1, \ b_2 \cdots$$

$$a_{\iota_1}, \ a_{\iota_2} \cdots b_{\kappa}$$

has a first element.

269. 1. *Every partial set $\mathfrak{B}$ of a*
ordered.

For $\mathfrak{B}$ has a first element, since
ordered. If $\mathfrak{C}$ is a part of $\mathfrak{B}$, it is
a first element.

2. *If a is not the last element of*
is an element of $\mathfrak{A}$ immediately follo

For let $\mathfrak{B}$ be the part of $\mathfrak{A}$ form
has a first element b since $\mathfrak{A}$ is wel

For example, let

$$\mathfrak{A} = a_1 a_2 \cdots b_1 b_2 \cdots$$

Then
$$a_n + 1 = a_{n+1} \quad , \quad b_m + 1 = b_{m+1}$$
$$a_n - 1 = a_{n-1} \quad , \quad b_m - 1 = b_{m-1}.$$

There is, however, no $b_1 - 1$.

270. 1. *If $\mathfrak{A}$ is well ordered, it is impossible to pick out an te sequence of the type*

$$a_1 > a_2 > a_3 > \cdots$$

For
$$\mathfrak{B} = \cdots a_3, a_2, a_1$$

part of $\mathfrak{A}$ whose elements occur in the same relative orde $\mathfrak{A}$, and $\mathfrak{B}$ has no first element.

2. A sequence as 1) may be called a *decreasing sequence*, w

$$a_1 < a_2 < a_3 \cdots$$

y be called *increasing*.

In every infinite well ordered aggregate there exist increas uences.

3. *Let $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \cdots$ be a well ordered set. Let $\mathfrak{A} = \{a\}$ be ered in the a's, $\mathfrak{B} = \{b\}$ be well ordered in the b's, etc. The*

$$\mathfrak{U} = \mathfrak{A}, \mathfrak{B}, \mathfrak{C} \cdots$$

vell ordered with regard to the little letters $a, b \cdots$

For $\mathfrak{U}$ has a first element in the little letters, viz. the first nt of $\mathfrak{A}$. Moreover, any part of $\mathfrak{U}$, as $\mathfrak{B}$, has a first elemen little letters. For if it has not, there exists in $\mathfrak{B}$ an infi reasing sequence

$$t > s > r > \cdots$$

Let each $\mathfrak{A}$, $\mathfrak{B}$, $\cdots$ *be well ordered.*

Let
$$\mathfrak{B} = \mathfrak{A} + B, \qquad \mathfrak{C} = \mathfrak{B} +$$

Then
$$\mathfrak{S} = \mathfrak{A} + B + C + \cdots$$

is a well ordered set, $\mathfrak{S}$ *preserving the relati*
intact.

For $\mathfrak{S}$ has a first element, viz. the first e
part S of $\mathfrak{S}$ has a first element. For, if no
an infinite decreasing sequence

$$r > q > p > \cdots$$

Now r lies in some set of 1) as $\mathfrak{R}$. Henc
$\mathfrak{R}$. But in $\mathfrak{R}$ there is no sequence as 2).

5. Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\cdots$ be an ordered set of
gates, no two of which have an element in co
must guard against assuming that $\mathfrak{A} + \mathfrak{B} + ($
relative order intact, is necessarily well order
For let us modify Ex. 5 in 265 by taking
points on each L_y only a well ordered set whi
Then the sum
$$\mathfrak{A} = \Sigma\mathfrak{A}_y$$

has a definite meaning. The elements of $\mathfrak{A}$ w
as in Ex. 5 of 265.
Obviously $\mathfrak{A}$ is not well ordered.

Sections

271. We now introduce a notion which in

Example 1. Let
$$\mathfrak{A} = 1, 2, 3, \cdots$$
Then
$$S100 = 1, 2, \cdots 99$$
is the section of $\mathfrak{A}$ generated by the element 100.

Example 2. Let
$$\mathfrak{A} = a_1, a_2 \cdots b_1, b_2 \cdots$$
Then
$$Sb_5 = a_1 a_2 \cdots b_1 b_2 b_3 b_4$$
is the section generated by b_5.
$$Sb_1 = a_1 a_2 \cdots$$
that generated by b_1, etc.

272. 1. *Every section of a well ordered aggregate*
For each section of $\mathfrak{A}$ is a partial aggregate o
well ordered by 269, 1.

2. *In the well ordered set $\mathfrak{A}$, let $a < b$. Then*
of Sb.

3. *Let $\mathfrak{S}$ denote the aggregate of sections of a*
ordered set $\mathfrak{A}$. If we order $\mathfrak{S}$ such that $Sa < Sb$ in
$\mathfrak{A}$, $\mathfrak{S}$ is well ordered.

For the correspondence between $\mathfrak{A}$ and $\mathfrak{S}$ is unifo

273. *Let $\mathfrak{A}$, $\mathfrak{B}$ be well ordered and $\mathfrak{A} \simeq \mathfrak{B}$.*
$Sa \simeq Sb$.

For in $\mathfrak{A}$ let $a'' < a' > a$. Let $b' \sim a'$ and b'
$\mathfrak{A} \simeq \mathfrak{B}$, we have
$$b'' < b' < b;$$
hence the theorem.

275. *Let $\mathfrak{A}, \mathfrak{B}$ be well ordered and $\mathfrak{A} \simeq \mathfrak{B}$. Then to Sa in $\mathfrak{A}$ c… correspond two sections Sb, $S\beta$ each $\simeq Sa$.*

For let $b < \beta$, and $Sa \simeq Sb$, $Sa \simeq S\beta$. Then

$$Sb \simeq S\beta, \qquad \text{by } 267.$$

But 1) contradicts 274.

276. *Let $\mathfrak{A}, \mathfrak{B}$ be two well ordered aggregates. It is imposs… establish a uniform and similar correspondence between $\mathfrak{A}$ and… more than one way.*

For say $Sa \simeq Sb$ in one correspondence, and $Sa \simeq S\beta$ in… er, b, β being different elements of $\mathfrak{B}$. Then

$$Sb \simeq S\beta, \qquad \text{by } 267.$$

This contradicts 275.

277. 1. We can now prove the following theorem, whic… converse of 273.

Let $\mathfrak{A}, \mathfrak{B}$ be well ordered. If to each section of $\mathfrak{A}$ corresponds … ilar section of $\mathfrak{B}$, and conversely, then $\mathfrak{B} \simeq \mathfrak{A}$.

Let us first show that $\mathfrak{A} \sim \mathfrak{B}$. Since to any Sa of $\mathfrak{A}$ co… nds a similar section Sb in $\mathfrak{B}$, let us set $a \sim b$. No ot… $\sim b$, and no other $b' \sim a$, as then $Sa' \simeq Sb$ or $Sb' \simeq Sa$, wh… tradicts 274. Let the first element of $\mathfrak{A}$ correspond to … t of $\mathfrak{B}$. Thus the correspondence we have set up betwee… $\mathfrak{B}$ is uniform and $\mathfrak{A} \sim \mathfrak{B}$.

We show now that this correspondence is *similar*. For let

$$a \sim b \text{ and } a' \sim b', \qquad a' < a.$$

…n $b' < b$. For a' lies in $Sa \simeq Sb$ and $b' \sim a'$ lies in Sb.

Let us begin by ordering the sections of $\mathfrak{A}$ and $\mathfrak{B}$
Let B denote the aggregate of sections of $\mathfrak{B}$ to whic
tions of $\mathfrak{A}$ do not correspond. Then B is well order
first section, say Sb. Let $\beta < b$. Then to $S\beta$ in $\mathfrak{B}$
by hypothesis a similar section $S\alpha$ in $\mathfrak{A}$. On the c
any section Sa' of $\mathfrak{A}$ corresponds a similar section S
viously $b' < b$. Thus to any section of $\mathfrak{A}$ correspo
section of Sb and conversely. Hence $\mathfrak{A} \simeq Sb$ by 277

279. *Let $\mathfrak{A}$, $\mathfrak{B}$ be well ordered. Either $\mathfrak{A}$ is simila*
is similar to a section of the other.

For either:

1° To each section of $\mathfrak{A}$ corresponds a similar
and conversely;

or 2° To each section of one corresponds a simi
the other but not conversely;

or 3° There is at least one section in both $\mathfrak{A}$ and $\mathfrak{B}$
similar section corresponds in the other.

If 1° holds, $\mathfrak{A} \simeq \mathfrak{B}$ by 277, 1. If 2° holds, either $\mathfrak{A}$
to a section of the other.

We conclude by showing 3° is impossible.

For let A be the set of sections of $\mathfrak{A}$ to which no s
in $\mathfrak{B}$ corresponds. Let B have the same meaning f
suppose $\mathfrak{A}$, $\mathfrak{B}$ ordered as in 272, 3, A will have a fir
$S\alpha$, and B a first section $S\beta$.

Let $a < \alpha$. Then to Sa in $\mathfrak{A}$ corresponds by hyp
tion Sb of $S\beta$ as in 278. Similarly if $b' < \beta$, to Sb
sponds a section Sa' of $S\alpha$. But then $S\alpha \simeq S\beta$ by 2
contradicts the hypothesis.

3° If a section of $\mathfrak{A}$ is $\simeq \mathfrak{B}$, the ordinal number of $\mathfrak{A}$ is *gre*
than that of $\mathfrak{B}$.

The ordinal number of $\mathfrak{A}$ may be denoted by

$$\text{Ord } \mathfrak{A},$$

when no ambiguity can arise, by the corresponding small lett
any two well ordered aggregates $\mathfrak{A}$, $\mathfrak{B}$ fall under one and
of the three preceding cases, any two ordinal numbers
sfy one of the three following relations, and only one, viz.

$$\mathfrak{a} = \mathfrak{b} \quad , \quad \mathfrak{a} < \mathfrak{b} \quad , \quad \mathfrak{a} > \mathfrak{b},$$

if $\mathfrak{a} < \mathfrak{b}$, it follows that $\mathfrak{b} > \mathfrak{a}$.

Obviously they enjoy also the following properties.

. *If* $\qquad \mathfrak{a} = \mathfrak{b} \quad , \quad \mathfrak{b} = \mathfrak{c} \quad , \quad$ *then* $\mathfrak{a} = \mathfrak{c}$.

For if $\mathfrak{c} = \text{Ord } \mathfrak{C}$, the first two relations state that

$$\mathfrak{A} \simeq \mathfrak{B} \quad , \quad \mathfrak{B} \simeq \mathfrak{C}.$$

But then $\qquad \mathfrak{A} \simeq \mathfrak{C} \quad , \quad$ by 267.

Hence

$$\mathfrak{a} = \mathfrak{c}.$$

. *If* $\qquad \mathfrak{a} > \mathfrak{b} \quad , \quad \mathfrak{b} > \mathfrak{c} \quad , \quad$ *then* $\mathfrak{a} > \mathfrak{c}$.

81. 1. Let $\mathfrak{A}$ be a finite aggregate, embracing say n elemen
en we set

$$\text{Ord } \mathfrak{A} = n.$$

us the ordinal number of a finite aggregate has exactly sim
perties to those of finite cardinal numbers. The ordinal n
of a finite aggregate is called *finite*, otherwise *transfinite*.

. The cardinal number of a set $\mathfrak{A}$ is independent of the o
which the elements of $\mathfrak{A}$ occur. This is not so in general
inal numbers.

'or example, let
$$\mathfrak{A} = 1, 2, 3, \cdots$$
$$\mathfrak{B} = 1, 3, 5, \cdots 2, 4, 6, \cdots$$
Iere
$$\text{Card } \mathfrak{A} = \text{Card } \mathfrak{B} = \aleph_0.$$
But
$$\text{Ord } \mathfrak{A} < \text{Ord } \mathfrak{B},$$
ce $\mathfrak{A}$ is similar to a section of $\mathfrak{B}$, viz. the set of odd numb
, 5, ...

82. 1. *Addition of Ordinals.* Let $\mathfrak{A}$, $\mathfrak{B}$ be well ordered
hout common elements. Let $\mathfrak{C}$ be the aggregate formed
cing the elements of $\mathfrak{B}$ after those of $\mathfrak{A}$, leaving the order i
erwise unchanged. Then the ordinal number of $\mathfrak{C}$ is called
of the ordinal numbers of $\mathfrak{A}$ and $\mathfrak{B}$, or
$$\text{Ord } \mathfrak{C} = \text{Ord } \mathfrak{A} + \text{Ord } \mathfrak{B},$$
$$\mathfrak{c} = \mathfrak{a} + \mathfrak{b}.$$

'he extension of this definition to any set of well-ordered ag
es such that the result is well ordered is obvious.

. *We note that*
$$\mathfrak{a} + \mathfrak{b} > \mathfrak{a}, \qquad \mathfrak{a} + \mathfrak{b} \geq \mathfrak{b}.$$
$\mathfrak{A}$ is similar to a section of $\mathfrak{C}$, and $\mathfrak{B}$ is equivalent to a
$\mathfrak{C}$.

. *The addition of ordinal numbers is associative.*

'his is an immediate consequence of the definition of additi

3ut $\mathfrak{A} \simeq$ a section of $\mathfrak{C}$, viz.: $\simeq Sb_1$, while $\mathfrak{D} \simeq \mathfrak{A}$. Hence

$$\omega < \mathfrak{c} \quad , \quad \omega = \mathfrak{d},$$

$$\omega + n > \omega \quad , \quad n + \omega = \omega.$$

. *If* $\mathfrak{a} > \mathfrak{b}$, *then* $\mathfrak{c} + \mathfrak{a} > \mathfrak{c} + \mathfrak{b}$, *and* $\mathfrak{a} + \mathfrak{c} \geq \mathfrak{b} + \mathfrak{c}$.

'or let
$$\mathfrak{a} = \text{Ord } \mathfrak{A}, \quad \mathfrak{b} = \text{Ord } \mathfrak{B}, \quad \mathfrak{c} = \text{Ord } \mathfrak{C}.$$

ce $\mathfrak{a} > \mathfrak{b}$, we can take for $\mathfrak{B}$ a section Sb of $\mathfrak{A}$. Then $\mathfrak{c} +$
ordinal number of
$$\mathfrak{C} + \mathfrak{A},$$

$\mathfrak{c} + \mathfrak{b}$ is the ordinal number of

$$\mathfrak{C} + Sb,$$

serving the relative order of the elements.

3ut 2) is a section of 1), and hence $\mathfrak{c} + \mathfrak{a} > \mathfrak{c} + \mathfrak{b}$.

'he proof of the rest of the theorem is obvious.

83. 1. *The ordinal number immediately following* $\mathfrak{a}$ *is* $\mathfrak{a} + 1$

'or let $\mathfrak{a} = \text{Ord } \mathfrak{A}$. Let $\mathfrak{B}$ be a set formed by adding afte
elements of $\mathfrak{A}$ another element b. Then

$$\mathfrak{a} + 1 = \text{Ord } \mathfrak{B} = \mathfrak{b}.$$

uppose now
$$\mathfrak{a} < \mathfrak{c} < \mathfrak{b} \quad , \quad \mathfrak{c} = \text{Ord } \mathfrak{C}.$$

'hen $\mathfrak{C}$ is similar to a section of $\mathfrak{B}$. But the greatest sec
$\mathfrak{B}$ is $Sb = \mathfrak{A}$. Hence

$$\mathfrak{c} \leq \mathfrak{a},$$

ch contradicts 1).

. *Let* $\mathfrak{a} > \mathfrak{b}$. *Then there is one and only one ordinal numb*
h that
$$\mathfrak{a} = \mathfrak{b} + \mathfrak{d}.$$

'or let
$$\mathfrak{a} = \text{Ord } \mathfrak{A} \quad , \quad \mathfrak{b} = \text{Ord } \mathfrak{B}.$$

284. 1. *Multiplication of Ordinals.* Let $\mathfrak{A}$, $\mathfrak{B}$ be aggregates having $\mathfrak{a}$, $\mathfrak{b}$ as ordinal numbers. Let us element of $\mathfrak{A}$ by an aggregate $\simeq \mathfrak{B}$. The resulting we denote by

$$\mathfrak{B} \cdot \mathfrak{A}.$$

As $\mathfrak{C}$ is a well-ordered set by 270, 3 it has an ordi
We define now the *product* $\mathfrak{b} \cdot \mathfrak{a}$ to be $\mathfrak{c}$, and write

$$\mathfrak{b} \cdot \mathfrak{a} = \mathfrak{c}.$$

We say $\mathfrak{c}$ is the result of *multiplying* $\mathfrak{a}$ *by* $\mathfrak{b}$, and ca
We write

$$\mathfrak{a} \cdot \mathfrak{a} = \mathfrak{a}^2 \quad , \quad \mathfrak{a} \cdot \mathfrak{a} \cdot \mathfrak{a} = \mathfrak{a}^3 \quad , \quad \text{etc.}$$

2. *Multiplication is associative.*

This is an immediate consequence of the definitio

3. *Multiplication is not always commutative.*

For example, let

$$\mathfrak{A} = (a_1 a_2),$$

$$\mathfrak{B} = (1, 2, 3 \cdots \text{in inf.}).$$

Then

$$\mathfrak{B} \cdot \mathfrak{A} = (b_1 b_2 b_3 \cdots, \quad c_1 c_2 c_3 \cdots).$$

$$\mathfrak{A} \cdot \mathfrak{B} = (b_1, c_1, \quad b_2, c_2, \quad b_3, c_3$$

Hence

$$\text{Ord}\,(\mathfrak{B} \cdot \mathfrak{A}) = \omega \cdot 2 > \omega,$$

$$\text{Ord}\,(\mathfrak{A} \cdot \mathfrak{B}) = 2\,\omega = \omega.$$

4. *If* $\mathfrak{a} < \mathfrak{b}$, *then* $\mathfrak{c}\mathfrak{a} < \mathfrak{c}\mathfrak{b}$.

For $\mathfrak{C} \cdot \mathfrak{A}$ is a section of $\mathfrak{C} \cdot \mathfrak{B}$.

Since $\alpha_{n-1} < \alpha_n$, $\mathfrak{A}_{n-1}$ is similar to a section of $\mathfrak{A}_n$. For simpl
may take $\mathfrak{A}_{n-1}$ to be a section of $\mathfrak{A}_n$. Let, therefore,

$$\mathfrak{A}_n = \mathfrak{A}_{n-1} + \mathfrak{B}_n.$$

Consider now
$$\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{B}_2 + \mathfrak{B}_3 + \cdots$$

ping the relative order of the elements intact. Then $\mathfrak{A}$ is
ered and has an ordinal number α.

As any $\mathfrak{A}_n$ is a section of $\mathfrak{A}$,

$$\alpha_n < \alpha.$$

Moreover any number $\beta < \alpha$ is also $<$ some α_m. For if $\mathfrak{B}$
ordinal number β, $\mathfrak{B}$ must be similar to a section of $\mathfrak{A}$.
re is no last section of $\mathfrak{A}$.

. The number α we have just determined is called the *lim*
sequence 1). We write

$$\alpha = \lim \alpha_n \quad , \quad \text{or } \alpha_n \doteq \alpha.$$

We also say that α *corresponds to the sequence* 1).

All numbers corresponding to infinite enumerable increa
uences of ordinal numbers are called *limitary*.

. *If every* α_n *in* 1) *is* $< \beta$, *then* $\alpha \leq \beta$.

For if $\beta < \alpha$, α is not the least ordinal number greater t
ry α_n.

. *If* $\beta < \alpha$, β *is* $<$ *some* α_n.

86. *In order that*

$$\alpha_1 < \alpha_2 < \cdots$$

$$\beta_1 < \beta_2 < \cdots$$

287. *Cantor's Principles of Generating Ordinals.* two methods of generating ordinal numbers. First to any ordinal number α. In this way we get

$$\alpha,\ \alpha+1,\ \alpha+2,\ \cdots$$

Secondly, by taking the limit of an infinite enume ing sequence of ordinal numbers, as

$$\alpha_1 < \alpha_2 < \alpha_3 < \cdots$$

Cantor calls these two methods the *first* and *seco* of generating ordinal numbers.

Starting with the ordinal number 1, we get by suc cations of the first principle the numbers

$$1,\ 2,\ 3,\ 4,\ \cdots$$

The limit of this sequence is ω by 285, 1. Using ciple alone, this number would not be attained; to g the application of the second principle. Making u principle again, we obtain

$$\omega+1, \qquad \omega+2, \qquad \omega+3,\ \cdots$$

The second principle gives now the limitary numb by 285, 1. From this we get, using the first princi

$$\omega 2+1, \qquad \omega 2+2, \qquad \omega 2+3,\ \cdots$$

whose limit is $\omega 3$. In this way we may obtain the

$$\omega m+n \quad , \quad m,\ n \text{ finite.}$$

The limit of any increasing sequence of these nun

$$\omega \quad , \quad \omega 2 \quad , \quad \omega 3 \quad , \quad \omega 4,\ \cdots$$

But here the process does not end. For th

$$\omega < \omega^2 < \omega^3 < \cdots$$

has a limit which we denote by ω^ω.

Continuing we obtain

$$\omega^{\omega^\omega}, \quad \omega^{\omega^{\omega^\omega}}, \quad \text{etc.}$$

288. It is interesting to see how we may sets of points whose ordinal numbers are th sidered.

In the unit interval $\mathfrak{A} = (0, 1)$, let us take

$$\tfrac{1}{2} \;,\; \tfrac{3}{4} \;,\; \tfrac{7}{8} \;,\; \tfrac{15}{16} \cdots$$

These form a well ordered set whose ordinal n

The points 1) divided $\mathfrak{A}$ into a set of interv

$$\mathfrak{A}_1 \;,\; \mathfrak{A}_2 \;,\; \mathfrak{A}_3 \cdots$$

In m of these intervals, let us take a set gives us a set whose ordinal number is ωm.

In each interval 2), let us take a set simila us a set whose ordinal number is ω^2. The divide $\mathfrak{A}$ into a set of ω^2 intervals. In each let us take a set of points similar to 1). points whose ordinal number is ω^3, etc.

Let us now put in $\mathfrak{A}_1$ a set of points $\mathfrak{B}_1$ wh is ω. In $\mathfrak{A}_2$ let us put a set $\mathfrak{B}_2$ whose ordina so on, for the other intervals of 2).

We thus get in $\mathfrak{A}$ the well ordered set

$$\mathfrak{B} = \mathfrak{B}_1 + \mathfrak{B}_2 + \mathfrak{B}_3 + \cdots$$

Classes of Ordinals

289. Cantor has divided the ordinal numbers
Class 1, denoted by Z_1, embraces all finite ord
Class 2, denoted by Z_2, embraces all transfinit
corresponding to well ordered enumerable sets
whose cardinal number is $\aleph_0$. For this reason v

$$Z_2 = Z(\aleph_0).$$

It will be shown in 293, 1 that Z_2 is not enume
if we set

$$\aleph_1 = \text{Card } Z_2,$$

there is no cardinal number between $\aleph_0$ and $\aleph_1$ a
294. We are thus justified in saying that Cl
Z_3 or $Z(\aleph_1)$, embraces all ordinal numbers corr
ordered sets whose cardinal number is $\aleph_1$, etc.

Let $\beta = \text{Ord } \mathfrak{B}$ be any ordinal number. The
$\alpha < \beta$ correspond to sections of $\mathfrak{B}$. These sec
ordered set by 272, 3. Therefore if we arra
$\alpha < \beta$ in an order such that α' precedes α when
well ordered. We shall call this the natural
first number in Z_1 is 1, the first number of Z
number in Z_3 is denoted by Ω.

290. As the numbers in Class 1 are the posit
need no comment here. Let us therefore turn t

If α is in Z_2, so is $\alpha + 1$.

For let $\alpha = \text{Ord } \mathfrak{A}$. Let $\mathfrak{B}$ be the well ord
by placing an element b after all the elements of

r using the notation employed in the proof of 285, 1, α is
al number of

$$\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{B}_1 + \mathfrak{B}_2 + \cdots$$

t $\mathfrak{A}_1, \mathfrak{B}_1, \mathfrak{B}_2 \cdots$ are each enumerable.
ence $\mathfrak{A}$ is enumerable by 235, 1, and α lies in Z_2.

2. We prove now the converse of 290 and 291.

ach number α in Z_2, except ω, is obtained by adding 1 *to s*
ber in Z_2; or it is the limit of an infinite enumerable increa
numbers in Z_2.

r, let $\alpha = \text{Ord}\,\mathfrak{A}$. Suppose *first*, that $\mathfrak{A}$ has a last elem
. Since $\mathfrak{A}$ is enumerable, so is Sa. Hence

$$\beta = \text{Ord} \cdot Sa$$

Z_2. Then

$$\alpha = \beta + 1.$$

ppose *secondly*, that $\mathfrak{A}$ has no last element. All the num
in Z_2 belong to sections of $\mathfrak{A}$. Since $\mathfrak{A}$ is enumerable,
bers β are enumerable. Let them be arranged in a seque

$$\beta_1,\ \beta_2,\ \beta_3 \cdots$$

nce they have no greatest, let β_1' be the first number i
, let β_2' be the first number in it $> \beta_1'$, etc. We get thus
ence

$$\beta_1 < \beta_1' < \beta_2' < \cdots$$

e limit is λ, say.

en $\lambda = \alpha$. For λ is $>$ any number in 1), which embraces
numbers of $Z_2 < \alpha$. Moreover it is the least number wh
ys this property.

3. 1. *The numbers of Z_2 are not enumerable.*

Let $\alpha'_n \doteq \alpha'$. Then α' lies in Z_2 by 291. On
285, α' is $>$ any number in 2), and therefore
1). But 1) embraces all the numbers of Z_2, b
are thus led to a contradiction.

2. We set

$$\aleph_1 = \text{Card } Z_2.$$

294. *There is no cardinal number between* $\aleph_0$ *a*

For let $\alpha = \text{Card } \mathfrak{A}$ be such a number. Then
partial aggregate of Z_2, which without loss of
taken to be a section of Z_2. But every such
able. Hence $\mathfrak{A}$ is enumerable and $\alpha = \aleph_0$, which

295. We have just seen that the numbers in
able. Let us order them so that each numbe
succeeding number. We shall call this the *nat*

1. *The numbers of* Z_2 *when arranged in their*
a well ordered set.

For Z_2 has a first element ω. Moreover any
relative order being preserved, has a first elem
not, there exists an infinite enumerable decreas

$$\alpha > \beta > \gamma > \cdots$$

This, however, is not possible. For $\beta, \gamma, \cdots$
which is well ordered.

There is thus one well ordered set having $\aleph_1$
ber. Let

$$\Omega = \text{Ord } Z_2.$$

Let now $\mathfrak{A}$ be an enumerable well ordered
number is α. The set

296. 1. *An aggregate formed of an $\aleph_1$ set of $\aleph_1$ se*
Consider the set

$$A = \begin{array}{lllllllll} a_{11}, & a_{12}, & a_{13} & \cdots & a_{1\omega} & \cdots & a_{1\alpha} & \cdots \\ a_{21}, & a_{22}, & a_{23} & \cdots & a_{2\omega} & \cdots & a_{2\alpha} & \cdots \\ & & & & & & & \\ a_{\omega 1}, & a_{\omega 2}, & a_{\omega 3} & \cdots & a_{\omega\omega} & \cdots & a_{\omega\alpha} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{\alpha 1}, & a_{\alpha 2}, & a_{\alpha 3} & \cdots & a_{\alpha\omega} & \cdots & a_{\alpha\alpha} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

Here each row is an $\aleph_1$ set. As there are an $\aleph_1$
is an $\aleph_1$ set of $\aleph_1$ sets. To show that A is an $\aleph_1$ se
each $a_{\iota\kappa}$ with some number in the first two number
In the first place the elements $a_{\iota\kappa}$ where ι, $\kappa < \omega$
ated with the numbers 1, 2, 3, $\cdots < \omega$. The el
lying just inside the ω^{th} square and which ar
by the condition that $\iota = 1, 2, \cdots \omega$; $\kappa = 1, 2 \cdots$
enumerable set and may therefore be associated wi
ω, $\omega + 1$, $\cdots < \omega 2$. For the same reason the elem
the $\omega + 1^{\text{st}}$ square may be associated with the ordi
$\cdots < \omega 3$. In this way we may continue. For
arrived at the α^{th} row and column (edge of the
have only used up an enumerable set of numbers i

$$1, 2, \cdots \omega \cdots < \Omega$$

in our process of association. There are thus stil
in 1) to continue the process of association.

2. As a corollary of 1 we have:

The ordinal numbers

$$\Omega^2, \qquad \Omega^3, \qquad \Omega^4, \cdots$$

lie in Z_2

Since $\alpha < \beta$ we may take $\mathfrak{A}$ to be a section
we may suppose $\mathfrak{B}$ is a section of $\mathfrak{C}$, etc.

Let now

$$\mathfrak{B} = \mathfrak{A} + B, \qquad \mathfrak{C} = \mathfrak{B} + C,$$

Consider now

$$\mathfrak{L} = \mathfrak{A} + B + C + \cdots$$

keeping the relative order intact. Then $\mathfrak{L}$ is
270, 4. Let

$$\lambda = \text{Ord}\, \mathfrak{L}.$$

Since Card $\mathfrak{L} = \aleph_1$, by 296, 1, λ lies in Z_3.

As any $\mathfrak{A}, \mathfrak{B}, \cdots$ is a section of $\mathfrak{L}$,

$$\alpha < \beta < \cdots < \lambda.$$

Moreover, any number $\mu < \lambda$ is also $<$ some
$\mathfrak{M}$ has ordinal number μ, $\mathfrak{M}$ must be similar
But there is no last section in $\mathfrak{L}$.

2. We shall call sequences of the type 1)
The number λ whose existence we have just est
call the *limit of* 1). We shall also write

$$\alpha < \beta < \gamma \cdots \doteq \lambda$$

to indicate that $\alpha, \beta, \cdots$ is an $\aleph_1$ sequence whose

298. 1. The preceding theorem gives us a
generating ordinal numbers. We call it the *th*

We have seen that the first and second princi
erate the numbers of the first two classes of ord
do not suffice to generate even the first number,
prove now the following fundamental theorem :

2. *The three principles already described are*

where β lies in Z_2. In the latter case, reasoni
shows that we can pick out an $\aleph_1$ increasing se

$$\beta_1 < \beta_2' < \beta_3' \cdots < \alpha.$$

299. 1. *The numbers of Z_3 form a set whos* *is $> \aleph_1$.*

The proof is entirely similar to 293, 1. Su $\alpha = \aleph_1$. Let us arrange the elements of Z_3 in

$$\alpha_1 < \alpha_2 \cdots \alpha_\omega \cdots \alpha_\Omega \cdots$$

As in 292, there exists in this sequence an $\aleph_1$ i

$$\alpha_1' < \alpha_2' < \cdots < \alpha'.$$

Then α' lies in Z_3 by 297, 1. On the other han any number in 2) and hence greater than a But 1) embraces all the numbers in Z_3 by hy thus led to a contradiction.

2. We set

$$\aleph_2 = \text{Card } Z_3.$$

3. *There is no cardinal number between $\aleph_1$ an*

For let $\alpha = \text{Card } \mathfrak{A}$ be such a number. Then a section of Z_3. But every such section has ber $\aleph_1$.

300. The reasoning of the preceding para once generalized. The ordinal numbers of Z well ordered sets of cardinal number $\aleph_{n-2}$ form having a greater cardinal number α than $\aleph_{n-2}$. no cardinal lying between $\aleph_{n-2}$ and α. We propriately denote α by $\aleph$. The $\aleph$ se

CHAPTER X

POINT SETS

Pantaxis

301. 1. (*Borel.*) *Let each point of the limited*
$\mathfrak{A}$ *lie at the center of a cube* $\mathfrak{C}$. *Then there exists a*
of non-overlapping cubes $\{\mathfrak{c}\}$ *such that each* $\mathfrak{c}$ *lies wi*
each point of $\mathfrak{A}$ *lies in some* $\mathfrak{c}$. *If* $\mathfrak{A}$ *is limited and*
is a finite set $\{\mathfrak{c}\}$ *having this property.*

For let $D_1, D_2 \cdots$ be a sequence of superposed c
of norms $\doteq 0$. Any cell of D_1 which lies withi
which contains a point of $\mathfrak{A}$ we call a black cell;
of D we call white. The black cells are not furt
The division D_2 divides each white cell. Any of t
cells which lies within some $\mathfrak{C}$ and contains a poin
black cell, the others are white. Continuing we
able set of non-overlapping cubical cells $\{\mathfrak{c}\}$.

Each point a of $\mathfrak{A}$ lies within some $\mathfrak{c}$. For a
some $\mathfrak{C}$. But when n is taken sufficiently large, a
D_n, which cell lies within $\mathfrak{C}$.

Let now $\mathfrak{A}$ *be limited and complete.* Each a lies
or on the faces of a finite number of these $\mathfrak{c}$. W
ate the diagonal δ of the smallest of these cu

cells of some D_n, n sufficiently large, which surr
cube c, lying within $\mathfrak{C}$. Thus the points of $\mathfrak{A}$
enumerable set of cells $\{c\}$, each c lying within
cells c of course will in general overlap. Obviou
complete, the points of $\mathfrak{A}$ will lie within a fin
these c's.

302. *If $\mathfrak{A}$ is dense, $\mathfrak{A}'$ is perfect.*

For, in the first place, $\mathfrak{A}'$ is dense. In fact, let
$\mathfrak{A}'$. Then in any $D^*(\alpha)$ there are points of $\mathfrak{A}$.
point. Since $\mathfrak{A}$ is dense, it is a limiting point of $\mathfrak{A}$
point of $\mathfrak{A}'$. Thus in any $D^*(\alpha)$ there are points
Secondly, $\mathfrak{A}'$ is complete, by I, 266.

303. *Let $\mathfrak{B}$ be a complete partial set of the perf*
Then $\mathfrak{C} = \mathfrak{A} - \mathfrak{B}$ is dense.

For if $\mathfrak{C}$ contains the isolated point c, all the points
lie in $\mathfrak{B}$, if r is taken sufficiently small. But
plete, c must then lie in $\mathfrak{B}$.

Remark. We take this occasion to note that a fi
regarded as complete.

304. 1. *If $\mathfrak{A}$ does not embrace all $\mathfrak{R}_n$, it has at l*
point in $\mathfrak{R}_n$.

For let a be a point of $\mathfrak{A}$, and b a point of $\mathfrak{R}_n$
points on the join of a, b have coördinates

$$x_\iota = a_\iota + \theta(b_\iota - a_\iota) = x_\iota(\theta), \quad 0 \leq \theta \leq 1, \quad \iota = 1,$$

Let θ' be the maximum of those θ's such that x
$\mathfrak{A}$ if $\theta < \theta'$. Then $x(\theta')$ is a frontier point of $\mathfrak{A}$.

Example 1. Let $\mathfrak{A}$ be the unit interval (
tional points in $\mathfrak{A}$. Then $\mathfrak{B}$ is pantactic in $\mathfrak{A}$.

Example 2. Let $\mathfrak{A}$ be the interval $(0, 1)$, a
of I, 272. Then $\mathfrak{B}$ is apantactic in $\mathfrak{A}$.

2. *If* $\mathfrak{B} < \mathfrak{A}$ is *pantactic in* $\mathfrak{A}$, $\mathfrak{A}$ *contains*
in $\mathfrak{B}$.

For let a be a point of $\mathfrak{A}$ not in $\mathfrak{B}$. Then
$D_\delta(a)$ there is a point of $\mathfrak{B}$. Hence there are
of $\mathfrak{B}$ in this domain. Hence a is a limiting p

306. Let $\mathfrak{A}$ be complete. We say $\mathfrak{B} \leq \mathfrak{A}$
in $\mathfrak{A}$, if $\mathfrak{B}$ is the union of an enumerable s
in $\mathfrak{A}$.

If $\mathfrak{B}$ is not of the 1° category, we say it is
Sets of the 1° category may be called *Baire*

Example. Let $\mathfrak{A}$ be the unit interval,
points in it. Then $\mathfrak{B}$ is of the 1° category.

For $\mathfrak{B}$ being enumerable, let $\mathfrak{B} = \{b_n\}$. Bu
point and is thus apantactic in $\mathfrak{A}$.

The same reasoning shows that if $\mathfrak{B}$ is an
$\mathfrak{A}$, then $\mathfrak{B}$ is of the 1° category.

307. 1. *If* $\mathfrak{B}$ *is of the* 1° *category in* $\mathfrak{A}$, $\mathfrak{A}$ –

For since $\mathfrak{B}$ is of the 1° category in $\mathfrak{A}$, it
enumerable set of apantactic sets $\{\mathfrak{B}_n\}$. The
exist points $a_1, a_2, \cdots$ in $\mathfrak{A}$ such that

$$D_{\delta_1}(a_1) > D_{\delta_2}(a_2) > \cdots \quad , \quad \delta_n$$

3. *Let $\mathfrak{B}$ be of the* 1° *category in* $\mathfrak{A}$. *Then*
2° *category in* $\mathfrak{A}$.

For otherwise $\mathfrak{B} + B$ would be of the 1° cat

$$\mathfrak{A} - (\mathfrak{B} + B) = 0,$$

and this violates 1.

4. It is now easy to give examples of sets
For instance, the irrational points in the int
set of the 2° category.

308. *Let $\mathfrak{A}$ be a set of the* 1° *category in*
$A = \mathfrak{Q} - \mathfrak{A}$ *has the cardinal number* $\mathfrak{c}$.

If A has an inner point, $D_\delta(a)$, for sufficient
As Card $D_\delta = \mathfrak{c}$, the theorem is proved.

Suppose that A has no inner point. Let $\mathfrak{A}$
apantactic sets $\mathfrak{A}_1 < \mathfrak{A}_2 < \cdots$ in $\mathfrak{Q}$. Let $A_n =$
the maximum of the sides of the cubes lying
viously $q_n \doteq 0$, since by hypothesis A has no in
be a cube lying in A_1. As $q_n \doteq 0$, there exists
has at least two cubes lying in A_{n_1}; call them
ists an $n_2 > n_1$ such that Q_0, Q_1 each have tw
them

$$Q_{0,0} \quad , \quad Q_{0,1} \quad ; \quad Q_{1,0} \quad , \quad Q_1$$

or more shortly Q_{ι_1, ι_2}.

Each of these gives rise similarly to two
which may be denoted by $Q_{\iota_1, \iota_2, \iota_3}$, where the in
the values 0, 1. In this way we may continue

$$Q_{\iota_1} \quad , \quad Q_{\iota_1 \iota_2} \quad , \quad Q_{\iota_1 \iota_2 \iota_3} \cdots$$

Let α be a point lying in a sequence of the
ously does not lie in $\mathfrak{A}$, if the indices are not,
all 0 or all 1. This point α is characterized by

For let $D_1 > D_2 > \cdots$ be a sequence of supe
of $\mathfrak{Q}$, whose norms $\delta_n \doteq 0$. Let

$$d_{11},\ d_{12},\ d_{13} \cdots$$

be the cells of D_1 containing no point of $\mathfrak{A}$ wit

$$d_{21},\ d_{22},\ d_{23} \cdots$$

denote those cells of D_2 containing no point of
not lying in a cell of 1). In this way we may
quence of cells $\mathfrak{D} = \{d_{mn}\}$, where for each m, t
is finite, and $m \doteq \infty$. Each point a of A lies in
being complete, Dist $(a, \mathfrak{A}) > 0$. As the norm
in some cell of D_n, for a sufficiently large n.
theorem is now obvious.

310. *Let $\mathfrak{B}$ be pantactic in $\mathfrak{A}$. Then there e*
set $\mathfrak{C} \leq \mathfrak{B}$ which is pantactic in $\mathfrak{A}$.

For let $D_1 > D_2 > \cdots$ be a set of superimpose
of norms $d_n \doteq 0$. In any cell of D_1 containin
of $\mathfrak{A}$, there is at least one point of $\mathfrak{B}$. If the
the face of two or more cells, the foregoing s
for at least one of the cells. Let us now take
in each of these cells; this gives an enumer
same holds for the cells of D_2. Let us take
these cells, taking when possible points of $\mathfrak{C}_1$.
points of this set not in $\mathfrak{C}_1$. Continuing in thi

$$\mathfrak{C} = \mathfrak{C}_1 + \mathfrak{C}_2 + \cdots$$

Then $\mathfrak{C}$ is pantactic in $\mathfrak{A}$, and is enumerable,

Corollary. In any set $\mathfrak{A}$, finite or infinite, the

As f is continuous,
$$f(c_n) \doteq f(c).$$
Hence
$$f(c) = g,$$
and c lies in $\mathfrak{E}$.

2. *Let $f(x_1 \cdots x_m)$ be continuous in the limite*
If the value of f is known in an enumerable p
which contains all the isolated points of $\mathfrak{A}$, in c
value of f is known at every point of $\mathfrak{A}$.

For let a be a limiting point of $\mathfrak{A}$ not in $\mathfrak{E}$.
in $\mathfrak{A}$, there exists a sequence of points e_1, e_2
Since f is continuous, $f(e_n) \doteq f(a)$. As f is
it is known at a.

3. *Let $\mathfrak{F} = \{f\}$ be the class of one-valued*
defined over a limited point set $\mathfrak{A}$. Then
$$\mathfrak{f} = \text{Card}\, \mathfrak{F} = \mathfrak{c}.$$

For let $\mathfrak{R}_\infty$ be a space of an infinite enu
dimensions, and let
$$y = (y_1, y_2, \cdots)$$
denote one of its points. Let f have the valu
η_2 at $e_2 \cdots$ for the points of $\mathfrak{E}$ defined in 2. T
$$(\eta_1, \eta_2 \cdots)$$
completely determines f. But this complex
point η in $\mathfrak{R}_\infty$ whose coördinates are η_n. We
η. Hence
$$\mathfrak{f} \leq \text{Card}\, \mathfrak{R}_\infty = \mathfrak{c}.$$

On the other hand, $\mathfrak{f} \geq \mathfrak{c}$, since in $\mathfrak{F}$ th
$f(x_1 \cdots x_m) = g$ in $\mathfrak{A}$, where g is any real numb

312. *Let $\mathfrak{P}$ denote the class of complete or ne*

Obviously the cardinal number of the class
1) is $\mathfrak{e}^{\mathfrak{e}} = \mathfrak{c}$. But

$$(a, a_{\iota_1}, a_{\iota_2}, a_{\iota_3} \cdots)$$

is a complete set in $\mathfrak{A}$. Hence $\mathfrak{b} \geq \mathfrak{c}$. On the
For let

$$D_1 > D_2 > \cdots$$

be a sequence of superimposed cubical divis
Each D_n embraces an enumerable set of cells.
divisions gives an enumerable set of cells. Ea
assigned to it, for a given set in $\mathfrak{B}$, the sign +
$\mathfrak{B}$ is exterior to this cell or not. This determ
of two things over an enumerable set of compar

The cardinal number of the class of these dis
But each $\mathfrak{B}$ determines a distribution. Hence

Transfinite Derivatives

313. 1. We have seen, I, 266, that

$$\mathfrak{A}' \geq \mathfrak{A}'' \geq \mathfrak{A}''' \geq \cdots$$

Thus

$$\mathfrak{A}^{(n)} = Dv(\mathfrak{A}', \mathfrak{A}'' \cdots \mathfrak{A}^{(n)}).$$

Let now $\mathfrak{A}$ be a limited point aggregate of t
It has then derivatives of every finite order.

$$Dv(\mathfrak{A}', \mathfrak{A}'', \mathfrak{A}''', \cdots)$$

contains at least one point, and in analogy wit
set 2) *the derivative of order ω of* $\mathfrak{A}$, and denote i

$$\mathfrak{A}^{(\omega)},$$

or more shortly by

$$\mathfrak{A}^{\omega}.$$

Now we may reason on $\mathfrak{A}^{\omega}$ as on any point set

In order that the point set $\mathfrak{A}$ *is of the first spe*
and sufficient that $\mathfrak{A}^{(\omega)} = 0$.

2. We have seen in 18 that $\mathfrak{A}^{\omega}$ is complete.
I, 266 shows *that* $\mathfrak{A}^{\omega+1}, \mathfrak{A}^{\omega+2}, \cdots$, *when they exist*
Then 18 shows that, *if* $\mathfrak{A}^{\omega+n}$ $\quad n = 1, 2, \cdots$ *exis*

$$Dv(\mathfrak{A}^{\omega} \geq \mathfrak{A}^{\omega+1} \geq \mathfrak{A}^{\omega+2} \geq \cdots)$$

exists and is complete. The set 3) is called *the*
$\omega \cdot 2$ and is denoted by

$$\mathfrak{A}^{(\omega 2)} \text{ or } \mathfrak{A}^{\omega 2}.$$

Obviously we may continue in this way in
reach a derivative of order α containing only
points. Then

$$\mathfrak{A}^{\alpha+1} = 0.$$

That this process of derivation may never sto
taking for $\mathfrak{A}$ any limited perfect set, for then

$$\mathfrak{A} = \mathfrak{A}' = \mathfrak{A}'' = \cdots = \mathfrak{A}^{\omega} = \mathfrak{A}^{\omega \cdot 2} =$$

3. We may generalize as follows: Let α den
nal number. If each $\mathfrak{A}^{\beta} > 0$, $\beta < \alpha$, we set

$$\mathfrak{A}^{(\alpha)} = \mathfrak{A}^{\alpha} = Dv\{\mathfrak{A}^{\beta}\}$$

when it exists.

4. If $\mathfrak{A}^{\alpha} > 0$, while $\mathfrak{A}^{\ +1} = 0$, we say $\mathfrak{A}$ is of *o*

314. 1. Let a be a limiting point of $\mathfrak{A}$. Le

$$\alpha_{\delta} = \text{Card } V_{\delta}(a).$$

Obviously α_{δ} is monotone decreasing with
there exists an α and a $\delta_0 > 0$, such that for all

2. *Let* $\mathfrak{A}$ *be a limited aggregate of cardinal number*
is at least one limiting point of $\mathfrak{A}$, *of rank* α.

The demonstration is entirely similar to I, 2
$\delta_2 > \cdots \doteq 0$. Let us effect a cubical division of $\mathfrak{A}$
at least one cell lies an aggregate $\mathfrak{A}_1$ having the
ber α. Let us effect a cubical division of $\mathfrak{A}_1$ of n
least one cell lies an aggregate $\mathfrak{A}_2$ having the card
etc. These cells converge to a point a, such that

$$\text{Card } V_\delta(a) = \alpha,$$

however small δ is taken.

3. *If* Card $\mathfrak{A} > \mathfrak{e}$, *there exists a limiting point of*

The demonstration is similar to that of 2.

4. *If there is no limiting point of* $\mathfrak{A}$ *of rank* $> \mathfrak{e}$, $\mathfrak{A}$
This follows from 3.

5. *Let* Card $\mathfrak{A}$ *be* $> \mathfrak{e}$. *Let* $\mathfrak{B}$ *denote the limiti*
whose ranks are $> \mathfrak{e}$. *Then* $\mathfrak{B}$ *is perfect.*

For obviously $\mathfrak{B}$ is complete. We need therefor
that it is dense. To this end let b be a point of $\mathfrak{B}$
us describe a sequence of concentric spheres of radii
spheres determine a sequence of spherical shells
which have a point in common. If $\mathfrak{A}_n$ denote the po
we have

$$V = V_{r_1}^*(b) = \mathfrak{A}_1 + \mathfrak{A}_2 + \mathfrak{A}_3 + \cdots$$

Thus if each $\mathfrak{A}_m$ were enumerable, V is enumer
Rank b is not $> \mathfrak{e}$. Thus there is one set $\mathfrak{A}_m$ whi
merable, and hence by 3 there exists a point of $\mathfrak{B}$ in
there are points of $\mathfrak{B}$ in any $V_r^*(b)$, and b is not is

6. *A set* $\mathfrak{A}$ *which contains no dense component is e*

complete. The derivatives of $\mathfrak{A}$ of order $\leq$
plete, form a well ordered set and have ther
$\mathfrak{A}^\beta$, where β is necessarily a limitary number.

$$\mathfrak{A}^\beta = Dv(\mathfrak{A}^\gamma) \quad , \quad \gamma < \beta.$$

But every point of $\mathfrak{A}^\beta$ lies in each $\mathfrak{A}^\gamma$. H
point of $\mathfrak{A}^\beta$ is a limiting point of each $\mathfrak{A}^\gamma$ a
Hence $\mathfrak{A}^\beta$ is complete, which is a contradictio

316. *Let α be a limitary number in Z_n.*
$\beta < \alpha$, $\mathfrak{A}^\alpha$ exists.

For there exists an $\aleph_m$, $m \leq n - 2$, seque

$$\gamma < \delta < \epsilon < \eta < \cdots \doteq \alpha.$$

Let c be a point of $\mathfrak{A}^\gamma$, d a point of $\mathfrak{A}^\delta$, e
Then the set

$$(c, d, e, f, \cdots)$$

has at least one limiting point l of rank $\aleph_m$.
in 1). Then l is a limiting point of rank $\aleph_m$

$$(e, f, \cdots).$$

Thus l is a limiting point of every $\mathfrak{A}^\beta$, $\beta < \alpha$,

317. Let us show how we may form poin
is any number in Z_1 or Z_2.

We take the unit interval $\mathfrak{A} = (0, 1)$ as
siderations.

In $\mathfrak{A}$, take the points

$$\mathfrak{B}_1 = \tfrac{1}{2} \quad , \quad \tfrac{3}{4} \quad , \quad \tfrac{7}{8} \quad , \quad \tfrac{15}{16}$$

Obviously $\qquad \mathfrak{B}_2' = \frac{1}{2} \quad , \quad \frac{3}{4} \quad , \quad \frac{7}{8} \quad ,$

Hence $\qquad \mathfrak{B}_2'' = \mathfrak{B}_1' = 1$ and $\mathfrak{B}_2''' =$

Thus $\mathfrak{B}_2$ is of order 2.

In each of the intervals 2) we may pl
to $\mathfrak{B}_2$, such that the right-hand end poi
limiting point of the set. The resulting

This shows that we may form sets of

Let us now place a set of order 1 in
etc. The resulting set $\mathfrak{B}_\omega$ is of order ω
in $\mathfrak{A}_1, \mathfrak{A}_2 \cdots \mathfrak{A}_{n-1}$, while the point 1 lies

Thus
$$\mathfrak{B}_\omega^{(\omega)} = 1.$$
Hence
$$\mathfrak{B}_\omega^{(\omega+1)} = 0,$$
and $\mathfrak{B}_\omega$ is of order ω.

Let us now place in each $\mathfrak{A}_n$ a set si
right-hand end point of $\mathfrak{A}_n$ as limiting
$\mathfrak{B}_{\omega+1}$ is of order $\omega + 1$. In this way we
of order $\omega + 2, \omega + 3, \cdots$ just as we did
may also form now a set of order $\omega 2$, a
of order ω.

Thus we may form sets of order
$$\omega \quad , \quad \omega \cdot 2 \quad , \quad \omega \cdot 3 \quad ,$$
and hence of order ω^2, etc.

318. 1. *Let* $\mathfrak{A}$ *be limited or not, and*
points of $\mathfrak{A}^\beta$. *Then*
$$\mathfrak{A}' = \Sigma \mathfrak{A}_\iota^{(\beta)} + \mathfrak{A}^\Omega \quad , \quad \beta = 1$$

On $\mathfrak{A}^{\omega}$ we can reason as on $\mathfrak{A}'$, and in general f
have

$$\mathfrak{A}' = \sum_{\beta<\alpha} \mathfrak{A}_{\iota}^{(\beta)} + \mathfrak{A}^{(\alpha)},$$

which gives 1).

2. *If* $\mathfrak{A}^{\Omega} = 0$, $\mathfrak{A}$ *and* $\mathfrak{A}'$ *are enumerable.*

For not every $\qquad \mathfrak{A}^{(\alpha)} > 0 \qquad \alpha < \Omega$, by 316.

Hence there is a first α, call it γ, such that
reduces to

$$\mathfrak{A}' = \sum_{\beta} \mathfrak{A}_{\iota}^{(\beta)} \quad , \quad \beta = 1, 2, \cdots < \gamma.$$

But the summation extends over an enumer
each of which is enumerable by 289. Hence
But then $\mathfrak{A}$ is also enumerable by 237, 2.

3. *Conversely, if* $\mathfrak{A}'$ *is enumerable,* $\mathfrak{A}^{\Omega} = 0$.

For if $\mathfrak{A}^{\Omega} > 0$, there is a non-enumerable set
no $\mathfrak{A}^{(\beta)}$ is perfect; and as each term contains a
$\mathfrak{A}'$ is not enumerable. If some $\mathfrak{A}^{(\beta)}$ is perfect,
fect partial set and is therefore not enumerable

4. From 2, 3, we have:

For $\mathfrak{A}'$ *to be enumerable, it is necessary and s
exists a number* α *in* Z_1 *or* Z_2 *such that* $\mathfrak{A}^{\alpha} = 0$.

5. *If* $\mathfrak{A}$ *is complete, it is necessary and suffici
be enumerable, that there exists an* α *in* Z_1 *or* Z_2 *s*

For

$$\mathfrak{A} = \mathfrak{A}_{\iota} + \mathfrak{A}',$$

and the first term is enumerable.

6. If $\mathfrak{A}^{\beta} = 0$ for some $\beta < \Omega$, we say $\mathfrak{A}$ is *redu*
is *irreducible*.

Obviously $\mathfrak{B}_2' = \frac{1}{2} \;,\; \frac{3}{4} \;,\; \frac{7}{8} \;,\; \cdots = \mathfrak{B}_1 .$

Hence $\mathfrak{B}_2'' = \mathfrak{B}_1' = 1$ and $\mathfrak{B}_2''' = 0$

Thus $\mathfrak{B}_2$ is of order 2.

In each of the intervals 2) we may place a set of
to $\mathfrak{B}_2$, such that the right-hand end point of each i
limiting point of the set. The resulting set $\mathfrak{B}_3$ is o

This shows that we may form sets of every finite

Let us now place a set of order 1 in $\mathfrak{A}_1$, a set of
etc. The resulting set $\mathfrak{B}_\omega$ is of order ω. For $\mathfrak{B}_\omega^{(n}$
in $\mathfrak{A}_1, \mathfrak{A}_2 \cdots \mathfrak{A}_{n-1}$, while the point 1 lies in every $\mathfrak{B}_\omega^{(}$

Thus
$$\mathfrak{B}_\omega^{(\omega)} = 1.$$
Hence
$$\mathfrak{B}_\omega^{(\omega+1)} = 0,$$
and $\mathfrak{B}_\omega$ is of order ω.

Let us now place in each $\mathfrak{A}_n$ a set similar to $\mathfrak{B}$
right-hand end point of $\mathfrak{A}_n$ as limiting point. Th
$\mathfrak{B}_{\omega+1}$ is of order $\omega + 1$. In this way we may proce
of order $\omega + 2, \omega + 3, \cdots$ just as we did for order
may also form now a set of order $\omega 2$, as we befor
of order ω.

Thus we may form sets of order

$$\omega \;,\; \omega \cdot 2 \;,\; \omega \cdot 3 \;,\; \omega \cdot 4 \;\cdots$$

and hence of order ω^2, etc.

318. 1. *Let $\mathfrak{A}$ be limited or not, and let $\mathfrak{A}_\iota^{(\beta)}$ den*
points of $\mathfrak{A}^\beta$. Then

$$\mathfrak{A}' = \Sigma \mathfrak{A}_\iota^{(\beta)} + \mathfrak{A}^\Omega \quad , \quad \beta = 1, 2, \cdots < \Omega$$

On $\mathfrak{A}^{\omega}$ we can reason as on $\mathfrak{A}'$, and in general
have

$$\mathfrak{A}' = \sum_{\beta<\alpha} \mathfrak{A}_{\iota}^{(\beta)} + \mathfrak{A}^{(\alpha)},$$

which gives 1).

2. *If* $\mathfrak{A}^{\Omega} = 0$, $\mathfrak{A}$ *and* $\mathfrak{A}'$ *are enumerable.*

For not every

$$\mathfrak{A}^{(\alpha)} > 0 \qquad \alpha < \Omega, \text{ by } 316.$$

Hence there is a first α, call it γ, such that
reduces to

$$\mathfrak{A}' = \sum_{\beta} \mathfrak{A}_{\iota}^{(\beta)} \quad , \quad \beta = 1, 2, \cdots < \gamma.$$

But the summation extends over an enume
each of which is enumerable by 289. Hence
But then $\mathfrak{A}$ is also enumerable by 237, 2.

3. *Conversely, if* $\mathfrak{A}'$ *is enumerable*, $\mathfrak{A}^{\Omega} = 0$.

For if $\mathfrak{A}^{\Omega} > 0$, there is a non-enumerable set
no $\mathfrak{A}^{(\beta)}$ is perfect; and as each term contains
$\mathfrak{A}'$ is not enumerable. If some $\mathfrak{A}^{(\beta)}$ is perfect,
fect partial set and is therefore not enumerable

4. From 2, 3, we have:

For $\mathfrak{A}'$ *to be enumerable, it is necessary and*
exists a number α *in* Z_1 *or* Z_2 *such that* $\mathfrak{A}^{\alpha} = 0$.

5. *If* $\mathfrak{A}$ *is complete, it is necessary and suffici*
be enumerable, that there exists an α *in* Z_1 *or* Z_2

For

$$\mathfrak{A} = \mathfrak{A}_{\iota} + \mathfrak{A}',$$

and the first term is enumerable.

6. If $\mathfrak{A}^{\beta} = 0$ for some $\beta < \Omega$, we say $\mathfrak{A}$ is *red*
is *irreducible.*

Let S_n denote a sphere about a of radius r_n. Le
points of $\mathfrak{B}$ lying between S_{n-1}, S_n, including th
may lie on S_{n-1}. Then

$$\mathfrak{B} = \mathfrak{B}_1 + \mathfrak{B}_2 + \mathfrak{B}_3 + \cdots + a.$$

Each $\mathfrak{B}_m$ is enumerable. For any point of $\mathfrak{A}$
$\mathfrak{B}^\Omega = a$. Hence $\mathfrak{B}_m^\Omega = 0$ and $\mathfrak{B}_m$ is enumerable by
Thus $\mathfrak{B}$ is enumerable. This, however, is
$\mathfrak{B}^\Omega = a$, and is thus > 0.

320. 1. *In the relation*

$$\mathfrak{A}' = \sum_\beta \mathfrak{A}_\iota^{(\beta)} + \mathfrak{A}^\Omega \qquad \beta = 1, 2, \cdots < \Omega$$

the first term on the right is enumerable.

For let us set

$$\mathfrak{B} = \sum_\beta \mathfrak{A}_\iota^{(\beta)};$$

also let

$$r_1 > r_2 > \cdots \doteq 0.$$

Let $\mathfrak{B}_n$ denote the points of $\mathfrak{B}$ whose distance δ
fies the relation

$$r_n \geq \delta > r_{n+1}.$$

Then the distance of any point of $\mathfrak{B}_n'$ from $\mathfrak{A}^\Omega$ i
includes all points of $\mathfrak{B}$ whose distance from $\mathfrak{A}^\Omega$ i

$$\mathfrak{B} = \mathfrak{B}_0 + \mathfrak{B}_1 + \mathfrak{B}_2 + \cdots$$

Each $\mathfrak{B}_n$ is enumerable. For if not, $\mathfrak{B}_n^\Omega > 0.$
$\mathfrak{B}_n^\Omega$ as b lies in $\mathfrak{A}^\Omega$. Hence

$$\text{Dist}\,(b, \mathfrak{A}^\Omega) = 0.$$

On the other hand, as b lies in $\mathfrak{B}_n'$, its dista
$\geq r_{n+1}$, which is a contradiction.

2. *If $\mathfrak{A}'$ is not enumerable, there exists a first n*
Z such that $\mathfrak{A}^a$ is perfect.

Complete Sets

321. Let us study now some of the properties of
sets. We begin by considering limited perfect r
Let $\mathfrak{A}$ be such a set. It has a first point a and a la
therefore lies in the interval $I=(a, b)$. If $\mathfrak{A}$ is pa
partial interval $J=(\alpha, \beta)$ of I, $\mathfrak{A}$ embraces all th
since $\mathfrak{A}$ is perfect. Let us therefore suppose that
in I. An example of such sets is the Cantor set of

Let $D=\{\delta\}$ be a set of intervals no two of whi
in common. We say D is *pantactic* in an interval
tains no interval which does not contain some in
least a part of some δ.

It is *separated* when no two of its intervals ha
common.

322. 1. *Every limited rectilinear apantactic perf
mines an enumerable pantactic set of separated int
whose end points alone lie in* $\mathfrak{A}$.

For let $\mathfrak{A}$ lie in $I=(\alpha, \beta)$, where α, β are the
points of $\mathfrak{A}$. Let $\mathfrak{B}=I-\mathfrak{A}$. Each point b of $\mathfrak{B}$ f
terval δ whose end points lie in $\mathfrak{A}$. For other
approach b as near as we chose, ranging over a set
But then b is a point of $\mathfrak{A}$, as this is perfect. L
take these intervals as large as possible and call th

The intervals δ are pantactic in I, for otherwise
apantactic. They are enumerable, for but a finit
lengths $> I/n+1$ and $\leq I/n$, $n=1, 2 \cdots$

It is separated, since $\mathfrak{A}$ contains no isolated poin

$\mathfrak{A}$ contains no other points. For let a be a point o
E'. Let α be another point of $\mathfrak{A}$. In the interval
end point e of some interval of D. In the interval (
other end point e_1. In the interval (a, e_1) lies anoth
e_2, etc. The set of points $e, e_1, e_2 \cdots \doteq a$. Hence
which is a contradiction.

324. *Conversely, the end points $E = \{e\}$ and the limi
the end points of a pantactic enumerable set of separ
$D = \{\delta\}$ form a perfect apantactic set $\mathfrak{A}$.*

For in the first place, $\mathfrak{A}$ is complete, since $\mathfrak{A} = (E,$
contain no isolated points, since the intervals δ a
Hence $\mathfrak{A}$ is perfect. It is apantactic, since otherwise
brace all the points of some interval, which is impos
pantactic.

325. Since the adjoint set of intervals $D = \{\delta\}$ is e
can be arranged in a 1, 2, 3, $\cdots$ order *according to siz*

Let δ be the largest interval, or if several are equa
of them. The interval δ causes I to fall into two oth
The interval to the left of δ, call I_0, that to the right
The largest interval in I_0, call δ_0, that in I_1, call δ_1.
we may continue without end, getting a sequence of

$$\delta, \delta_0, \delta_1, \delta_{00}, \delta_{01}, \delta_{10}, \delta_{11} \cdots$$

and a similar series of intervals

$$I, I_0, I_1, I_{00}, I_{01} \cdots$$

The lengths of the intervals in 1) form a monoton
sequence which $\doteq 0$.

If ν denote a complex of indices $\iota\jmath\kappa \cdots$

Let $D = \{\delta_\nu\}$ be its adjoint set of intervals, a
Let $\mathfrak{E}$ be the Cantor set of I, 272. Let its adjo
be $H = \{\eta_\nu\}$, arranged also as in 325. If we s
$D \simeq H$. Hence

$$\text{Card } \mathfrak{A} = \text{Card } \mathfrak{E}.$$

But Card $\mathfrak{E} = \mathfrak{c}$ by 244, 4.

2. *The cardinal number of every limited rectil*
is either $\mathfrak{e}$ *or* $\mathfrak{c}$.

For we have seen, 320, 4, that

$$\mathfrak{A} = \mathfrak{E} + \mathfrak{P}, \qquad \mathfrak{P} \geq 0,$$

where $\mathfrak{E}$ is enumerable and $\mathfrak{P}$ is perfect.

$$\text{If } \mathfrak{P} = 0, \qquad \text{Card } \mathfrak{A} = \mathfrak{e}.$$

$$\text{If } \mathfrak{P} > 0, \qquad \text{Card } \mathfrak{A} = \mathfrak{c}.$$

For Card $\mathfrak{A}$ = Card $\mathfrak{E}$ + Card $\mathfrak{P} = \mathfrak{e} + \mathfrak{c} = \mathfrak{c}$.

327. *The cardinal number of every limited com*
either $\mathfrak{e}$ *or* $\mathfrak{c}$. *It is* $\mathfrak{c}$, *if* $\mathfrak{A}$ *has a perfect componen*

The proof may be made by induction.

For simplicity take $m = 2$. By a transformat
we may bring $\mathfrak{A}$ into a unit square S. Let us
$\mathfrak{A}$ were in S originally. Then Card $\mathfrak{A} \leq \mathfrak{c}$ by 2

Let $\mathfrak{C}$ be the projection of $\mathfrak{A}$ on one of the sid
points of $\mathfrak{A}$ lying on a parallel to the other side
point of $\mathfrak{C}$. If $\mathfrak{B}$ has a perfect component, Car
Card $\mathfrak{A} = \mathfrak{c}$. If $\mathfrak{B}$ does not have a perfect comp
number of each $\mathfrak{B}$ is $\mathfrak{e}$. Now $\mathfrak{C}$ is complete by
if $\mathfrak{C}$ contains a perfect component, Card $\mathfrak{C} =$

complete, any point a of A is an inner point of A
lies in A, for some ρ sufficiently small. Hence a
We have thus the result:

Any limited complete set is uniquely determined set of cubes $\{d_n\}$, *each of which is exterior to it.*

We may call $\mathfrak{B} = \{d_n\}$ the *border of* $\mathfrak{A}$, and the *cells.*

2. *The totality of all limited perfect or complete dinal number* $\mathfrak{c}$.

For any limited complete set $\mathfrak{C}$ is completely border $\{d_n\}$. The totality of such sets has a $\leq \mathfrak{c}^{\mathfrak{e}} = \mathfrak{c}$. Hence Card $\{\mathfrak{C}\} \leq \mathfrak{c}$. Since among th of segments, Card $\mathfrak{C} \geq \mathfrak{c}$.

329. If $\mathfrak{A}_\iota$ denote the isolated points of $\mathfrak{A}$, a limiting points, we may write

$$\mathfrak{A} = \mathfrak{A}_\iota + \mathfrak{A}_\lambda.$$

Similarly we have

$$\mathfrak{A}_\lambda = \mathfrak{A}_{\lambda\iota} + \mathfrak{A}_{\lambda^2},$$

$$\mathfrak{A}_{\lambda^2} = \mathfrak{A}_{\lambda^2\iota} + \mathfrak{A}_{\lambda^3}, \text{ etc.}$$

We thus have

$$\mathfrak{A} = \mathfrak{A}_\iota + \mathfrak{A}_{\lambda\iota} + \mathfrak{A}_{\lambda^2\iota} + \cdots + \mathfrak{A}_{\lambda^{n-1}\iota} + \mathfrak{A}$$

At the end of each step, certain points of $\mathfrak{A}$ are may be considered as *adhering* loosely to $\mathfrak{A}$, whil remains may be regarded as *cohering* more closely we may call $\mathfrak{A}_{\lambda^{n-1}\iota}$, the n^{th} *adherent*, and $\mathfrak{A}_{\lambda^n}$ the n

If α is a limitary number, defined by

$$\alpha_1 < \alpha_2 < \alpha_3 \cdots \doteq \alpha,$$

we set

$$\mathfrak{A}_\alpha = Dv\{\mathfrak{A}_\lambda{}^{\alpha n}\}$$

and call it, when it exists, the *coherent of orde*

write

$$\mathfrak{A} = \sum_\alpha \mathfrak{A}_{\lambda^{\alpha}\iota} + \mathfrak{A}_{\lambda^\beta} \qquad \alpha = 1, 2, \cdots <$$

where β is a number in Z_2.

330. 1. *When $\mathfrak{A}$ is enumerable,*

$$\mathfrak{A} = \sum_\alpha \mathfrak{A}_{\lambda^{\alpha}\iota} + \mathfrak{A}_{\lambda^\beta} \qquad \alpha = 1, 2,$$

$$= \mathfrak{J} + \mathfrak{D};$$

where $\mathfrak{J}$ is the sum of an enumerable set of isolat it exists, is dense.

For the adherences of different orders have r with those of any other order. They are thus sum $\mathfrak{J}$ can contain but an enumerable set of a wise $\mathfrak{A}$ could not be enumerable. Thus ther number β for which

$$\mathfrak{A}_{\lambda^{\beta}\iota} = 0.$$

As now in general

$$\mathfrak{A}_{\lambda^\beta} = \mathfrak{A}_{\lambda^{\beta}\iota} + \mathfrak{A}_{\lambda^{\beta+1}},$$

we have

$$\mathfrak{A}_{\lambda^\beta} = \mathfrak{A}_\lambda{}^{\beta+1} = \mathfrak{A}_\lambda{}^{\beta+2} = \cdots$$

As $\mathfrak{A}_{\lambda^\beta}$ thus contains no isolated points, it is by I, 270.

2. *When $\mathfrak{A}$ is not enumerable,* $\mathfrak{D} > 0$. For if

these may be consolidated with the cells of D
vision Δ of norm δ which in general will not b

$$\overline{\mathfrak{A}}_\Delta = \overline{\mathfrak{A}}'_\Delta + \overline{\mathfrak{A}}_\Delta{}^*.$$

The last term is formed of cells that contain o
of points of $\mathfrak{A}$. These cells may be subdivi
division E such that in

$$\overline{\mathfrak{A}}_E = \overline{\mathfrak{A}}'_E + \overline{\mathfrak{A}}_E{}^*$$

the last term is $< \epsilon/3$. Now if δ is sufficient

$$\overline{\mathfrak{A}}_\Delta - \overline{\mathfrak{A}} < \frac{\epsilon}{3} \quad , \quad \overline{\mathfrak{A}}'_\Delta - \overline{\mathfrak{A}}' <$$

Hence from 2), 3) we have 1).

332. *If* $\overline{\mathfrak{A}} > 0$, Card $\mathfrak{A} = \mathfrak{c}$.

For let $\mathfrak{B}$ denote the sifted set of $\mathfrak{A}$ [I, 71
fect. Hence Card $\mathfrak{B} = \mathfrak{c}$, hence Card $\mathfrak{A} = \mathfrak{c}$.

333. *Let* $\mathfrak{A} = \{\mathfrak{a}\}$, *where each* $\mathfrak{a}$ *is metric and*
two of the $\mathfrak{a}$*'s have more than their frontiers*
enumerable set in the $\mathfrak{a}$*'s.* $\mathfrak{A}$ *may be unlimited.*

Let us first suppose that $\mathfrak{A}$ lies in a cube $\mathfrak{Q}$.
removing its proper frontier points. Then n
a point in common. Let

$$q_1 > q_2 > \cdots \doteq 0,$$

where the first term $q_1 = \widehat{\mathfrak{Q}}$. There can be bu
sets α, such that their contents lie between
For if

$$\widehat{\alpha}_{\iota_1} \quad , \quad \widehat{\alpha}_{\iota_2} \cdots \geq q_s$$

CHAPTER XI

MEASURE

Upper Measure

334. 1. Let $\mathfrak{A}$ be a limited point set. An
metric sets $D = \{d_\iota\}$, such that each point of $\mathfrak{A}$
called an *enclosure* of $\mathfrak{A}$. If each point of $\mathfrak{A}$ lie
is called an *outer* enclosure. The sets d_ι are cal
enclosure corresponds the finite or infinite serie

$$\Sigma \bar{d}_\iota$$

which may or may not converge. In any case t
the numbers 1) is finite and ≤ 0. For let Δ be
of space, $\bar{\mathfrak{A}}_\Delta$ is obviously an enclosure and the
1) is also $\bar{\mathfrak{A}}_\Delta$, since we have agreed to read this
as a point set or as its content.

We call

$$\text{Min}\, \Sigma d_\iota,$$

with respect to the class of all possible enclos
measure of $\mathfrak{A}$, and write

$$\bar{\bar{\mathfrak{A}}} = \text{Meas}\, \mathfrak{A} = \underset{D}{\text{Min}}\, \Sigma d_\iota.$$

2. *The minimum of the sums* 1) *is the same w*

As ϵ is small at pleasure, Min Σd_ι o
closures = Min Σd_ι over the class of all

3. Two metric sets whose common p
are called *non-overlapping*. The enclos
overlapping, when any two of its cells a

Any enclosure D may be replaced by a

For let $$U(d_1, d_2) = d_1 + e$$

$$U(d_1, d_2, d_3) = d_1 + e$$

$$U(d_1\ d_2\ d_3\ d_4) = d_1 + e$$

Obviously each e_n is metric. For un
Then $E = \{e_n\}$ is a non-overlapping enc

$$\Sigma \widehat{e}_n \leq \Sigma \widehat{d}_n$$

we see that *the minimum of the sums* 1)
ourselves to the class of non-overlapping e

Obviously we may adjoin to any
improper limiting points.

4. In the enclosure $E = \{e_n\}$ found
have a point in common. Such enclosu

335. 1. *Let* $D = \{d_\iota\}$, $E = \{e_\kappa\}$ *be two*
of $\mathfrak{A}$. *Let*

$$\delta_{\iota\kappa} = Dv\,(d_\iota, e$$

Then

$$\Delta = \{\delta_{\iota\kappa}\}, \qquad \iota, \kappa =$$

is a non-overlapping enclosure of $\mathfrak{A}$.

For $\delta_{\iota\kappa}$ is metric by 22, 2. Two of t
overlapping. Each point of $\mathfrak{A}$ lies in
hence a lies in δ .

As
$$\Sigma \widehat{d}_\iota \leq \Sigma \widehat{e}_\iota,$$
we have
$$\underset{\Delta}{\mathrm{Min}}\, \Sigma \delta_\iota \leq \underset{D}{\mathrm{Min}}\, \Sigma d_\iota \leq \underset{E}{\mathrm{Min}}\, \Sigma \widehat{e}_\iota$$
from which 1) follows at once.

337. *If* $\mathfrak{A}$ *is metric,*
$$\bar{\bar{\mathfrak{A}}} = \widehat{\mathfrak{A}}.$$

For let D be a cubical division of space such
$$\bar{\mathfrak{A}}_D - \widehat{\mathfrak{A}} < \epsilon \quad , \quad \widehat{\mathfrak{A}} - \underline{\mathfrak{A}}_D < \epsilon.$$

Let us set $\mathfrak{B} = \underline{\mathfrak{A}}_D$. Let $E = \{e_\iota\}$ be an oute
Since $\mathfrak{B}$ is complete, there exists a finite set of
contain all the points of $\mathfrak{B}$ by 301. The vol
obviously $> \widehat{\mathfrak{B}}$; hence *a fortiori*
$$\Sigma \widehat{e}_\iota > \widehat{\mathfrak{B}}.$$
Hence
$$\bar{\bar{\mathfrak{B}}} > \widehat{\mathfrak{B}}.$$
But
$$\bar{\bar{\mathfrak{A}}} > \bar{\bar{\mathfrak{B}}}, \text{ by } 336,$$
$$\geq \widehat{\mathfrak{B}} = \underline{\mathfrak{A}}_D$$
$$> \widehat{\mathfrak{A}} - \epsilon, \text{ by } 2).$$
On the other hand,
$$\bar{\bar{\mathfrak{A}}} < \bar{\mathfrak{A}}_D < \widehat{\mathfrak{A}} + \epsilon, \text{ by } 2).$$

From 3), 4) we have 1), since ϵ is arbitrarily

338. *If* $\mathfrak{A}$ *is complete,*
$$\bar{\bar{\mathfrak{A}}} = \bar{\mathfrak{A}}.$$
For by definition
$$\bar{\bar{\mathfrak{A}}} = \mathrm{Min}\, \Sigma d_\iota,$$

339. *Let the limited set $\mathfrak{A} = \{\mathfrak{A}_n\}$ be the union*
enumerable set of sets $\mathfrak{A}_n$. Then

$$\overline{\overline{\mathfrak{A}}} \leq \Sigma \overline{\overline{\mathfrak{A}}}_n.$$

For to each $\mathfrak{A}_n$ corresponds an enclosure D_n

$$\sum_\iota \widehat{d}_{n\iota} < \overline{\overline{\mathfrak{A}}}_n + \frac{\epsilon}{2^n} \quad , \quad \epsilon > 0, \text{ arbitraril}$$

But the cells of all the enclosures D_n, also
Hence

$$\overline{\overline{\mathfrak{A}}} \leq \sum_{\iota, n} \widehat{d}_{n\iota} < \Sigma\left(\overline{\overline{\mathfrak{A}}}_n + \frac{\epsilon}{2^n}\right)$$

$$\leq \Sigma \overline{\overline{\mathfrak{A}}}_n + \epsilon.$$

This gives 1), as ϵ is small at pleasure.

340. *Let $\mathfrak{A}$ lie in the metric set $\mathfrak{M}$. Let*
complementary set. Then

$$\overline{\overline{\mathfrak{A}}} + \overline{\overline{A}} \geq \widehat{\mathfrak{M}}.$$

For from

$$\mathfrak{M} = \mathfrak{A} + A,$$

follows

$$\overline{\overline{\mathfrak{M}}} \leq \overline{\overline{\mathfrak{A}}} + \overline{\overline{A}}, \qquad \text{by 339.}$$

But

$$\overline{\overline{\mathfrak{M}}} = \widehat{\mathfrak{M}}, \qquad \text{by 337.}$$

341. *If $\mathfrak{A} = \mathfrak{B} + \mathfrak{C}$, and $\mathfrak{B}$, $\mathfrak{C}$ are exterior to*

$$\overline{\overline{\mathfrak{A}}} = \overline{\overline{\mathfrak{B}}} + \overline{\overline{\mathfrak{C}}}.$$

For, if any enclosure $D = \{d_\iota\}$ of $\mathfrak{A}$ embrac
a point of $\mathfrak{B}$ and $\mathfrak{C}$, it may be split up into

By properly choosing D, we may crowd th
down toward its minimum. Now the class o
included in the class of all enclosures of $\mathfrak{B}$, an
holds for D''.

Thus from 2) follows that

$$\overline{\mathfrak{A}} > \overline{\mathfrak{B}} + \overline{\mathfrak{C}}.$$

This with 339 gives 1).

342. *If* $\mathfrak{A} = \mathfrak{B} + \mathfrak{M}$, $\mathfrak{M}$ *being metric,*

$$\mathfrak{A} = \mathfrak{B} + \widehat{\mathfrak{M}}.$$

For let D be a cubical division of norm d. I
of $\mathfrak{M}$ in the cells containing points of Front
the other points of $\mathfrak{M}$. Then $\mathfrak{m}$ and $\mathfrak{B}$ are exte
and by 337 and 341,

$$\overline{\mathrm{Meas}}\,(\mathfrak{B} + \mathfrak{m}) = \overline{\mathfrak{B}} + \widehat{\mathfrak{m}}.$$

As

$$\mathfrak{A} = \mathfrak{B} + \mathfrak{m} + \mathfrak{n},$$

$$\overline{\mathrm{Meas}}\,(\mathfrak{B} + \mathfrak{m}) < \overline{\mathfrak{A}}$$

Also

$$\overline{\mathfrak{A}} < \overline{\mathfrak{B}} + \widehat{\mathfrak{m}} + \widehat{\mathfrak{n}} \qquad \text{by}$$

Thus

$$\overline{\mathfrak{B}} + \widehat{\mathfrak{m}} < \overline{\mathfrak{A}} < \overline{\mathfrak{B}} + \widehat{\mathfrak{m}} + \widehat{\mathfrak{n}}.$$

Now if d is sufficiently small,

$$\widehat{\mathfrak{M}} - \epsilon < \widehat{\mathfrak{m}} \quad ; \quad \mathfrak{n} < \epsilon.$$

Thus 2) gives, as $\widehat{\mathfrak{m}} < \widehat{\mathfrak{M}}$,

$$\overline{\mathfrak{B}} + \widehat{\mathfrak{M}} - \epsilon < \overline{\mathfrak{A}} < \overline{\mathfrak{B}} + \widehat{\mathfrak{M}} + \epsilon,$$

which gives 1), as $\epsilon > 0$ is arbitrarily small.

Thus

$$\widehat{\mathfrak{B}} - \overline{\overline{B}} = \widehat{\mathfrak{D}} + \widehat{\mathfrak{B}}_1 - (\widehat{\mathfrak{B}}_1 + \overline{\overline{D}}) =$$

$$\widehat{\mathfrak{C}} - \overline{\overline{C}} = \widehat{\mathfrak{D}} + \widehat{\mathfrak{C}}_1 - (\widehat{\mathfrak{C}}_1 + \overline{\overline{D}}) =$$

2. If $\mathfrak{A} < \mathfrak{B}$, the complement of $\mathfrak{A}$ with frequently be denoted by the corresponding E

$$A = C(\mathfrak{A}), \qquad \text{M}$$
$$= \mathfrak{B} - \mathfrak{A}.$$

Lower Measure

344. 1. We are now in position to define measure. Let $\mathfrak{A}$ lie in a metric set $\mathfrak{M}$. Th $A = \mathfrak{M} - \mathfrak{A}$ has an upper measure $\overline{\overline{A}}$. We s is the *lower measure* of $\mathfrak{A}$, and write

$$\underline{\underline{\mathfrak{A}}} = \underline{\text{Meas}}\, \mathfrak{A} = \widehat{\mathfrak{M}} - \overline{\overline{A}}.$$

By 343 this definition is independent of the

When
$$\overline{\overline{\mathfrak{A}}} = \underline{\underline{\mathfrak{A}}}$$

we say $\mathfrak{A}$ is *measurable*, and write

$$\widehat{\mathfrak{A}} = \overline{\overline{\mathfrak{A}}} = \underline{\underline{\mathfrak{A}}}.$$

A set whose measure is 0 is called a *null* se

2. *Let $E = \{e_\iota\}$ be an enclosure of A.*

Then $$\underline{\underline{\mathfrak{A}}} = \text{Max}\,(\widehat{\mathfrak{M}} - \Sigma \widehat{e}_\iota),$$

345. 1.
$$\underline{\underline{\mathfrak{A}}} \geq 0.$$

For let $\mathfrak{A}$ lie in the metric set $\mathfrak{M}$.

Then
$$\underline{\underline{\mathfrak{A}}} = \widehat{\mathfrak{M}} - \overline{\overline{A}}.$$

But by 336,
$$\overline{\overline{A}} \leq \widehat{\mathfrak{M}},$$

hence
$$\widehat{\mathfrak{M}} - \overline{\overline{A}} \geq 0.$$

2.
$$\underline{\underline{\mathfrak{A}}} \leq \overline{\overline{\mathfrak{A}}}.$$

For let $\mathfrak{A}$ lie in the metric set $\mathfrak{M}$.

Then
$$\overline{\overline{\mathfrak{A}}} + \overline{\overline{A}} \geq \widehat{\mathfrak{M}} \qquad \text{by 340.}$$

Hence
$$\underline{\underline{\mathfrak{A}}} = \widehat{\mathfrak{M}} - \overline{\overline{A}} \leq \overline{\overline{\mathfrak{A}}}.$$

346. 1. *For any limited set* $\mathfrak{A}$,
$$\underline{\mathfrak{A}} \leq \underline{\underline{\mathfrak{A}}} \leq \overline{\overline{\mathfrak{A}}} \leq \overline{\mathfrak{A}}.$$

For let $D = \{d_\iota\}$ be an enclosure of $\mathfrak{A}$. Then
$$\overline{\mathfrak{A}} = \operatorname*{Min}_{D} \Sigma \widehat{d_\iota},$$

when D ranges over the class F of all *finite* enclosu
other hand,
$$\overline{\overline{\mathfrak{A}}} = \operatorname*{Min}_{D} \Sigma \widehat{d_\iota}$$

when D ranges over the class E of all enumerabl
But the class E includes the class F. Hence $\overline{\overline{\mathfrak{A}}} \leq \overline{\mathfrak{A}}$.

2. *If $\mathfrak{A}$ is metric, it is measurable, and*

$$\hat{\hat{\mathfrak{A}}} = \hat{\hat{\mathfrak{A}}}.$$

This follows at once from 1).

347. *Let $\mathfrak{A}$ be measurable and lie in the metric is measurable, and*

$$\hat{\hat{\mathfrak{A}}} + \hat{\hat{A}} = \widehat{\mathfrak{M}}.$$

For

$$\underline{\underline{A}} = \widehat{\mathfrak{M}} - \hat{\hat{\mathfrak{A}}}.$$

$$\underline{\underline{\mathfrak{A}}} = \widehat{\mathfrak{M}} - \overline{\overline{A}} = \hat{\hat{\mathfrak{A}}},$$

since $\mathfrak{A}$ is measurable. This last gives

$$\overline{\overline{A}} = \widehat{\mathfrak{M}} - \hat{\hat{\mathfrak{A}}}.$$

This with 2) shows that $\overline{\overline{A}} = \underline{\underline{A}}$; hence A is m
2) now follows 1).

348. *If* $\mathfrak{A} < \mathfrak{B}$, *then* $\underline{\underline{\mathfrak{A}}} \le \underline{\underline{\mathfrak{B}}}$.

For as usual let A, B be the complements of
to a metric set $\mathfrak{M}$. Since $\mathfrak{A} < \mathfrak{B}$, $A > B$.

Hence, by 336,

$$\overline{\overline{A}} \geq \overline{\overline{B}}.$$

Thus,

$$\widehat{\mathfrak{M}} - \overline{\overline{A}} \leq \widehat{\mathfrak{M}} - \overline{\overline{B}},$$

which gives 1).

349. *For $\mathfrak{A}$ to be measurable, it is necessary and*

$$\overline{\overline{\mathfrak{A}}} + \overline{\overline{A}} = \widehat{\mathfrak{M}}$$

where $\mathfrak{M}$ is any metric set $> \mathfrak{A}$, and $A = \mathfrak{M} - \mathfrak{A}$.

It is sufficient, for then 1) shows that

Let S denote the infinite series on the right of 1
let S_n denote the sum of the first n terms. Let $\mathfrak{A}_n$
Then $\mathfrak{A}_n \leq \mathfrak{A}$ and by 336,

$$\widehat{\mathfrak{A}}_n = S_n \leq \overline{\overline{\mathfrak{A}}} \quad , \quad \text{for any } n.$$

Thus S is convergent and

$$S \leq \overline{\overline{\mathfrak{A}}}.$$

On the other hand, by 339,

$$\overline{\overline{\mathfrak{A}}} \leq S.$$

From 3), 4) follows that

$$S = \overline{\overline{\mathfrak{A}}} = \lim S_n = \lim \widehat{\mathfrak{A}}_n.$$

We show now that $\mathfrak{A}$ is measurable. To this en
metric set $> \mathfrak{A}$, and $\mathfrak{A}_n + A_n = \mathfrak{M}$ as usual.

Then

$$\widehat{\mathfrak{A}}_n + \widehat{A}_n = \widehat{\mathfrak{M}}.$$

But $\qquad A \leq A_n \quad , \quad$ hence $\overline{\overline{A}} \leq \widehat{A}_n$.

Thus 6) gives

$$\overline{\overline{A}} + \widehat{\mathfrak{A}}_n \leq \widehat{\mathfrak{M}},$$

for any n. Hence

$$\overline{\overline{A}} + \lim \widehat{\mathfrak{A}}_n \leq \widehat{\mathfrak{M}};$$

or using 5),

$$\overline{\overline{A}} + \overline{\overline{\mathfrak{A}}} \leq \widehat{\mathfrak{M}}.$$

Hence by 339,

$$\overline{\overline{A}} + \overline{\overline{\mathfrak{A}}} = \widehat{\mathfrak{M}}.$$

Thus by 349, $\mathfrak{A}$ is measurable.

351. *Let*

$$\mathfrak{A} = \mathfrak{B} + \mathfrak{C};$$

then

$$\underline{\underline{\mathfrak{B}}} + \underline{\underline{\mathfrak{C}}} \leq \underline{\underline{\mathfrak{A}}}.$$

As all the points of A are in B, and also in C, E and F, and hence in the cells of D, which thu enclosure of A. Let

$$\gamma_m = (d_{m1}, d_{m2} \cdots) \quad , \quad \eta_n = (d_{1n}, d_{2n}$$

Let us set

$$e_m = \gamma_m + g_m \quad , \quad f_n = \eta_n + h_n.$$

Then by 350,

$$\widehat{\gamma}_m = \sum_n \widehat{d}_{mn} \quad , \quad \widehat{\eta}_n = \sum_m \widehat{d}_{mn}.$$

By 347,

$$\widehat{e}_m = \widehat{\gamma}_m + \widehat{g}_m \quad , \quad \widehat{f}_n = \widehat{\eta}_n + \widehat{h}_n.$$

Hence

$$\widehat{\mathfrak{M}} - \sum_m \widehat{e}_m = \widehat{\mathfrak{M}} - \sum_{m,n} \widehat{d}_{mn} - \sum_m \widehat{g}_m,$$

$$\widehat{\mathfrak{M}} - \sum_n \widehat{f}_n = \widehat{\mathfrak{M}} - \sum_{m,n} \widehat{d}_{mn} - \sum_n \widehat{h}_n.$$

Hence adding,

$$(\widehat{\mathfrak{M}} - \Sigma \widehat{e}_m) + (\widehat{\mathfrak{M}} - \Sigma \widehat{f}_n)$$

$$= \widehat{\mathfrak{M}} - \Sigma \widehat{d}_{mn} + [\widehat{\mathfrak{M}} - (\Sigma \widehat{g}_m + \Sigma \widehat{h},$$

Now

$$\mathfrak{M} = U\{g_m, h_n, d_{mn}\} \qquad m, n = 1,$$

Thus by 339, the term in [ ] is ≤ 0. Thus

$$(\widehat{\mathfrak{M}} - \Sigma \widehat{e}_m) + (\widehat{\mathfrak{M}} - \Sigma \widehat{f}_n) \leq \widehat{\mathfrak{M}} - \Sigma \widehat{d}_{mn}$$

But

$$\underline{\underline{\mathfrak{B}}} = \operatorname{Max}(\widehat{\mathfrak{M}} - \Sigma \widehat{e}_m)$$

$$\underline{\underline{\mathfrak{C}}} = \operatorname{Max}(\widehat{\mathfrak{M}} - \Sigma \widehat{f}_n).$$

Thus 3) gives 1) at once.

Measurable Sets

352. 1. *Let* $\mathfrak{A} = \mathfrak{B} + \mathfrak{C}$. *If* $\mathfrak{B}$, $\mathfrak{C}$ *are meas*

2. *Let* $\mathfrak{A} = \mathfrak{B} + \mathfrak{C}$. *If* $\mathfrak{A}$, $\mathfrak{B}$ *are measurable, so is*

$$\widehat{\mathfrak{C}} = \widehat{\mathfrak{A}} - \widehat{\mathfrak{B}}.$$

For let $\mathfrak{A}$ lie in the metric set $\mathfrak{M}$. Then

$$\mathfrak{M} - \mathfrak{A} = \mathfrak{M} - (\mathfrak{B} + \mathfrak{C}) = (\mathfrak{M} - \mathfrak{C}) - \mathfrak{B}$$

Thus

$$A = C - \mathfrak{B};$$

Hence

$$C = \mathfrak{B} + A.$$

Thus C is measurable by 1. Hence $\mathfrak{C}$ is measu
and

$$\widehat{\mathfrak{A}} = \widehat{\mathfrak{B}} + \widehat{\mathfrak{C}}.$$

From this follows 2) at once.

353. 1. *Let* $\mathfrak{A} = \Sigma\mathfrak{A}_n$ *be the sum of an enumerable able sets. Then* $\mathfrak{A}$ *is measurable and*

$$\widehat{\mathfrak{A}} = \Sigma\widehat{\mathfrak{A}}_n.$$

If $\mathfrak{A}$ is the sum of a finite number of sets, the th ously true by 352, 1. In case $\mathfrak{A}$ embraces an infin sets, the reasoning of 350 may be employed.

2. *Let* $\mathfrak{N} = \{\mathfrak{N}_n\}$ *be the union of an enumerable s Then* $\mathfrak{N}$ *is a null set.*

Follows at once from 1.

3. *Let* $\mathfrak{A} = \{\mathfrak{A}_n\}$ *be the union of an enumerable set sets whose common points two and two, form null set measurable and*

$$\widehat{\mathfrak{A}} = \Sigma\widehat{\mathfrak{A}}_n.$$

4. *Let* $\mathfrak{E} = \{\mathfrak{e}_n\}$ *be a non-overlapping enclosure of*
measurable and

6. *If* $\mathfrak{F} = \{\mathfrak{f}_n\}$ *is another non-overlapping enclos*
then

$$\mathfrak{D} = Dv(\mathfrak{E}, \mathfrak{F})$$

is measurable.

For the cells of $\mathfrak{D}$ are

$$\delta_{\iota\kappa} = Dv(\mathfrak{e}_\iota, \mathfrak{f}_k).$$

Thus $\delta_{\iota\kappa}$ is metric, and

$$\widehat{\mathfrak{D}} = \Sigma \widehat{\delta}_{\iota\kappa}.$$

354. 1. *Harnack Sets.* Let $\mathfrak{A}$ be an interval of

$$\lambda = l_1 + l_2 + \cdots$$

be a positive term series whose sum $\lambda > 0$ is $\leq l$.
Cantor's set, I, 272, let us place a *black* interval of
middle of $\mathfrak{A}$. In a similar manner let us place in
maining or *white* intervals, a black interval, whos
$= l_2$. Let us continue in this way; we get an enu
black intervals $\mathfrak{B}$, and obviously

$$\widehat{\mathfrak{B}} = \lambda.$$

If we omit the end points from each of the black i
a set $\mathfrak{B}^*$, and obviously

$$\widehat{\mathfrak{B}}^* = \lambda.$$

The set

$$\mathfrak{H} = \mathfrak{A} - \mathfrak{B}^*$$

we call a *Harnack set.* This is complete by 324; a

$$\overline{\mathfrak{H}} = \widehat{\mathfrak{H}} = l - \lambda.$$

When $\lambda = l$, $\mathfrak{H}$ is discrete, and the set reduces t
to Cantor's set. When $\lambda < l$, we get an apanta

3. It is easy to extend Harnack sets to $\mathfrak{R}_n$. For example, in
S be the unit square. On two of its adjacent sides let us p
ngruent Harnack sets $\mathfrak{H}$. We now draw lines through the
nts of the black intervals parallel to the sides. There res
enumerable set of black squares $\mathfrak{S} = \{S_n\}$. The sides of
uares $\mathfrak{S}$ and their limiting points form obviously an apanta
rfect set $\mathfrak{K}$.

Let

$$a_1^2 + a_2^2 + \cdots = m$$

a series whose sum $0 < m \leq 1$.

We can choose $\mathfrak{H}$ such that the square corresponding to its l
black interval has the area a_1^2; the four squares correspond
the next two largest black intervals have the total area a_2^2,

Then

$$\overline{\overline{\mathfrak{S}}} = \Sigma a_n^2 = m.$$

Hence

$$\overline{\overline{\mathfrak{K}}} = 1 - m = \overline{\mathfrak{K}}.$$

355. 1. If $\mathfrak{E} = \{\mathfrak{e}_m\}$ is an enclosure of $\mathfrak{A}$ such that

$$\Sigma \widehat{\mathfrak{e}}_m - \overline{\overline{\mathfrak{A}}} < \epsilon,$$

is called an *ϵ-enclosure.* Let A be the complement of $\mathfrak{A}$ w
pect to the metric set $\mathfrak{M}$. Let $E = \{e_n\}$ be an ϵ-enclosure o
e call $\mathfrak{E}$, E *complementarg ϵ-enclosures belonging to* $\mathfrak{A}$.

2. *If* $\mathfrak{A}$ *is measurable, then each pair of complementary*
rmal enclosures $\mathfrak{E}$, E, *whose divisor* $\mathfrak{D} = Dv(\mathfrak{E}, E)$, *is such th*

$$\widehat{\widehat{\mathfrak{D}}} < \epsilon, \qquad \epsilon \textit{ small at pleasure.}$$

For let $\mathfrak{E}$, E be any pair of complementary $\epsilon/2$ normal en
res. Then

$$\widehat{\widehat{\mathfrak{E}}} - \widehat{\widehat{\mathfrak{A}}} < \frac{\epsilon}{2} \quad , \quad \widehat{\widehat{E}} - \widehat{\widehat{A}} < \frac{\epsilon}{2}.$$

356. 1. Up to the present we have used only
of a set $\mathfrak{A}$. If the cells enclosing $\mathfrak{A}$ are measu
enclosure *measurable.*

Let $\mathfrak{E} = \{\mathfrak{e}_n\}$ be a measurable enclosure. If th
to any two of its cells form a null set, we
overlapping. The terms distinct, normal, g
change.

2. *We prove now that* $\overline{\overline{\mathfrak{A}}} = \text{Min}\, \Sigma\widehat{\mathfrak{e}}_n,$

with respect to the class of non-overlapping measura

For, as in 339, there exists a metric enclosu
each $\mathfrak{e}_n$ such that $\sum_\kappa \widehat{d}_{n\kappa}$ differs from $\widehat{\mathfrak{e}}_n$ by $< \epsilon/2$
$\{\mathfrak{m}_n\}$ forms a metric enclosure of $\mathfrak{A}$. Thus

$$\overline{\overline{\mathfrak{A}}} \leq \sum_{n,\kappa} \widehat{d}_{n,\kappa} < \Sigma\left(\widehat{\mathfrak{e}}_n + \frac{\epsilon}{2^n}\right) = \Sigma\widehat{\mathfrak{e}}_n + \epsilon,$$

which establishes 1).

357. *Let $\mathfrak{E}$ be a distinct measurable enclosure of*
those cells containing points of the complement A.
there exists an $\mathfrak{E}$ such that $\widehat{\mathfrak{f}} < \epsilon$, then $\mathfrak{A}$ is measurab

For let $\mathfrak{E} = \mathfrak{e} + \mathfrak{f}$. Then $\mathfrak{e} \leq \mathfrak{A}$. Hence $\mathfrak{e} \leq \underline{\underline{\mathfrak{A}}}$

$$\overline{\overline{\mathfrak{A}}} \leq \widehat{\mathfrak{E}} = \widehat{\mathfrak{e}} + \widehat{\mathfrak{f}} \leq \underline{\underline{\mathfrak{A}}} + \epsilon.$$

Hence

$$\overline{\overline{\mathfrak{A}}} - \underline{\underline{\mathfrak{A}}} < \epsilon,$$

and thus

$$\overline{\overline{\mathfrak{A}}} = \underline{\underline{\mathfrak{A}}}.$$

358. 1. *The divisor* $\mathfrak{D}$ *of two measurable*

Now $\mathfrak{G} = Dv(\mathfrak{E}, \mathfrak{F})$ is a normal metric enclosure over its cells $\mathfrak{g}$ which contain points of $\mathfrak{D}$ and $C($ the cells of $\mathfrak{e}$, $\mathfrak{f}$. Hence

$$\widehat{\mathfrak{g}} \leq \widehat{\mathfrak{e}} + \widehat{\mathfrak{f}} < \frac{\epsilon}{2} + \frac{\epsilon}{2}.$$

Thus by 357, $\mathfrak{D}$ is measurable.

2. *Let* $\mathfrak{A}$, $\mathfrak{B}$ *be measurable.*

Let $$\mathfrak{D} = Dv(\mathfrak{A}, \mathfrak{B}) \quad , \quad \mathfrak{U} = (\mathfrak{A}, \mathfrak{B}).$$

Then $$\widehat{\mathfrak{A}} + \widehat{\mathfrak{B}} = \widehat{\mathfrak{U}} + \widehat{\mathfrak{D}}.$$

For $$\mathfrak{U} = \mathfrak{A} + (\mathfrak{B} - \mathfrak{D}).$$

Hence $$\widehat{\mathfrak{U}} = \widehat{\mathfrak{A}} + \text{Meas}\,(\mathfrak{B} - \mathfrak{D})$$
$$= \widehat{\mathfrak{A}} + \widehat{\mathfrak{B}} - \widehat{\mathfrak{D}}.$$

359. *Let* $\mathfrak{A} = U\{\mathfrak{A}_m\}$ *be the union of an enu measurable cells; moreover let* $\mathfrak{A}$ *be limited. Then* $\mathfrak{A}$ *If we set* $$\mathfrak{B}_1 = \mathfrak{A}_1 \quad , \quad (\mathfrak{A}_1, \mathfrak{A}_2) = \mathfrak{B}_1 + \mathfrak{B}_2,$$
$$(\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3) = \mathfrak{B}_1 + \mathfrak{B}_2 + \mathfrak{B}_3, \textit{ etc.},$$
then $$\widehat{\mathfrak{A}} = \Sigma \widehat{\mathfrak{B}}_m.$$

For $\mathfrak{D} = Dv(\mathfrak{A}_1, \mathfrak{A}_2)$ is measurable by 358.

Let $$\mathfrak{A}_1 = \mathfrak{D} + \mathfrak{a}_1 \quad , \quad \mathfrak{A}_2 = \mathfrak{D} + \mathfrak{a}_2.$$

Then $\mathfrak{a}_1$, $\mathfrak{a}_2$ are measurable by 352, 2.

As $$\mathfrak{U} = (\mathfrak{A}_1, \mathfrak{A}_2) = \mathfrak{D} + \mathfrak{a}_1 + \mathfrak{a}_2,$$

$\mathfrak{U}$ is measurable. As $\mathfrak{U}$ and $\mathfrak{B}_1$ are measurable, s similar manner we show that $\mathfrak{B}_3$, $\mathfrak{B}_4 \cdots$ are measura

For let
$$\mathfrak{a}_2 = \mathfrak{A}_2 - \mathfrak{A}_1 \quad , \quad \mathfrak{a}_3 = \mathfrak{A}_3 - \mathfrak{A}_2 \cdots$$

For uniformity let us set $\mathfrak{a}_1 = \mathfrak{A}$. Then
$$\mathfrak{A} = \Sigma \mathfrak{a}_m.$$

As each $\mathfrak{a}_n$ is measurable
$$\widehat{\mathfrak{A}} = \Sigma \widehat{\mathfrak{a}}_m$$
$$= \lim_{n=\infty} (\widehat{\mathfrak{a}}_1 + \cdots + \widehat{\mathfrak{a}}_n$$
$$= \lim \widehat{\mathfrak{A}}_n .$$

361. *Let* $\mathfrak{A}_1, \mathfrak{A}_2 \cdots$ *be measurable and their unio* $\mathfrak{D} = Dv \{\mathfrak{A}_n\} > 0$, *it is measurable.*

For let $\mathfrak{A}$ lie in the metric set $\mathfrak{M}$;

let
$$\mathfrak{D} + D = \mathfrak{M} \quad , \quad \mathfrak{A}_n + A_n = \mathfrak{M}$$
as usual.

Now $\mathfrak{D}$ denoting the points common to all the D can lie in all of the $\mathfrak{A}_n$, hence it lies in some on A_n. Thus
$$D \leq \{A_n\}.$$

On the other hand, a point of $\{A_n\}$ lies in so does not lie in $\mathfrak{A}_m$. Hence it does not lie in $\mathfrak{D}$. D. Hence
$$\{A_n\} \leq D.$$

From 1), 2) we have
$$D = \{A_n\}.$$

As each A_n is measurable, so is D. Hence $\mathfrak{D}$ is

362. *If* $\mathfrak{A}_1 \geq \mathfrak{A}_2 \geq \cdots$ *is an enumerable set of m gates, their divisor* $\mathfrak{D}$ *is measurable, and*

As $$\mathfrak{D} = \mathfrak{M} - D,$$

we have
$$\widehat{\mathfrak{D}} = \widehat{\mathfrak{M}} - \widehat{D}$$
$$= \lim (\widehat{\mathfrak{M}} - \widehat{A}_n)$$
$$= \lim \widehat{\mathfrak{A}}_n.$$

363. 1. The points $x = (x_1 \cdots x_m)$ such that

$$a_1 \leq x_1 \leq b_1 \quad , \quad \cdots \quad , \quad a_m \leq x_m \leq$$

form a *standard rectangular cell*, whose edges h

$$e_1 = b_1 - a_1 \quad , \quad \cdots \quad , \quad e_m = b_m -$$

When $e_1 = e_2 = \cdots = e_m$, the cell is a *standar* enclosure of the limited set $\mathfrak{A}$, whose cells $\mathfrak{E} =$ cells, is called a *standard enclosure.*

2. *For each* $\epsilon > 0$, *there are standard* ϵ-*enclos* *set* $\mathfrak{A}$.

For let $\mathfrak{E} = \{\mathfrak{e}_n\}$ be any η-enclosure of $\mathfrak{A}$. T

$$\Sigma \widehat{\mathfrak{e}}_n - \overline{\overline{\mathfrak{A}}} < \eta.$$

Each $\mathfrak{e}_n$ being metric, may be enclosed in t standard outer enclosure F_n, such that

$$\widehat{F}_n - \widehat{\mathfrak{e}}_n < \eta / 2^n \quad , \quad n = 1, 2, \cdot\cdot$$

Then $\mathfrak{F} = \{F_n\}$ is an enclosure of $\mathfrak{A}$, and

$$\Sigma \widehat{F}_n < \Sigma (\widehat{\mathfrak{e}}_n + \eta / 2^n) = \Sigma \widehat{\mathfrak{e}}_n + \eta$$

$$< \overline{\overline{\mathfrak{A}}} + 2\eta, \qquad \text{by 2).}$$

But the enclosure F can be replaced by

Let
$$\mathfrak{e}_m = (\mathfrak{e}_{m,1}, \ \mathfrak{e}_{m,2}, \ \mathfrak{e}_{m,3} \cdots) + e_m$$

then e_m is measurable. By this process the me
cell $\mathfrak{e}_m$ falls into an enumerable set of non-ov
able cells, as indicated in 3). If we suppose th
take place for each cell of $\mathfrak{C}$, we shall say we ha
on $\mathfrak{C}$.

364. (*W. H. Young.*) *Let* $\mathfrak{C}$ *be any comple*
Then
$$\underline{\underline{\mathfrak{A}}} = \text{Max } \overline{\mathfrak{C}}.$$

For let $\mathfrak{A}$ lie within a cube $\mathfrak{M}$, and let $A =$
be as usual the complementary sets.

Let $\mathfrak{B} = \{\mathfrak{b}_n\}$ be a border set of $\mathfrak{C}$ [328].
overlapping enclosure of C; we may suppose i
closure of C. Let E be a standard ϵ-enclosu
superimpose E on $\mathfrak{B}$, getting a measurable enc
and A. Then
$$C = C_\Delta \geq A_\Delta.$$

Hence
$$\mathfrak{C} = \mathfrak{M} - C = \mathfrak{M} - C_\Delta \leq \mathfrak{M} - A$$

Thus
$$\overline{\mathfrak{C}} = \overline{\overline{\mathfrak{C}}}, \qquad \text{by } 338$$
$$\leq \overline{\text{Meas}} \ (\mathfrak{M} - A_\Delta)$$
$$\leq \widehat{\mathfrak{M}} - \widehat{A}_\Delta, \qquad \text{by}$$
$$\leq \widehat{\mathfrak{M}} - \overline{\overline{A}}.$$

Hence
$$\overline{\mathfrak{C}} \leq \underline{\underline{\mathfrak{A}}},$$

and thus

Let $$\mathfrak{K} = \mathfrak{M} - A_D + \mathfrak{F},$$

where $\mathfrak{F}$ denotes the frontier points of A_D lying
$\mathfrak{K}$ is complete. Since each face of D is a null s
Thus each set on the right of 4) is measurable, h

$$\widehat{\mathfrak{K}} = \widehat{\mathfrak{M}} - \widehat{A}_D + \widehat{\mathfrak{F}}$$
$$= \widehat{\mathfrak{M}} - \widehat{A}_D$$
$$= \widehat{\mathfrak{M}} - \underline{A} - \epsilon' \quad , \quad 0 \leq$$
$$= \underline{\mathfrak{A}} - \epsilon'.$$

Thus $$\text{Max } \mathfrak{C} \geq \mathfrak{K} = \widehat{\mathfrak{K}} > \underline{\mathfrak{A}} - \epsilon,$$

from which follows 3), since ϵ is small at pleasur

365. 1. *If $\mathfrak{A}$ is complete, it is measurable, and*

$$\widehat{\mathfrak{A}} = \mathfrak{A}.$$

For by 364,

$$\underline{\mathfrak{A}} = \mathfrak{A}.$$

On the other hand,

$$\overline{\mathfrak{A}} = \mathfrak{A}, \quad \text{by 338.}$$

2. *Let $\mathfrak{B}$ be any measurable set in the limited s*

$$\underline{\mathfrak{A}} = \text{Max } \widehat{\mathfrak{B}}.$$

For $$\underline{\mathfrak{A}} \geq \underline{\mathfrak{B}} = \widehat{\mathfrak{B}}.$$

Hence, $$\underline{\mathfrak{A}} \geq \text{Max } \widehat{\mathfrak{B}}.$$

But the class of measurable components of
class of complete components $\mathfrak{C}$, since each $\mathfrak{C}$ is

We throw the points $\mathfrak{J}$ into two classes $\mathfrak{A} = \{a\}$,
the following properties:

1° To each a corresponds a point b symmetrica
to M, and conversely.

2° If a falls in the segment δ of D_n, each of
ments δ' of D_n shall contain a point a' of $\mathfrak{A}$ such tha
in δ' as a is situated in δ.

3° Each δ of D_n shall contain a point a' of $\mathfrak{A}$
situated in δ, as any given point a of $\mathfrak{A}$ is situated i

4° $\mathfrak{A}$ shall contain a point a situated in $\mathfrak{C}$ as a
a' of $\mathfrak{A}$ is in any δ_n.

The 1° condition states that $\mathfrak{A}$ goes over into $\mathfrak{B}$
about M. The 2° condition states that $\mathfrak{A}$ falls in
$2^3, \cdots$ congruent subsets. The 3° condition states
$\mathfrak{A}_n$ of $\mathfrak{A}$ in δ_n goes over into $\mathfrak{A}$ on stretching it in t
The condition 4° states that $\mathfrak{A}$ goes over into $\mathfrak{A}_n$ on
in the ratio $1 : 2^n$.

We show now that $\mathfrak{A}$, and therefore $\mathfrak{B}$ are not m
the first place, we note that

$$\bar{\bar{\mathfrak{A}}} = \bar{\bar{\mathfrak{B}}},$$

by 1°. As $\mathfrak{J} = \mathfrak{A} + \mathfrak{B}$, if $\mathfrak{A}$ or $\mathfrak{B}$ were measurable, t
be, and

$$\widehat{\mathfrak{A}} = \widehat{\mathfrak{B}} = \tfrac{1}{2}.$$

Thus if we show $\bar{\bar{\mathfrak{A}}}$ or $\bar{\bar{\mathfrak{B}}} = 1$, neither $\mathfrak{A}$ nor $\mathfrak{B}$
We show this by proving that if $\bar{\bar{\mathfrak{A}}} = \alpha < 1$, then $\mathfrak{B}$ i
set, and $\widehat{\mathfrak{B}} = 1$. But when $\mathfrak{B}$ is measurable, $\widehat{\mathfrak{B}} = \frac{1}{2}$
we are led to a contradiction.

Let $\epsilon = \epsilon_1 + \epsilon_2 + \cdots$ be a positive term series w
small at pleasure. Let $\mathfrak{E}_1 = \{e_n\}$ be a non-overlappi

Each interval e_n contains one or more interv
some D_s, such that

$$\sum_m \widehat{\eta}_{nm} = \widehat{e}_n - \sigma_n \quad , \quad 0 \leq \sigma_n$$

where

$$\sigma = \Sigma \sigma_n$$

may be taken small at pleasure.

Now each η_{nm} has a subset $\mathfrak{A}_{nm}$ of $\mathfrak{A}$ enti
Hence there exists an enclosure $\mathfrak{E}_{nm}$ of $\mathfrak{A}_{nm}$, wh
such that

$$\frac{\alpha_{nm}}{\alpha_1} = \frac{\widehat{\eta}_{nm}}{1} \quad , \quad \text{or } \alpha_{nm} = \alpha_1 \widehat{\eta}_{nm} .$$

But $\mathfrak{E}_2 = \{\mathfrak{E}_{nm}\}$ is a non-overlapping enclo
measure

$$\alpha_2 = \alpha_1 \sum_{n,\, m} \widehat{\eta}_{nm} = \alpha_1 \sum_n (\widehat{e}_n - \sigma_n)$$

$$= \alpha_1^2 - \sigma \alpha_1 = \alpha^2 + \epsilon_2' \quad , \quad 0 \leq \epsilon_2'$$

if σ is taken sufficiently small.

Let $\mathfrak{B}_2$ denote the irrational points in $\mathfrak{E}_1 - \mathfrak{E}$
$\mathfrak{B}$, and $\mathfrak{B}_2$ has no point in common with $\mathfrak{B}_1$. W

$$\widehat{\mathfrak{B}}_2 = \widehat{\mathfrak{E}}_1 - \widehat{\mathfrak{E}}_2 = \alpha_1 - \alpha_2$$

$$= \alpha + \epsilon_1' - \alpha^2 - \epsilon_2'$$

$$> \alpha(1 - \alpha) - \epsilon_2 .$$

In this way we may continue. Thus $\mathfrak{B}$ contai
component

$$\mathfrak{B}_1 + \mathfrak{B}_2 + \cdots$$

whose measure is

$$> (1 - \alpha)\{1 + \alpha + \alpha^2 + \cdots\} - \Sigma\epsilon$$

$$> 1 - \epsilon .$$

For by 365, 2, there exists in the sets 1), m

$$\mathfrak{E}_1 \quad , \quad \mathfrak{E}_2 \quad , \quad \mathfrak{E}_3 \cdots$$

each of whose measures $\mathfrak{E}_n > \alpha$. Let us co
these sets, viz.:

$$\mathfrak{E}_1 \quad , \quad \mathfrak{E}_2 \quad \cdots \quad \mathfrak{E}_n.$$

The points common to any two of the sets
set $\mathfrak{D}_{\iota\kappa}$ by 358, 1. Hence the union $\mathfrak{E}_{1n} = \{\mathfrak{D}$
359. The difference of one of the sets 3), as
is a measurable set $\mathfrak{c}_1$ which contains no point
remaining sets of 3). Moreover

$$\hat{\mathfrak{c}}_1 > \alpha - \hat{\mathfrak{E}}_{1n}.$$

In the same way we may reason with the
of 3). Thus $\mathfrak{A}$ contains n measurable sets $\mathfrak{c}$
which have a common point.

Hence

$$\mathfrak{c} = \mathfrak{c}_1 + \cdots + \mathfrak{c}_n$$

is a measurable set and

$$\overline{\mathfrak{A}} \geq \hat{\mathfrak{c}} > n(\alpha - \hat{\mathfrak{E}}_{1n}).$$

The first and last members give

$$\hat{\mathfrak{E}}_{1n} > \alpha - \frac{1}{n}\overline{\mathfrak{A}}.$$

Thus however small $\alpha > 0$ may be, there exist

$$\hat{\mathfrak{E}}_{1,\,\mu} \quad \left(1 - \frac{\epsilon}{2}\right)\alpha.$$

Let us now group the sets 2) in sets of μ.

such that the points of each set in 6) lie in at le
5), and hence in at least 2^2 of the sets 1), an
measures are.

$$>\left(1-\frac{\epsilon}{2}\right)\left(1-\frac{\epsilon}{2^2}\right)\alpha > (1-\epsilon)\alpha$$

In this way we may continue indefinitely. I
union of all the points of $\mathfrak{A}$, common to at least
Let $\mathfrak{B}_2$ be the union of the points of $\mathfrak{A}$ commo
the sets 1), etc. In this way we get the sequen

$$\mathfrak{B}_1 \geq \mathfrak{B}_2 \geq \cdots$$

each of which contains a measurable set w
$>(1-\epsilon)\alpha$.

We have now only to apply 25 and 364.

368. As corollaries of 367 we have:

1. *Let $\mathfrak{D}_1, \mathfrak{D}_2 \cdots$ be an infinite enumerable set
cubes whose union is limited. Let each $\bar{\mathfrak{D}}_n > \alpha$
exists a set of points $\mathfrak{d}$ whose cardinal number is c
ity of the $\mathfrak{D}_n$ and such that $\underline{\mathfrak{d}} \geq \alpha$.*

2. *(Arzelà.) Let $y_1, y_2 \cdots \doteq \eta$. On each lin
enumerable set of intervals of length δ_n. Should t
vals ν_n on the lines y_n be finite, let $\nu_n \doteq \infty$. In a
$n = 1, 2, \cdots$ and the projections of these intervals
Then there exists at least one point $x = \xi$ in $\mathfrak{A}$, suc
through ξ is cut by an infinity of these intervals.*

Associate Sets

369. 1. Let $\epsilon_1 > \epsilon_2 > \epsilon_3 \cdots \doteq 0.$

2. *Each outer associated set* $\mathfrak{A}_e$ *is measurable, and*

$$\overline{\overline{\mathfrak{A}}} = \widehat{\mathfrak{A}}_e = \lim_{n=\infty} \widehat{\mathfrak{E}}_n.$$

r each $\mathfrak{E}_n$ is measurable; hence $\mathfrak{A}_e$ is measurable by 362, and

$$\begin{aligned}\mathfrak{A}_e &= \lim \mathfrak{E}_n \\ &= \lim(\overline{\overline{\mathfrak{A}}} + \epsilon_n'), \qquad 0 \leq \epsilon_n' < \epsilon_n \\ &= \overline{\overline{\mathfrak{A}}}, \qquad \text{as } \epsilon_n \doteq 0.\end{aligned}$$

370. 1. Let A be the complement of $\mathfrak{A}$ with respect to s be $\mathfrak{Q}$ containing $\mathfrak{A}$. Let A_e be an outer associated set of nen

$$\mathfrak{A}_\iota = \mathfrak{Q} - A_e$$

called *an inner associated set of* $\mathfrak{A}$. Obviously

$$\mathfrak{A}_\iota \leq \mathfrak{A}.$$

2. *The inner associated set* $\mathfrak{A}_\iota$ *is measurable, and*

$$\widehat{\mathfrak{A}}_\iota = \underline{\underline{\mathfrak{A}}}.$$

For A_e is measurable by 369, 2. Hence $\mathfrak{A}_\iota = \mathfrak{Q} - A_e$ is me able. But

$$\widehat{A}_e = \overline{\overline{A}}$$

369, 2. Hence

$$\widehat{\mathfrak{A}}_\iota = \widehat{\mathfrak{Q}} - \widehat{A}_e = \widehat{\mathfrak{Q}} - \overline{\overline{A}} = \underline{\underline{\mathfrak{A}}}.$$

Separated Sets

371. Let $\mathfrak{A}$, $\mathfrak{B}$ be two limited point sets. If there e asurable enclosures $\mathfrak{E}$, $\mathfrak{F}$ of $\mathfrak{A}$, $\mathfrak{B}$ such that $\mathfrak{D} = Dv(\mathfrak{E}, \mathfrak{F})$

372. *For $\mathfrak{A}$, $\mathfrak{B}$ to be separated, it is necessary and*

$$\mathfrak{D} = Dv(\mathfrak{A}_e, \mathfrak{B}_e)$$

is a null set.

It is sufficient. For let

$$\mathfrak{C} = (\mathfrak{A}, \mathfrak{B}) \quad , \quad \mathfrak{A}_e = \mathfrak{D} + \mathfrak{a}, \qquad \mathfrak{B}_e = \mathfrak{D} +$$

Then

$$\mathfrak{E} = (\mathfrak{a}, \mathfrak{b}, \mathfrak{D})$$

is a measurable enclosure of $\mathfrak{C}$, consisting of thr
cells. Of these only $\mathfrak{D}$ contains points of both $\mathfrak{A}$
hypothesis $\mathfrak{D}$ is a null set. Hence $\mathfrak{A}$, $\mathfrak{B}$ are separat

It is necessary. For let $\mathfrak{M}$ be a null distinct e
such that those of its cells $\mathfrak{N}$, containing points o
null set. Let us superimpose $\mathfrak{M}$ on the enclosure
ting an enclosure $\mathfrak{F}$ of $\mathfrak{A}$.

The cells of $\mathfrak{F}$ arising from $\mathfrak{a}$ contain no point of
the cells arising from $\mathfrak{b}$ contain no point of $\mathfrak{A}$.
hand, the cells arising from $\mathfrak{D}$, split up into three cl

$$\mathfrak{D}_a \quad , \quad \mathfrak{D}_b \quad , \quad \mathfrak{D}_{ab} .$$

The first contains no point of $\mathfrak{B}$, the second no p
cells of the last contain both points of $\mathfrak{A}$, $\mathfrak{B}$. As $\mathfrak{D}_a$

$$\widehat{\mathfrak{D}}_{a,\,b} = 0.$$

On the other hand,

$$\mathfrak{A}_e = \mathfrak{a} + \mathfrak{D} \geq \mathfrak{A} ;$$

hence

$$\mathfrak{a} + \mathfrak{D}_a + \mathfrak{D}_{ab} \geq \mathfrak{A}.$$

Thus

$$\widehat{\mathfrak{a}} + \widehat{\mathfrak{D}}_a \geq \overline{\overline{\mathfrak{A}}},$$

by 1). Also

$$\widehat{\mathfrak{A}}_e = \widehat{\mathfrak{a}} + \widehat{\mathfrak{D}} = \overline{\overline{\mathfrak{A}}} \qquad \text{by 369,}$$

This with 2) gives

373. 1. *If* $\mathfrak{A}$, $\mathfrak{B}$ *are separated, then* $\mathfrak{D} = Dv(\mathfrak{A},$
For $\mathfrak{D}_e = Dv(\mathfrak{A}_e, \mathfrak{B}_e)$ is a null set by 372. But

2. Let $\mathfrak{A}$, $\mathfrak{B}$ be the Van Vleck sets in 366. W
$\bar{\bar{\mathfrak{A}}} = \bar{\bar{\mathfrak{B}}} = 1$. Then by 369, 2, $\widehat{\mathfrak{A}}_e = \widehat{\mathfrak{B}}_e = 1$. The di
not a null set. Hence by 372, $\mathfrak{A}$, $\mathfrak{B}$ are not separ
condition that $\mathfrak{D}$ *be a null set is necessary, but not s*

374. 1. *Let* $\{\mathfrak{A}_n\}$, $\{\mathfrak{B}_n\}$ *be separated divisio*
$\mathfrak{C}_{\iota\kappa} = Dv(\mathfrak{A}_\iota, \mathfrak{B}_\kappa)$. *Then* $\{\mathfrak{C}_{\iota\kappa}\}$ *is a separated divis*

We have to show there exists a null enclosure o
sets $\mathfrak{C}_{\iota\kappa}$, $\mathfrak{C}_{mn}$. Now $\mathfrak{C}_{\iota\kappa}$ lies in $\mathfrak{A}_\iota$ and $\mathfrak{B}_\kappa$; also $\mathfrak{C}_m$
By hypothesis there exists a null enclosure $\mathfrak{E}$ of $\mathfrak{A}$
enclosure $\mathfrak{F}$ of $\mathfrak{B}_\kappa$, $\mathfrak{B}_n$. Then $\mathfrak{G} = Dv(\mathfrak{E}, \mathfrak{F})$ is a
$\mathfrak{A}_\iota$, $\mathfrak{A}_m$ and of $\mathfrak{B}_\kappa$, $\mathfrak{B}_n$. Thus those cells of $\mathfrak{G}$, ca
taining points of both $\mathfrak{A}_\iota$, $\mathfrak{A}_m$ form a null set; and
$\mathfrak{G}_b$, containing points of both $\mathfrak{B}_\kappa$, $\mathfrak{B}_n$ also form a n

Let $G = \{g\}$ denote the cells of $\mathfrak{G}$ that contain
$\mathfrak{C}_{\iota\kappa}$, $\mathfrak{C}_{mn}$. Then a cell g contains points of $\mathfrak{A}_\iota\, \mathfrak{A}_m$
lies in $\mathfrak{G}_a$ or $\mathfrak{G}_b$. Thus in either case G is a null s
form a separated division of $\mathfrak{A}$.

2. Let D be a separated division of $\mathfrak{A}$ into th
Let E be another separated division of $\mathfrak{A}$ into t
We have seen that $F = \{f_{\iota\kappa}\}$ where $f_{\iota\kappa} = Dv(d_\iota, e$
rated division of $\mathfrak{A}$. We shall say that F is obtai
posing E on D or D on E, and write $F = D + E =$

3. Let E be a separated division of the separat
of $\mathfrak{A}$, while D is a separated division of $\mathfrak{A}$. If d_ι i
a cell of E, and $d_{\iota\kappa} = Dv(d_\iota, e_\kappa)$, then

$$d_\iota = (d_{\iota 1}, d_{\iota 2}, \cdots) + \delta_\iota.$$

For let $\mathfrak{D}$ be a null enclosure of $\mathfrak{A}_m$, $\mathfrak{A}_n$. Let
cells of $\mathfrak{D}$ containing points of both $\mathfrak{A}_m$, $\mathfrak{A}_n$. Le
cells of $\mathfrak{D}$ containing points of $\mathfrak{B}$; let $\mathfrak{E}_{a,b}$ denote
taining points of both $\mathfrak{B}_m$, $\mathfrak{B}_n$. Then

$$\mathfrak{E}_{ab} \leq \mathfrak{D}_{ab}.$$

As $\mathfrak{D}_{ab}$ is a null set, so is $\mathfrak{E}_{ab}$.

376. 1. *Let* $\mathfrak{A} = (\mathfrak{B}, \mathfrak{C})$ *be a separated division of*

$$\overline{\overline{\mathfrak{A}}} = \overline{\overline{\mathfrak{B}}} + \overline{\overline{\mathfrak{C}}}.$$

For let $\epsilon_1 > \epsilon_2 > \cdots \doteq 0$. There exist ϵ_n-measur
of $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$; call them respectively A_n, B_n, C_n. T
$B_n + C_n$ is an ϵ_n-enclosure of $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ simultaneousl

Since $\mathfrak{B}$, $\mathfrak{C}$ are separated, there exist enclosure
such that those cells of $D = B + C$ containing p
and $\mathfrak{C}$ form a null set. Let us now superpose D
an ϵ_n-enclosure $E_n = \{e_{ns}\}$ of $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ simultanec
denote the cells of E_n containing points of $\mathfrak{B}$ a
cells containing only points of $\mathfrak{C}$; and e_{bc} those c
points of both $\mathfrak{B}$, $\mathfrak{C}$. Then

$$\sum_s \widehat{e}_{ns} = \Sigma \widehat{e}_{bn} + \Sigma \widehat{e}_{cn} + \Sigma \widehat{e}_{bc}.$$

As $\Sigma e_{bc} = 0$, we see that as $n \doteq \infty$,

$$\Sigma \widehat{e}_{ns} \doteq \overline{\overline{\mathfrak{A}}} \quad , \quad \Sigma \widehat{e}_{bn} \doteq \overline{\overline{\mathfrak{B}}} \quad , \quad \Sigma \widehat{e}_{cn} \doteq \overline{\overline{\mathfrak{C}}}.$$

Hence passing to the limit $n = \infty$, in 2) we get 1

2. *Let* $\mathfrak{A} = \{\mathfrak{B}_n\}$ *be a separated division of limitec*

On the other hand, by 1

$$\overline{\overline{\mathfrak{A}}}_n = \overline{\overline{\mathfrak{B}}}_1 + \cdots + \overline{\overline{\mathfrak{B}}}_n = B_n,$$

the sum of the first n terms of the series 2). Thus

$$B_n \leq \overline{\overline{\mathfrak{A}}},$$

and hence B is convergent by 80, 4. Thus

$$B \leq \overline{\overline{\mathfrak{A}}}.$$

On the other hand, by 339,

$$B \geq \overline{\overline{\mathfrak{A}}}.$$

The last two relations give 1).

CHAPTER XII

LEBESGUE INTEGRALS

General Theory

377. In the foregoing chapters we have developed a the
integration which rests on the notion of content. In this c
we propose to develop a theory of integration due to Leb
which rests on the notion of measure. The presentatio
given differs considerably from that of Lebesgue. As the
will see, the theory of Lebesgue integrals as here presented
from that of the theory of ordinary integrals only in emp
an infinite number of cells instead of a finite number.

378. In the following we shall suppose the field of integ
$\mathfrak{A}$ to be limited, as also the integrand $\mathfrak{A}$ lies in $\mathfrak{R}_m$ and for b
we set $f(x) = f(x_1 \cdots x_m)$. Let us effect a separated divi
$\mathfrak{A}$ into cells $\delta_1, \delta_2 \cdots$. If each cell δ_ι lies in a cube of side
shall say D is a *separated division of norm d.*

As before, let

$$M_\iota = \operatorname{Max} f \quad , \quad m_\iota = \operatorname{Min} f \quad , \quad \omega_\iota = \operatorname{Osc} f = M_\iota - m_\iota \quad \text{i}$$

Then
$$\overline{S}_D = \Sigma M_\iota \overline{\overline{\delta}}_\iota \quad , \quad \underline{S}_D = \Sigma m_\iota \overline{\overline{\delta}}_\iota ,$$

the summation extending over all the cells of $\mathfrak{A}$, are call
upper and lower sums of f over $\mathfrak{A}$ with respect to D.

Hence $$\Sigma m\bar{\bar{\delta}}_\iota \le \Sigma m_\iota \bar{\bar{\delta}}_\iota \le \Sigma M_\iota \bar{\bar{\delta}}_\iota \le \Sigma M \bar{\bar{\delta}}_\iota .$$

Thus $$m\Sigma\bar{\bar{\delta}}_\iota \le \underline{S}_D \le \overline{S}_D \le M\Sigma\bar{\bar{\delta}}_\iota .$$

But $$\Sigma\bar{\bar{\delta}}_\iota = \bar{\bar{\mathfrak{A}}},$$

by 376, 2.

380. 1. Since f is limited in $\mathfrak{A}$,

$$\text{Max } \underline{S}_D \quad , \quad \text{Min } \overline{S}_D$$

with respect to the class of all separated divisio
finite. We call them respectively the *lower and*
integrals of f over the field $\mathfrak{A}$, and write

$$\underline{\mathcal{L}\!\!\int_{\mathfrak{A}}} f = \text{Max } \underline{S}_D \quad ; \quad \overline{\mathcal{L}\!\!\int_{\mathfrak{A}}} f = \text{Min } \overline{S}_D .$$

In order to distinguish these new integrals fror
we have slightly modified the old symbol $\int$ to rese
script L, or $\mathcal{L}\!\!\int$, in honor of the author of these inte

If $$\underline{\mathcal{L}\!\!\int_{\mathfrak{A}}} f = \overline{\mathcal{L}\!\!\int_{\mathfrak{A}}} f$$

we say f is *L-integrable over* $\mathfrak{A}$, and denote the com

$$\mathcal{L}\!\!\int_{\mathfrak{A}} f,$$

which we call the *L-integral.*

The integrals treated of in Vol. I we will call
integrals in the sense of Riemann.

2. *Let f be limited over the null set $\mathfrak{A}$. Then f is*

For let $d_1, d_2 \cdots$ be an unmixed metric or complet
$\mathfrak{A}$ of norm d. Let each cell d_ι be split up into the s
$d_{\iota 1}, d_{\iota 2} \cdots$

Then since d_ι is complete or metric,

$$\overline{d}_\iota = \overline{\overline{d}}_\iota = \Sigma \overline{\overline{d}}_{\iota\kappa}.$$

Hence using the customary notation,

$$m_\iota \overline{\overline{d}}_{\iota\kappa} \leq m_{\iota\kappa} \overline{\overline{d}}_{\iota\kappa} \leq M_{\iota\kappa} \overline{\overline{d}}_{\iota\kappa} \leq M_\iota \overline{\overline{d}}_{\iota\kappa}.$$

Thus summing over κ,

$$m_\iota \overline{d}_\iota \leq \sum_\kappa m_{\iota\kappa} \overline{\overline{d}}_{\iota\kappa} \leq \sum_\kappa M_{\iota\kappa} \overline{\overline{d}}_{\iota\kappa} \leq M_\iota \overline{d}_\iota.$$

Summing over ι gives

$$\Sigma m_\iota \overline{d}_\iota \leq \sum_{\iota\kappa} m_{\iota\kappa} \overline{\overline{d}}_{\iota\kappa} \leq \sum_{\iota\kappa} M_{\iota\kappa} \overline{\overline{d}}_{\iota\kappa} \leq \Sigma M_\iota \overline{d}_\iota.$$

Thus by definition,

$$\Sigma m_\iota \overline{d}_\iota \leq \overline{\int_{\mathfrak{A}}} f \leq \Sigma M_\iota \overline{d}_\iota.$$

Letting now $d \doteq 0$, we get 1).

2. *Let $\mathfrak{A}$ be metric or complete. If f is R-integra*
L-integrable and

$$\int_{\mathfrak{A}} f = \underline{\int_{\mathfrak{A}}} f.$$

3. In case that $\mathfrak{A}$ is not metric or complete, the re
may not hold.

Example 1. Let $\mathfrak{A}$ denote the rational points in
$(0, 1)$.

Let

$$f = 1, \text{ for } x = \frac{m}{n}, \ n \text{ even}$$

Example 2. Let $f = 1$ at the rational points $\mathfrak{A}$ in

$$\overline{\int_{\mathfrak{A}}} f = 1 \quad , \quad \underline{\int_{\mathfrak{A}}} f = 0 \quad , \quad \text{and} \quad \underline{\int_{\mathfrak{A}}} f < \int$$

Let $g = -1$ in $\mathfrak{A}$. Then

$$\underline{\int_{\mathfrak{A}}} g = 0 \quad , \quad \int_{\mathfrak{A}} g = -1 \quad , \quad \text{and} \quad \int_{\mathfrak{A}} g <$$

Thus in 3) the L-integral is less than the R-int
4) it is greater.

Example 3. Let $f = 1$ at the irrational point
Then

$$\underline{\int_{\mathfrak{A}}} f = \int_{\mathfrak{A}} f,$$

although $\mathfrak{A}$ is neither metric nor complete.

382. *Let D, Δ be separated divisions of $\mathfrak{A}$. Let*

$$E = D + \Delta = \{e_\iota\}.$$

Then

$$\overline{S}_E \leq \overline{S}_D, \overline{S}_\Delta$$

$$\underline{S}_E \geq \underline{S}_D, \underline{S}_\Delta.$$

For any cell d_ι of D splits up into $d_{\iota 1}, d_{\iota 2} \cdots$ on
Δ, and

$$\overline{\overline{d}}_\iota = \sum_\kappa \overline{\overline{d}}_{\iota\kappa}.$$

But

$$M_{\iota\kappa} \overline{\overline{d}}_{\iota\kappa} \leq M_\iota \overline{\overline{d}}_{\iota\kappa},$$

and

$$m_{\iota\kappa} \overline{\overline{d}}_{\iota\kappa} \geq m_\iota \overline{\overline{d}}_{\iota\kappa}.$$

Thus

$$\overline{S}_E \leq \overline{S}_D \quad , \quad \underline{S}_E \geq \underline{S}_D.$$

383. 1. *Extremal Sequences.* There exists a seq

For let $\epsilon_1 > \epsilon_2 > \cdots \doteq 0$. For each ϵ_n, there ex
E_n such that

$$0 < \overline{S}_{E_n} - \overline{\int_{\mathfrak{A}}} < \epsilon_n.$$

Let
$$E_2 + D_1 = D_2 \quad , \quad E_3 + D_2 = D_3 , \cdot$$

and for uniformity set $E_1 = D_1$. Then by 382,

$$\overline{S}_{D_{n+1}} \leq \overline{S}_{D_n} \quad , \quad \overline{S}_{D_n} \leq \overline{S}_{E_n}.$$

Hence
$$0 \leq \overline{S}_{D_n} - \overline{\int_{\mathfrak{A}}} < \epsilon_n .$$

Letting $n \doteq \infty$ we get 2).

Thus there exists a sequence $\{D_n'\}$ of the type 1)
sequence $\{D_n''\}$ of the same type for 3). Let now I
Obviously 2), 3) hold simultaneously for the seque

2. The sequence 1) is called an *extremal sequence*

3. *Let* $\{D_n\}$ *be an extremal sequence, and* E *any* s
sion of $\mathfrak{A}$. *Let* $E_n = D_n + E$. *Then* $E_1, E_2 \cdots$
sequence also.

384. *Let* f *be L-integrable in* $\mathfrak{A}$. *Then for any ext*
$\{D_n\}$,

$$\int_{\mathfrak{A}} f = \lim_{n \doteq \infty} \Sigma f(\xi_\iota) \overline{\overline{d}}_\iota ,$$

where d_ι *are the cells of* D_n, *and* ξ_ι *any point of* $\mathfrak{A}$ *in*

For
$$m_\iota \leq f(\xi_\iota) \leq M_\iota .$$

Hence
$$S_{D_n} \leq \Sigma f(\xi_\iota) \overline{\overline{d}}_\iota \leq \overline{S}_{D_n}$$

2. *Let* $F = \text{Max}\,|f|$ *in* $\mathfrak{A}$, *then*

$$\left|\int_{\mathfrak{A}} f\right| \leq F\overline{\overline{\mathfrak{A}}}.$$

This follows from 1.

386. *In order that* $\bar{f}$ *be L-integrable in* $\mathfrak{A}$, *it is necessary that, for each extremal sequence* $\{D_n\}$,

$$\lim_{n=\infty} \Omega_{D_n} f = 0\,;$$

and it is sufficient if there exists a sequence of superimposed separated divisions $\{E_n\}$, *such that*

$$\lim_{n=\infty} \Omega_{E_n} f = 0.$$

It is *necessary*. For

$$\underline{\int_{\mathfrak{A}}} = \lim \underline{S}_{D_n} \quad , \quad \overline{\int_{\mathfrak{A}}} = \lim \overline{S}_{D_n}.$$

As f is L-integrable,

$$0 = \overline{\int_{\mathfrak{A}}} - \underline{\int_{\mathfrak{A}}} = \lim (\overline{S}_{D_n} - \underline{S}_{D_n}) = \lim \Omega_{D_n} f.$$

It is *sufficient*. For

$$\underline{S}_{E_n} \leq \overline{\int_{\mathfrak{A}}} \leq \overline{S}_{E_n}.$$

Both $\{\underline{S}_{E_n}\}$, $\{\overline{S}_{E_n}\}$ are limited monotone sequences. Their limits therefore exist. Hence

$$0 = \lim \Omega_{E_n} = \lim \overline{S}_{E_n} - \lim \underline{S}_{E_n}.$$

Thus

$$\underline{\int_{\mathfrak{A}}} = \overline{\int_{\mathfrak{A}}}.$$

387. *In order that f be L-integrable, it is necessary and sufficient that for each* $\epsilon > 0$, *there exists a separated division D of* $\mathfrak{A}$, *for which*

$$\Omega_D f < \epsilon. \qquad (1$$

It is *necessary*. For by 386, there exists an extremal sequence $\{D_n\}$, such that

$$0 \leq \Omega_{D_n} f < \epsilon \quad , \quad \text{for any } n \geq \text{some } m.$$

Thus we may take D_m for D.

It is *sufficient*. For let $\epsilon_1 > \epsilon_2 > \cdots \doteq 0$. Let $\{D_n\}$ be an extremal sequence for which

$$0 \leq \Omega_{D_n} f < \epsilon_n.$$

Let $\Delta_1 = D_1$, $\Delta_2 = \Delta_1 + D_2$, $\Delta_3 = \Delta_2 + D_3 \cdots$ Then $\{\Delta_n\}$ is a set of superimposed separated divisions, and obviously

$$\Omega_{\Delta_n} f < \epsilon_n \doteq 0.$$

Hence f is L-integrable by 386.

388. *In order that f be L-integrable, it is necessary and sufficient that, for each pair of positive numbers ω, σ there exists a separated division D of $\mathfrak{A}$, such that if $\eta_1, \eta_2, \cdots$ are those cells in which* Osc $f > \omega$, *then*

$$\Sigma \bar{\eta}_\iota < \sigma. \tag{1}$$

It is *necessary*. For by 387 there exists a separated division $D = \{\delta_\iota\}$ for which

$$\Omega_D f = \Sigma \omega_\iota \bar{\delta}_\iota < \omega\sigma. \tag{2}$$

If $\theta_1, \theta_2 \cdots$ denote the cells of D in which Osc $f \leq \omega$,

$$\Omega_D f = \Sigma \omega_\iota \bar{\eta}_\iota + \Sigma \omega_\iota \bar{\theta}_\iota \geq \omega \Sigma \bar{\eta}_\iota. \tag{3}$$

This in 2) gives 1).

It is *sufficient*. For taking $\epsilon > 0$ small at pleasure, let us then take

$$\sigma = \frac{\epsilon}{2\,\Omega}, \quad \omega = \frac{\epsilon}{2\,\bar{\mathfrak{A}}}, \tag{4}$$

where $\Omega =$ Osc f in $\mathfrak{A}$.

From 1), 3), and 4) we have, since $\omega_\iota \leq \Omega$,

$$\Omega_D f \leq \Sigma \Omega \bar{\eta}_\iota + \Sigma \omega_\iota \bar{\theta}_\iota < \sigma\Omega + \Sigma \omega \bar{\theta}_\iota < \sigma\Omega + \omega \bar{\mathfrak{A}} = \epsilon.$$

We now apply 387.

389. 1. *If f is L-integrable in $\mathfrak{A}$, it is in $\mathfrak{B} < \mathfrak{A}$.*

For let $\{D_n\}$ be an extremal sequence of f relative to $\mathfrak{A}$. Then by 386,

$$\Omega_{D_n} f \doteq 0. \tag{1}$$

But the sequence $\{D_n\}$ defines a sequence of superposed separated divisions of $\mathfrak{B}$, which we denote by $\{E_n\}$. Obviously

$$\Omega_{E_n} f \leq \Omega_{D_n} f.$$

Hence by 1),

$$\Omega_{E_n} f \doteq 0,$$

and f is L-integrable in $\mathfrak{B}$ by 386.

2. *If f is L-integrable in $\mathfrak{A}$, so is $|f|$.*

The proof is analogous to I, 507, using an extremal sequence for f.

390. 1. *Let $\{\mathfrak{A}_n\}$ be a separated division of $\mathfrak{A}$ into a finite or infinite number of subsets. Let f be limited in $\mathfrak{A}$. Then*

$$\overline{\underline{\int}}_{\mathfrak{A}} f = \overline{\underline{\int}}_{\mathfrak{A}_1} f + \overline{\underline{\int}}_{\mathfrak{A}_2} f + \cdots \tag{1}$$

For *let us* 1° *suppose* that the subsets $\mathfrak{A}_1 \cdots \mathfrak{A}_r$ are finite in number. Let $\{D_n\}$ be an extremal sequence of f relative to $\mathfrak{A}$, and $\{D_{mn}\}$ an extremal sequence relative to $\mathfrak{A}_m$. Let

$$E_n = D_n + D_{1n} + \cdots + D_{rn}.$$

Then $\{E_n\}$ is an extremal sequence of f relative to $\mathfrak{A}$, and also relative to each $\mathfrak{A}_m$.

Now

$$\overline{\underline{S}}_{\mathfrak{A}, E_n} = \overline{\underline{S}}_{\mathfrak{A}_1, E_n} + \cdots + \overline{\underline{S}}_{\mathfrak{A}_r, E_n}.$$

Letting $n \doteq \infty$, we get 1), for this case.

Let now r be infinite. We have

$$\overline{\overline{\mathfrak{A}}} = \sum_1^\infty \overline{\overline{\mathfrak{A}}}_m. \tag{2}$$

Let

$$\mathfrak{B}_n = (\mathfrak{A}_1 \cdots \mathfrak{A}_n) \quad , \quad \mathfrak{C}_n = \mathfrak{A} - \mathfrak{B}_n.$$

Then $\mathfrak{B}_n$, $\mathfrak{C}_n$ form a separated division of $\mathfrak{A}$, and

$$\overline{\overline{\mathfrak{A}}} = \overline{\overline{\mathfrak{B}}}_n + \overline{\overline{\mathfrak{C}}}_n.$$

If ν is taken large enough, 2) shows that

$$\overline{\overline{\mathfrak{C}}}_n < \frac{\epsilon}{M} \quad , \quad n \geq \nu \quad , \quad M = \operatorname{Max} |f| \quad \text{in } \mathfrak{A}.$$

Thus by case 1°,

$$\overline{\int_{\mathfrak{A}}} f = \overline{\int_{\mathfrak{B}_n}} f + \overline{\int_{\mathfrak{C}_n}} f$$

$$= \overline{\int_{\mathfrak{A}_1}} + \cdots + \overline{\int_{\mathfrak{A}_n}} + \epsilon', \tag{3}$$

where by 385, 2

$$|\epsilon'| \leq M\overline{\overline{\mathfrak{C}}}_n < \epsilon \quad , \quad n \geq \nu.$$

Thus 1) follows from 3) in this case.

2. *Let $\{\mathfrak{A}_n\}$ be a separated division of $\mathfrak{A}$. Then*

$$\int_{\mathfrak{A}} f = \Sigma \int_{\mathfrak{A}_n} f,$$

if f is L-integrable in $\mathfrak{A}$, or if it is in each $\mathfrak{A}_n$, and limited in $\mathfrak{A}$.

391. 1. *Let $f = g$ in $\mathfrak{A}$ except at the points of a null set $\mathfrak{N}$. Then*

$$\overline{\int_{\mathfrak{A}}} f = \overline{\int_{\mathfrak{A}}} g. \tag{1}$$

For let

$$\mathfrak{A} = \mathfrak{B} + \mathfrak{N}. \quad \text{Then}$$

$$\overline{\int_{\mathfrak{A}}} f = \overline{\int_{\mathfrak{B}}} f + \overline{\int_{\mathfrak{N}}} f = \overline{\int_{\mathfrak{B}}} f. \tag{2}$$

Similarly

$$\overline{\int_{\mathfrak{A}}} g = \overline{\int_{\mathfrak{B}}} g. \tag{3}$$

But $f = g$ in $\mathfrak{B}$. Thus 2), 3) give 1).

392. 1. *If $c > 0$;* $\quad \overline{\underline{\int}} cf = c \overline{\underline{\int}} f.$

If $c < 0$; $\quad \underline{\int} cf = c \overline{\int} f, \qquad \overline{\int} cf = c \underline{\int} f.$

The proof is similar to 3, 8, using extremal sequences.

2. *If f is L-integrable in $\mathfrak{A}$, so is cf, and*

$$\int_{\mathfrak{A}} cf = c \int_{\mathfrak{A}} f,$$

where c is a constant.

393. 1. *Let* $F(x) = f_1(x) + \cdots + f_n(x)$, *each* f_m *being limited in* $\mathfrak{A}$. *Then*

$$\sum_1^n \underline{\int_{\mathfrak{A}}} f_m \leq \underline{\int_{\mathfrak{A}}} F \leq \overline{\int_{\mathfrak{A}}} F \leq \sum_1^n \overline{\int_{\mathfrak{A}}} f_m. \tag{1}$$

For let $\{D_n\}$ be an extremal sequence common to $F, f_1, \cdots f_n$. In each cell

$$d_{n1} \quad , \quad d_{n2} \cdots$$

of D_n we have

$$\Sigma \operatorname{Min} f_m \leq \operatorname{Min} F \leq \operatorname{Max} F \leq \Sigma \operatorname{Max} f_m.$$

Multiplying by $\overline{\overline{d}}_{ns}$, summing over s and then letting $n \doteq \infty$, gives 1).

2. *If* $f_1(x), \cdots f_n(x)$ *are each L-integrable in* $\mathfrak{A}$, *so is*

$$F = c_1 f_1 + \cdots + c_n f_n,$$

and

$$\int_{\mathfrak{A}} F = c_1 \int_{\mathfrak{A}} f_1 + \cdots + c_n \int_{\mathfrak{A}} f_n.$$

394. 1. $$\underline{\int_{\mathfrak{A}}}(f+g) \leq \underline{\int_{\mathfrak{A}}} f + \overline{\int_{\mathfrak{A}}} g \leq \overline{\int_{\mathfrak{A}}}(f+g).$$

For using the notation of 393,

$$\operatorname{Min}(f+g) \leq \operatorname{Min} f + \operatorname{Max} g \leq \operatorname{Max}(f+g)$$

in each cell d_{ns} of D_n.

2. *If* g *is L-integrable in* $\mathfrak{A}$,

$$\overline{\underline{\int_{\mathfrak{A}}}}(f+g) = \overline{\underline{\int_{\mathfrak{A}}}} f + \int_{\mathfrak{A}} g.$$

Reasoning similar to 3, 4, using extremal sequences.

3. $$\underline{\int_{\mathfrak{A}}}(f-g) \leq \underline{\int_{\mathfrak{A}}} f - \underline{\int_{\mathfrak{A}}} g.$$

$$\overline{\int_{\mathfrak{A}}} f - \overline{\int_{\mathfrak{A}}} g \leq \overline{\int_{\mathfrak{A}}}(f-g).$$

For
$$\underline{\int}_{\mathfrak{A}}(f-g) < \underline{\int}_{\mathfrak{A}} f + \overline{\int}_{\mathfrak{A}}(-g) \leq \underline{\int}_{\mathfrak{A}} f - \underline{\int}_{\mathfrak{A}} g;$$
etc.

4. *If f, g are L-integrable in $\mathfrak{A}$, so is $f-g$, and*
$$\int_{\mathfrak{A}}(f-g) = \int_{\mathfrak{A}} f - \int_{\mathfrak{A}} g.$$

395. *If f, g are L-integrable in $\mathfrak{A}$, so is $f \cdot g$.*

Also their quotient f/g is L-integrable provided it is limited in $\mathfrak{A}$.

The proof of the first part of the theorem is analogous to I, 505, using extremal sequences common to both f and g. The proof of the second half is obvious and is left to the reader.

396. 1. *Let f, g be limited in $\mathfrak{A}$, and $f \leq g$, except possibly in a null set $\mathfrak{N}$. Then*
$$\overline{\underline{\int}}_{\mathfrak{A}} f \leq \overline{\underline{\int}}_{\mathfrak{A}} g. \tag{1}$$

Let us *suppose first* that $f \leq g$ everywhere in $\mathfrak{A}$.

Let $\{D_n\}$ be an extremal sequence common to both f and g. Then
$$\underline{\overline{S}}_{D_n} f \leq \underline{\overline{S}}_{D_n} g.$$

Letting $n \doteq \infty$, we get 1).

We consider now the *general case*. Let $\mathfrak{A} = \mathfrak{B} + \mathfrak{N}$. Then
$$\overline{\underline{\int}}_{\mathfrak{A}} f = \overline{\underline{\int}}_{\mathfrak{B}} f \quad , \quad \overline{\underline{\int}}_{\mathfrak{A}} g = \overline{\underline{\int}}_{\mathfrak{B}} g,$$
since
$$\overline{\underline{\int}}_{\mathfrak{N}} f = \overline{\underline{\int}}_{\mathfrak{N}} g = 0.$$

But in $\mathfrak{B}$, $f \leq g$ without exception. We may therefore use the result of case 1°.

2. *Let $f \geq 0$ in $\mathfrak{A}$. Then*
$$\operatorname{Min} g \cdot \underline{\int}_{\mathfrak{A}} f \leq \overline{\underline{\int}}_{\mathfrak{A}} f \cdot g \leq \operatorname{Max} g \cdot \overline{\underline{\int}}_{\mathfrak{A}} f.$$

For
$$f \cdot \operatorname{Min} g \leq fg \leq f \operatorname{Max} g.$$

397. *The relations of 4 also hold for L-integrals, viz.:*

$$\left|\overline{\underline{\int}}_{\mathfrak{A}} f\right| \leq \overline{\int}_{\mathfrak{A}} |f|. \tag{1}$$

$$\overline{\underline{\int}}_{\mathfrak{A}} f \leq \left|\overline{\int}_{\mathfrak{A}} f\right|. \tag{2}$$

$$-\underline{\int}_{\mathfrak{A}} |f| \leq \overline{\int}_{\mathfrak{A}} f < \overline{\int}_{\mathfrak{A}} |f|. \tag{3}$$

$$-\overline{\int}_{\mathfrak{A}} |f| \leq \underline{\int}_{\mathfrak{A}} f \leq \underline{\int}_{\mathfrak{A}} |f|. \tag{4}$$

The proof is analogous to that employed for the R-integrals, using extremal sequences.

398. *Let $\mathfrak{A} = (\mathfrak{B}_u, \mathfrak{C}_u)$ be a separated division for each $u \doteq 0$. Let $\overline{\overline{\mathfrak{C}}}_u \doteq 0$. Then*

$$\lim_{u=0} \overline{\underline{\int}}_{\mathfrak{B}_u} f = \overline{\underline{\int}}_{\mathfrak{A}} f.$$

For by 390, 1,

$$\overline{\underline{\int}}_{\mathfrak{A}} = \overline{\underline{\int}}_{\mathfrak{B}_u} + \overline{\underline{\int}}_{\mathfrak{C}_u}.$$

But by 385, 2, the last integral $\doteq 0$, since $\overline{\overline{\mathfrak{C}}}_u \doteq 0$, and since f is limited.

399. *Let f be limited and continuous in $\mathfrak{A}$, except possibly at the points of a null set $\mathfrak{N}$. Then f is L-integrable in $\mathfrak{A}$.*

Let us *first* take $\mathfrak{N} = 0$. Then f is continuous in $\mathfrak{A}$. Let $\mathfrak{A}$ lie in a standard cube $\mathfrak{Q}$. If $\text{Osc}\, f$ is not $< \epsilon$ in $\mathfrak{A}$, let us divide $\mathfrak{Q}$ into 2^n cubes. If in one of these cubes

$$\text{Osc}\, f < \epsilon, \tag{1}$$

let us call it a *black cube*. A cube in which 1) does not hold we will call *white*. Each white cube we now divide in 2^n cubes. These we call black or white according as 1) holds for them or does not. In this way we continue until we reach a stage where all cubes are black, or if not we continue indefinitely. In the latter case, we get an infinite enumerable set of cubes

$$q_1, q_2, q_3 \cdots \tag{2}$$

Each point a of $\mathfrak{A}$ lies in at least one cube 2). For since f is continuous at $x = a$,

$$|f(x) - f(a)| < \epsilon/2 \quad , \quad x \text{ in } V_\delta(a).$$

Thus when the process of division has been carried so far that the diagonals of the corresponding cubes are $< \delta$, the inequality 1) holds for a cube containing a. This cube is a black cube.

Thus, in either case, each point of $\mathfrak{A}$ lies in a black cube.

Now the cubes 2) effect a separated division D of $\mathfrak{A}$, and in each of its cells 1) holds. Hence f is L-integrable in $\mathfrak{A}$.

Let us now suppose $\mathfrak{N} > 0$. We set

$$\mathfrak{A} = \mathfrak{C} + \mathfrak{N}.$$

Then f is L-integrable in $\mathfrak{C}$ by case 1°. It is L-integrable in $\mathfrak{N}$ by 380, 2. Then it is L-integrable in $\mathfrak{A}$ by 390, 1.

2. If f is L-integrable in $\mathfrak{A}$, we cannot say that the points of discontinuity of f form a null set.

Example. Let $f = 1$ at the irrational points $\mathfrak{J}$, in $\mathfrak{A} = (0, 1)$;

$$= 0 \text{ at the other points } \mathfrak{R}, \text{ in } \mathfrak{A}.$$

Then each point of $\mathfrak{A}$ is a point of discontinuity. But here

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{R}} + \int_{\mathfrak{J}} = \int_{\mathfrak{J}} = 1,$$

since $\mathfrak{R}$ is a null set. Thus f is L-integrable.

400. *If $f(x_1 \cdots x_m)$ has limited variation in $\mathfrak{A}$, it is L-integrable.*

For let D be a cubical division of space of norm d. Then by I, 709, there exists a fixed number V, such that

$$\Sigma \omega_\iota d^{m-1} \leq V$$

for any D. Let ω, σ be any pair of positive numbers. We take d such that

$$d < \frac{\sigma\omega}{V}. \tag{1}$$

Let d'_ι denote those cells in which $\operatorname{Osc} f \geq \omega$, and let the number of these cells be ν. Let η_ι denote the points of $\mathfrak{A}$ in d'_ι. Then

$$\nu\omega d^{m-1} \leq \Sigma \omega_\iota d^{m-1} \leq V.$$

Hence
$$\nu \leq \frac{V}{\omega d^{m-1}}. \qquad (2$$

Thus
$$\Sigma \overline{\overline{\eta}}_\iota \leq \nu d^m \leq \frac{V d^m}{\omega d^{m-1}} \quad , \quad \text{by 2)},$$
$$\leq \frac{Vd}{\omega} < \sigma \quad , \quad \text{by 1)}.$$

Hence f is L-integrable by 388.

401. *Let*
$$\phi = f, \ in \ \mathfrak{A} < \mathfrak{B};$$
$$= 0, \ in \ A = \mathfrak{B} - \mathfrak{A}.$$

Then
$$\int_{\mathfrak{A}} f = \int_{\mathfrak{B}} \phi, \qquad (1$$

if 1°, ϕ *is L-integrable in* $\mathfrak{B}$; *or* 2°, f *is L-integrable in* $\mathfrak{A}$, *and* $\mathfrak{A}$, A *are separated parts of* $\mathfrak{B}$.

On the 1° *hypothesis* let $\{\mathfrak{E}_s\}$ be an extremal sequence of ϕ. Let the cells of $\mathfrak{E}_s$ be $e_1, e_2 \cdots$ They effect a separated division of $\mathfrak{A}$ into cells $d_1, d_2 \cdots$ Let m_ι, M_ι be the extremes of f in d_ι and n_ι, N_ι the extremes of ϕ in e_ι. Then for those cells containing at least a point of $\mathfrak{A}$,
$$n_\iota \overline{\overline{e}}_\iota \leq m_\iota \overline{\overline{d}}_\iota \leq M_\iota \overline{\overline{d}}_\iota \leq N_\iota \overline{\overline{e}}_\iota, \qquad (2$$
is obviously true when $e_\iota = d_\iota$. Let $d_\iota < e_\iota$. If $m_\iota \leq 0$,
$$n_\iota \overline{\overline{e}}_\iota \leq m_\iota \overline{\overline{d}}_\iota, \qquad \text{since } m_\iota = n_\iota. \qquad (3$$

If $m_\iota > 0$, $\qquad n_\iota = 0$, and 3) holds.

If $M_\iota \leq 0$, $\qquad M_\iota \overline{\overline{d}}_\iota \leq N_\iota \overline{\overline{e}}_\iota$, $\qquad$ since $N_\iota = 0$. $\qquad$ (4

If $M_\iota > 0$, 4) still holds, since $M_\iota = N_\iota$.

Thus 2) holds in all these cases. Summing 2) gives
$$\sum_{\mathfrak{B}} n_\iota \overline{\overline{e}}_\iota \leq \underline{\int_{\mathfrak{A}}} f \leq \overline{\int_{\mathfrak{A}}} f \leq \sum_{\mathfrak{B}} N_\iota e_\iota$$
for the division $\mathfrak{E}_s$, since in a cell e of $\mathfrak{E}$, containing no point of $\mathfrak{A}$, $\phi = 0$. Letting $s \doteq \infty$, we get 1), since the end members
$$\doteq \int_{\mathfrak{B}} \phi.$$

On the 2° hypothesis,

$$\int_{\mathfrak{B}} \phi = \int_{\mathfrak{A}} \phi + \int_{A} \phi = \int_{\mathfrak{A}} \phi = \int_{\mathfrak{A}} f,$$

since ϕ being $= 0$ in A, is L-integrable, and we can apply 390.

402. 1. If

$$\int_{\mathfrak{A}} f = 0,$$

we call f a *null function in* $\mathfrak{A}$.

2. *If $f \geq 0$ is a null function in $\mathfrak{A}$, the points $\mathfrak{P}$ where $f > 0$ form a null set.*

For let $\mathfrak{A} = \mathfrak{Z} + \mathfrak{P}$, so that $f = 0$ in $\mathfrak{Z}$.

By 401,

$$0 = \int_{\mathfrak{A}} f = \int_{\mathfrak{P}} f. \tag{1}$$

Let $\epsilon_1 > \epsilon_2 > \cdots \doteq 0$. Let $\mathfrak{P}_n$ denote the points of $\mathfrak{P}$ where $f \geq \epsilon_n$. Then

$$\int_{\mathfrak{P}} \geq \int_{\mathfrak{P}_n} = 0, \qquad \text{by 1)}.$$

Each $\mathfrak{P}_n$ is a null set. For

$$\int_{\mathfrak{P}_n} \geq \epsilon_n \overline{\overline{\mathfrak{P}}}_n = 0.$$

Hence $\overline{\overline{\mathfrak{P}}}_n = 0$.

Then
$$\mathfrak{P} = \{\mathfrak{P}_n\} = Q_1 + Q_2 + \cdots.$$

where $\quad Q_1 = \mathfrak{P}_1, \quad Q_2 = \mathfrak{P}_2 - \mathfrak{P}_1, \quad Q_3 = \mathfrak{P}_3 - Q_3 \cdots$

As each Q_n is a null set, $\mathfrak{P}$ is a null set.

Integrand Sets

403. Let $\mathfrak{A}$ be a limited point set lying in an m-way space $\Re_m$. Let $f(x_1 \cdots x_m)$ be a limited function defined over $\mathfrak{A}$. Any point of $\mathfrak{A}$ may be represented by

$$a = (a_1 \cdots a_m).$$

The point
$$x = (a_1 \cdots a_m x_{m+1})$$
lies in an $m+1$ way space $\mathfrak{R}_{m+1}$. The set of points $\{x\}$ in which x_{m+1} ranges from $-\infty$ to $+\infty$ is called an *ordinate through a*. If x_{m+1} is restricted by
$$0 \leq x_{m+1} \leq l,$$
we shall call the ordinate a *positive ordinate of length l*; if it is restricted by
$$-l \leq x_{m+1} \leq 0,$$
it is a *negative* ordinate. The set of ordinates through all the points a of $\mathfrak{A}$, each having a length $= f(a)$, and taken positively or negatively, as $f(a)$ is $\gtrless 0$, form a point set $\mathfrak{J}$ in $\mathfrak{R}_{m+1}$ which we call an *integrand set*. The points of $\mathfrak{J}$ for which x_{m+1} has a fixed value $x_{m+1} = c$ form a *section of* $\mathfrak{J}$, and is denoted by $\mathfrak{J}(c)$ or by $\mathfrak{J}_c$.

404. *Let $\mathfrak{A} = \{a\}$ be a limited point set in $\mathfrak{R}_m$. Through each point a, let us erect a positive ordinate of constant length l, getting a set $\mathfrak{O}$, in $\mathfrak{R}_{m+1}$. Then*
$$\overline{\overline{\underline{\underline{\mathfrak{O}}}}} = l\overline{\overline{\underline{\underline{\mathfrak{A}}}}}. \tag{1}$$

For let $\mathfrak{E}_1 > \mathfrak{E}_2 > \cdots$ form a standard sequence of enclosures of $\mathfrak{O}$, such that
$$\widehat{\mathfrak{E}}_n \doteq \overline{\overline{\mathfrak{O}}}. \tag{2}$$

Let us project each section of $\mathfrak{E}_n$ corresponding to a given value of x_{m+1} on $\mathfrak{R}_m$, and let $\mathfrak{A}_n$ be their divisor. Then $\mathfrak{A}_n \geq \mathfrak{A}$. Thus
$$\overline{\overline{\mathfrak{O}}} \leq \overline{\overline{\mathfrak{A}}}l \leq \overline{\overline{\mathfrak{A}}}_n l \leq \widehat{\mathfrak{E}}_n.$$
Letting $n \doteq \infty$, and using 2), we get
$$\overline{\overline{\mathfrak{O}}} = \overline{\overline{\mathfrak{A}}} \cdot l.$$

To prove the rest of 1), let O be the complement of $\mathfrak{O}$ with respect to some standard cube $\mathfrak{Q}$ in $\mathfrak{R}_{m+1}$, of base Q in $\mathfrak{R}_m$.

Then, as just shown,
$$\overline{\overline{O}} = l\overline{\overline{A}} \quad , \quad \text{where } A = Q - \mathfrak{A}.$$
Hence
$$\begin{aligned}\underline{\underline{\mathfrak{O}}} &= \widehat{\mathfrak{Q}} - \overline{\overline{O}} = \widehat{Q}l - l\overline{\overline{A}} \\ &= l\{\widehat{Q} - (\widehat{Q} - \underline{\underline{\mathfrak{A}}})\} \\ &= l\underline{\underline{\mathfrak{A}}}.\end{aligned}$$

405. *Let $f \geq 0$ be L-integrable in $\mathfrak{A}$. Then*

$$\int_{\mathfrak{A}} f = \overline{\overline{\mathfrak{J}}}, \tag{1}$$

where $\mathfrak{J}$ is the integrand set corresponding to f.

For let $\{\delta_\iota\}$ be a separated division D of $\mathfrak{A}$. On each cell δ_ι erect a cylinder $\mathfrak{C}_\iota$ of height $M_\iota = \text{Max } f$ in δ_ι. Then by 404,

$$\overline{\overline{\mathfrak{C}}}_\iota = \overline{\overline{\delta}}_\iota M_\iota.$$

Let $\mathfrak{C} = \{\mathfrak{C}_\iota\}$; the $\mathfrak{C}_\iota$ are separated. Hence, $\epsilon > 0$ being small at pleasure,

$$\overline{\overline{\mathfrak{J}}} \leq \overline{\overline{\mathfrak{C}}} = \Sigma \overline{\overline{\mathfrak{C}}}_\iota = \Sigma \overline{\overline{\delta}}_\iota M_\iota < \overline{\int_{\mathfrak{A}}} f + \epsilon,$$

for a properly chosen D. Thus

$$\overline{\overline{\mathfrak{J}}} \leq \overline{\int_{\mathfrak{A}}} f. \tag{2}$$

Similarly we find

$$\underline{\int_{\mathfrak{A}}} f \leq \overline{\overline{\mathfrak{J}}}. \tag{3}$$

From 2), 3) follows 1).

406. *Let $f \geq 0$ be L-integrable over the measurable field $\mathfrak{A}$. Then the corresponding integrand set $\mathfrak{J}$ is measurable, and*

$$\widehat{\mathfrak{J}} = \int_{\mathfrak{A}} f. \tag{1}$$

For by 2) in 405,

$$\overline{\overline{\mathfrak{J}}} \leq \overline{\int_{\mathfrak{A}}} f.$$

Using the notation of 405, let $\mathfrak{c}_n$ be a cylinder erected on δ_n of height $m_n = \text{Min } f$ in δ_n. Let $\mathfrak{c} = \{\mathfrak{c}_n\}$. Then $\mathfrak{c} \leq \mathfrak{J}$, and hence

$$\underline{\underline{\mathfrak{c}}} \leq \underline{\underline{\mathfrak{J}}}. \tag{2}$$

But $\mathfrak{A}$ being measurable, each $\mathfrak{c}_n$ is measurable, by 404. Hence $\mathfrak{c}$ is by 359. Thus 2) gives

$$\widehat{\mathfrak{c}} \leq \underline{\underline{\mathfrak{J}}}. \tag{3}$$

Now for a properly chosen D,

$$-\epsilon + \underline{\int_{\mathfrak{A}}} f < \Sigma m_\iota \widehat{\delta}_\iota = \widehat{\mathfrak{c}}.$$

Hence

$$\underline{\int}_{\mathfrak{A}} \leq \hat{\mathfrak{c}}, \tag{4}$$

as ϵ is arbitrarily small. From 2), 3), 4)

$$\underline{\int}_{\mathfrak{A}} f \leq \underline{\underline{\mathfrak{J}}} \leq \overline{\overline{\mathfrak{J}}} \leq \overline{\int}_{\mathfrak{A}} f,$$

from which follows 1).

Measurable Functions

407. Let $f(x_1 \cdots x_m)$ be limited in the limited measurable set $\mathfrak{A}$. Let $\mathfrak{A}_{\lambda\mu}$ denote the points of $\mathfrak{A}$ at which

$$\lambda \leq f < \mu.$$

If each $\mathfrak{A}_{\lambda\mu}$ is measurable, we say *f is measurable in $\mathfrak{A}$*.

We should bear in mind that when f is measurable in $\mathfrak{A}$, necessarily $\mathfrak{A}$ itself is measurable, by hypothesis.

408. 1. *If f is measurable in $\mathfrak{A}$, the points $\mathfrak{C}$ of $\mathfrak{A}$, at which $f = C$, form a measurable set.*

For let $\mathfrak{A}_n$ denote the points where

$$-\epsilon_n + C \leq f < C + \epsilon_n,$$

where

$$\epsilon_1 > \epsilon_2 > \cdots \doteq 0.$$

Then by hypothesis, $\mathfrak{A}_n$ is measurable. But $\mathfrak{C} = Dv\{\mathfrak{A}_n\}$. Hence $\mathfrak{C}$ is measurable by 361.

2. *If f is measurable in $\mathfrak{A}$, the set of points where*

$$\lambda \leq f \leq \mu$$

is measurable, and conversely.

Follows from 1, and 407.

3. *If the points $\mathfrak{A}_\lambda$ in $\mathfrak{A}$ where $f \geq \lambda$ form a measurable set for each λ, f is measurable in $\mathfrak{A}$.*

For $\mathfrak{A}_{\lambda\mu}$ having the same meaning as in 407,

$$\mathfrak{A}_{\lambda\mu} = \mathfrak{A}_\lambda - \mathfrak{A}_\mu.$$

Each set on the right being measurable, so is $\mathfrak{A}_{\lambda\mu}$.

409. 1. *If f is measurable in $\mathfrak{A}$, it is L-integrable.*

For setting $m = \mathrm{Min}\, f$, $M = \mathrm{Max}\, f$ in $\mathfrak{A}$, let us effect a division D of the interval $\mathfrak{F} = (m, M)$ of norm d, by interpolating a finite number of points

$$m_1 < m_2 < m_3 < \cdots$$

Let us call the resulting segments, as well as their lengths,

$$d_1, d_2, d_3 \cdots$$

Let $\mathfrak{A}_\iota$ denote the points of $\mathfrak{A}$ in which

$$m_{\iota-1} \leq f < m_\iota \quad , \quad \iota = 1, 2, \cdots; \ m_0 = m.$$

We now form the sums

$$\underline{s}_D = \Sigma m_{\iota-1} \widehat{\mathfrak{A}}_\iota \quad , \quad \bar{s}_D = \Sigma m_\iota \widehat{\mathfrak{A}}_\iota .$$

Obviously

$$\underline{s}_D \leq \underline{\int}_{\mathfrak{A}} f \leq \overline{\int}_{\mathfrak{A}} f \leq \bar{s}_D . \tag{1}$$

But
$$\begin{aligned}\bar{s}_D - \underline{s}_D &= m_1\widehat{\mathfrak{A}}_1 + m_2\widehat{\mathfrak{A}}_2 + \cdots - \{m\widehat{\mathfrak{A}}_1 + m_1\widehat{\mathfrak{A}}_2 + \cdots\} \\ &= \widehat{\mathfrak{A}}_1(m_1 - m) + \widehat{\mathfrak{A}}_2(m_2 - m_1) + \cdots \\ &\leq d\{\widehat{\mathfrak{A}}_1 + \widehat{\mathfrak{A}}_2 + \cdots\} \\ &\leq d\widehat{\mathfrak{A}} \\ &\doteq 0 \quad , \quad \text{as } d \doteq 0. \end{aligned} \tag{2}$$

We may now apply 387.

2. *If f is measurable in $\mathfrak{A}$*

$$\int_{\mathfrak{A}} f = \lim \Sigma m_{\iota-1} \widehat{\mathfrak{A}}_\iota = \lim \Sigma m_\iota \widehat{\mathfrak{A}}_\iota , \tag{3}$$

using the notation in 1.

This follows from 1), 2) in 1.

3. The relation 3) is taken by Lebesgue as definition of his integrals. His theory is restricted to measurable fields and to measurable functions. For Lebesgue's own development of his theory the reader is referred to his paper, *Intégrale, Longueur, Aire, Annali di Mat.*, Ser. 3, vol. 7 (1902); and to his book, *Leçons sur l'Intégration.* Paris, 1904. He may also consult the excellent account of it in *Hobson's* book, *The Theory of Functions of a Real Variable.* Cambridge, England, 1907.

Semi-Divisors and Quasi-Divisors

410. 1. The convergence of infinite series leads to the two following classes of point sets.

Let
$$F = \Sigma f_\iota(x_1 \cdots x_m) = \sum_1^n f_\iota + \sum_{n+1}^\infty f_\iota = F_n + \bar{F}_n, \tag{1}$$
each f_ι being defined in $\mathfrak{A}$.

Let us take $\epsilon > 0$ small at pleasure, and then fix it.

Let us denote by $\mathfrak{A}_n$ the points of $\mathfrak{A}$ at which
$$-\epsilon \leq \bar{F}_n(x) \leq \epsilon. \tag{2}$$

Of course $\mathfrak{A}_n$ may not exist. We are thus led in general to the sets
$$\mathfrak{A}_1 \quad , \quad \mathfrak{A}_2 \quad , \quad \mathfrak{A}_3 \cdots \tag{3}$$

The complementary set $A_n = \mathfrak{A} - \mathfrak{A}_n$ will denote the points where
$$| \bar{F}_n(x) | > \epsilon. \tag{4}$$

If now F is convergent at x, there exists a ν such that this point lies in
$$\mathfrak{A}_\nu \quad , \quad \mathfrak{A}_{\nu+1} \quad , \quad \mathfrak{A}_{\nu+2} \cdots \tag{5}$$

The totality of the points of convergence forms a set which has this property: corresponding to each of its points x, there exists a ν such that x lies in the set 5). A set having this property is called the *semi-divisor* of the sets 3), and is denoted by
$$\mathrm{Sdv}\, \{\mathfrak{A}_n\}.$$

Suppose now, on the other hand, that 1) does not converge at the point x in $\mathfrak{A}$. Then there exists an infinite set of indices
$$n_1 < n_2 < \cdots \doteq \infty,$$
such that
$$| \bar{F}_{n_s}(x) | > \epsilon.$$

Thus, the point x lies in an infinity of the sets
$$A_1 \quad , \quad A_2 \quad , \quad A_3 \cdots \tag{6}$$

The totality of points such that each lies in an infinity of the sets 6) is called the *quasi-divisor* of 6) and is denoted by
$$\mathrm{Qdv}\, \{A_n\}.$$

Obviously,
$$\mathrm{Sdv}\, \{\mathfrak{A}_n\} + \mathrm{Qdv}\, \{A_n\} = \mathfrak{A}. \tag{7}$$

We may generalize these remarks at once. Since $F(x)$ is nothing but

$$\lim F_n(x),$$

we can apply these notions to the case that the functions $f_\iota(x_1 \cdots x_m)$ are defined in $\mathfrak{A}$, and that

$$\lim f_\iota = \phi.$$

2. We may go still farther and proceed in the following abstract manner.

The divisor $\mathfrak{D}$ of the point sets

$$\mathfrak{A}_1 \quad , \quad \mathfrak{A}_2 \cdots \tag{1}$$

is the set of points lying in all the sets 1).

The totality of points each of which lies in an infinity of the sets 1) is called the quasi-divisor and is denoted by

$$\mathrm{Qdv}\,\{\mathfrak{A}_n\}. \tag{2}$$

The totality of points a, to each of which correspond an index m_a, such that a lies in

$$\mathfrak{A}_{m_a} \quad , \quad \mathfrak{A}_{m_a+1} , \cdots$$

forms a set called the semi-divisor of 1), and is denoted by

$$\mathrm{Sdv}\,\{\mathfrak{A}_n\}. \tag{3}$$

If we denote 2), 3) by $\mathfrak{Q}$ and $\mathfrak{S}$ respectively, we have, obviously,

$$\mathfrak{D} \leq \mathfrak{S} \leq \mathfrak{Q}. \tag{4}$$

3. In the special case that $\mathfrak{A}_1 > \mathfrak{A}_2 > \cdots$ we have

$$\mathfrak{Q} = \mathfrak{S} = \mathfrak{D}. \tag{5}$$

For denoting the complementary sets by the corresponding Roman letters, we have

$$D = A_1 + Dv\,(\mathfrak{A}_1, A_2) + Dv\,(\mathfrak{A}_2, A_3) + \cdots$$

But Q has precisely the same expression.

Thus $\mathfrak{Q} = \mathfrak{D}$, and hence by 4), $\mathfrak{S} = \mathfrak{D}$.

4. *Let* $\mathfrak{A}_n + A_n = \mathfrak{B}$, $n = 1, 2, \cdots$ *Then*

$$\text{Qdv}\,\{\mathfrak{A}_n\} + \text{Sdv}\,\{A_n\} = \mathfrak{B}.$$

For each point b of $\mathfrak{B}$ lies

either 1° only in a finite number of $\mathfrak{A}_n$, or in none at all,
or 2° in an infinite number of $\mathfrak{A}_n$.

In the 1° case, b does not lie in $\mathfrak{A}_s, \mathfrak{A}_{s+1} \cdots$; hence it lies in $A_s, A_{s+1} \cdots$ In the 2° case b lies obviously in $\text{Qdv}\,\{\mathfrak{A}_n\}$.

5. *If* $\mathfrak{A}_1, \mathfrak{A}_2 \cdots$ *are measurable, and their union is limited,*

$$\mathfrak{Q} = \text{Qdv}\,\{\mathfrak{A}_n\} \quad , \quad \mathfrak{S} = \text{Sdv}\,\{\mathfrak{A}_n\}$$

are measurable.

For let $\mathfrak{D}_n = Dv(\mathfrak{A}_n, \mathfrak{A}_{n+1} \cdots)$. Then $\mathfrak{S} = \{\mathfrak{D}_n\}$.

But $\mathfrak{S}$ is measurable, as each $\mathfrak{D}_n$ is. Thus $\text{Sdv}\,\{A_n\}$ is measurable, and hence $\mathfrak{Q}$ is by 4.

6. *Let* $\mathfrak{Q} = \text{Qdv}\,\{\mathfrak{A}_n\}$, *each* $\mathfrak{A}_n$ *being measurable, and their union limited. If there are an infinity of the* $\mathfrak{A}_n$, *say*

$$\mathfrak{A}_{\iota_1}, \mathfrak{A}_{\iota_2} \cdots \quad ; \quad \iota_1 < \iota_2 < \cdots$$

whose measure is $\geq \alpha$, *then*

$$\widehat{\overline{\mathfrak{Q}}} \geq \alpha. \tag{6}$$

For let $\mathfrak{B}_n = (\mathfrak{A}_{\iota_n}, \mathfrak{A}_{\iota_{n+1}} \cdots)$, then $\widehat{\overline{\mathfrak{B}}}_n \geq \alpha$.

Let $\mathfrak{B} = Dv\{\mathfrak{B}_n\}$. As $\mathfrak{B}_n \geq \mathfrak{B}_{n+1}$,

$$\widehat{\overline{\mathfrak{B}}} = \lim \widehat{\overline{\mathfrak{B}}}_n \geq \alpha \tag{7}$$

by 362. As $\mathfrak{Q} \geq \mathfrak{B}$ we have 6) at once, from 7).

Limit Functions

411. *Let*
$$\lim_{t=\tau} f(x_1 \cdots x_m, t_1 \cdots t_n) = \phi(x_1 \cdots x_m),$$

as x *ranges over* $\mathfrak{A}$, τ *finite or infinite. Let* f *be measurable in* $\mathfrak{A}$ *and numerically* $\leq M$, *for each* t *near* τ. *Then* ϕ *is measurable in* $\mathfrak{A}$ *also.*

To prove this we show that the points $\mathfrak{B}$ of $\mathfrak{A}$ where

$$\lambda \leq \phi \leq \mu \tag{1}$$

form a measurable set for each λ, μ. For simplicity let τ be finite. Let $t_1, t_2 \cdots \doteq \tau$; also let $\epsilon_1 > \epsilon_2 > \cdots \doteq 0$. Let $\mathfrak{E}_{n,s}$ denote the points of $\mathfrak{A}$ where

$$\lambda - \epsilon_n \leq f(x, t_s) \leq \mu + \epsilon_n. \tag{2}$$

Then for each point x of $\mathfrak{B}$, there is an s_0 such that 2) holds for any t_s, if $s \geq s_0$. Let $\mathfrak{E}_n = \underset{s}{\mathrm{Sdv}} \{\mathfrak{E}_{ns}\}$. Then $\mathfrak{B} < \mathfrak{E}_n$. But the $\mathfrak{E}_{ns}$ being measurable, $\mathfrak{E}_n$ is by 410, 5. Finally $\mathfrak{B} = Dv \{\mathfrak{E}_n\}$, and hence $\mathfrak{B}$ is measurable.

412. *Let* $\lim\limits_{t=\tau} f(x_1 \cdots x_m, t_1 \cdots t_n) = \phi(x_1 \cdots x_m)$,

for x in $\mathfrak{A}$, and τ finite or infinite. Let $t', t'' \cdots \doteq \tau$. Let each $f_s = f(x, t^{(s)})$ be measurable, and numerically $\leq M$. Let $\phi = f_s + g_s$. Let $\mathfrak{G}_s$ denote the points where

$$|g_s| > \epsilon.$$

Then for each $\epsilon > 0$,

$$\lim_{s=\infty} \widehat{\mathfrak{G}}_s = 0. \tag{1}$$

For by 411, ϕ is measurable, hence g_s is measurable in $\mathfrak{A}$, hence $\mathfrak{G}_s$ is measurable.

Suppose now that 1) does not hold. Then

$$\overline{\lim_{s=\infty}}\, \widehat{\mathfrak{G}}_s = l > 0.$$

Then there are an infinity of the $\mathfrak{G}_s$, as $\mathfrak{G}_{s_1}, \mathfrak{G}_{s_2} \cdots$ whose measures are $\geq \lambda > 0$. Then by 410, 6, the measure of

$$\mathfrak{G} = \mathrm{Qdv}\{\mathfrak{G}_s\}$$

is $\geq \lambda$. But this is not so, since $f_s \doteq \phi$, at each point of $\mathfrak{A}$.

413. 1. *Let* $\lim\limits_{t=\tau} f(x_1 \cdots x_m, t_1 \cdots t_n) = \phi(x_1 \cdots x_m)$,

for x in $\mathfrak{A}$, and τ finite or infinite.

Let

$$t', t'' \cdots \doteq \tau. \tag{1}$$

If each $f_s = f(x, t^{(s)})$ is measurable, and numerically $\leq M$ in $\mathfrak{A}$ for each sequence 1), *then*

$$\int_{\mathfrak{A}} \phi = \lim_{t=\tau} \int_{\mathfrak{A}} f(x, t). \tag{2}$$

For set
$$\phi = f_s + g_s,$$
and let
$$|g_s| \leq N \quad , \quad s = 1, 2 \cdots$$

Then as in 412, ϕ and g_s are measurable in $\mathfrak{A}$. Then by 409, they are L-integrable, and
$$\int_{\mathfrak{A}} \phi = \int_{\mathfrak{A}} f_s + \int_{\mathfrak{A}} g_s. \tag{3}$$

Let $\mathfrak{B}_s$ denote the points of $\mathfrak{A}$, at which
$$|g_s| \geq \epsilon;$$
and let $\mathfrak{B}_s + B_s = \mathfrak{A}$. Then $\mathfrak{B}_s$, B_s are measurable, since g_s is. Thus by 390,
$$\int_{\mathfrak{A}} g_s = \int_{\mathfrak{B}_s} g_s + \int_{B_s} g_s.$$

Hence
$$\left| \int_{\mathfrak{A}} g_s \right| \leq N\widehat{\widehat{\mathfrak{B}}}_s + \epsilon \widehat{\widehat{B}}_s \leq N\widehat{\widehat{\mathfrak{B}}}_s + \epsilon \widehat{\widehat{\mathfrak{A}}}.$$

By 412, $\widehat{\widehat{\mathfrak{B}}}_s \doteq 0$. Thus
$$\lim_{s=\infty} \int_{\mathfrak{A}} g_s = 0.$$

Hence passing to the limit in 3), we get 2), for the sequence 1). Since we can do this for every sequence of points t which $\doteq \tau$, the relation 2) holds.

2. *Let*
$$F = \Sigma f_{\iota_1 \cdots \iota_n}(x_1 \cdots x_m)$$
converge in $\mathfrak{A}$. *If each term* f_ι *is measurable, and each* $|F_\lambda| \leq M$, *then* F *is* L*-integrable, and*
$$\int_{\mathfrak{A}} F = \Sigma \int_{\mathfrak{A}} f_\iota.$$

Iterated Integrals

414. In Vol. I, 732, seq. we have seen that the relation,
$$\int_{\mathfrak{A}} f = \int_{\mathfrak{B}} \overline{\underline{\int}}_{\mathfrak{C}} f,$$
holds when f is R-integrable in the *metric* field $\mathfrak{A}$. This result was extended to *iterable* fields in 14 of the present volume. We

wish now to generalize still further to the case that f is L-integrable in the *measurable* field $\mathfrak{A}$. The method employed is due to *Dr. W. A. Wilson*,* and is essentially simpler than that employed by Lebesgue.

1. Let $x = (z_1 \cdots z_s)$ denote a point in s-way space $\mathfrak{R}_s$, $s = m + n$. If we denote the first m coördinates by $x_1 \cdots x_m$, and the remaining coördinates by $y_1 \cdots y_n$, we have

$$z = (x_1 \cdots x_m \, y_1 \cdots y_n).$$

The points

$$x = (x_1 \cdots x_m \, 0 \, 0 \cdots 0)$$

range over an m-way space $\mathfrak{R}_m$, when z ranges over $\mathfrak{R}_s$. We call *x the projection of z on $\mathfrak{R}_m$*.

Let z range over a point set $\mathfrak{A}$ lying in $\mathfrak{R}_s$, then x will range over a set $\mathfrak{B}$ in $\mathfrak{R}_m$, called *the projection of $\mathfrak{A}$ on $\mathfrak{R}_m$*. The points of $\mathfrak{A}$ whose projection is x is called *the section of $\mathfrak{A}$ corresponding to x*. We may denote it by

$$\mathfrak{A}(x), \text{ or more shortly by } \mathfrak{C}.$$

We write

$$\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$$

to denote that $\mathfrak{A}$ is conceived of as formed of the sections $\mathfrak{C}$, corresponding to the different points of its projection $\mathfrak{B}$.

2. Let $\mathfrak{Q}$ denote a standard cube containing $\mathfrak{A}$, let $\mathfrak{q}$ denote its projection on $\mathfrak{R}_m$. Then $\mathfrak{B} \leq \mathfrak{q}$. Suppose each section $\mathfrak{A}(x)$ is measurable. It will be convenient to let $\widehat{\mathfrak{A}}(x)$ denote a function of x defined over $\mathfrak{q}$ such that

$$\begin{aligned} \widehat{\mathfrak{A}}(x) = \text{Meas } \mathfrak{A}(x) = \widehat{\mathfrak{C}} \qquad & \text{when } x \text{ lies in } \mathfrak{B}, \\ = 0 \qquad & \text{when } x \text{ lies in } \mathfrak{q} - \mathfrak{B}. \end{aligned}$$

This function therefore is equal to the measure of the section of $\mathfrak{A}$ corresponding to the point x, when such a section exists; and when not, the function $= 0$.

When each section $\mathfrak{A}(x)$ is not measurable, we can introduce the functions

$$\overline{\overline{\mathfrak{A}}}(x) \quad , \quad \underline{\underline{\mathfrak{A}}}(x).$$

* Dr. Wilson's results were obtained in August, 1909, and were presented by me in the course of an address which I had the honor to give at the Second Decennial Celebration of Clark University, September, 1909.

Here the first $= \overline{\overline{\mathfrak{C}}}$ when a section exists, otherwise it $= 0$, in $\mathfrak{q}$. A similar definition holds for the other function.

3. Let us note that *the sections*

$$\mathfrak{A}_o(x) \quad , \quad \mathfrak{A}_\iota(x),$$

where $\mathfrak{A}_o$, $\mathfrak{A}_\iota$ are the outer and inner associated sets belonging to $\mathfrak{A}$, are always measurable.

For $\mathfrak{A}_o = Dv\{\mathfrak{E}_n\}$, where each $\mathfrak{E}_n$ is a standard enclosure, each of whose cells $\mathfrak{e}_{nm}$ is rectangular. But the sections $\mathfrak{e}_{nm}(x)$ are also rectangular. Hence

$$\mathfrak{A}_o(x) = Dv\{\mathfrak{e}_{nm}(x)\},$$

being the divisor of measurable sets, is measurable.

415. *Let $\mathfrak{A}_o$ be an outer associated set of $\mathfrak{A}$, both lying in the standard cube $\mathfrak{Q}$. Then $\widehat{\mathfrak{A}}_o(x)$ is L-integrable in $\mathfrak{q}$, and*

$$\overline{\overline{\mathfrak{A}}} = \int_{\mathfrak{q}} \widehat{\mathfrak{A}}_o(x). \tag{1}$$

For let $\{\mathfrak{E}_n\}$ be a sequence of standard enclosures of $\mathfrak{A}$, and $\mathfrak{E}_n = \{\mathfrak{e}_{nm}\}$. Then

$$\widehat{\mathfrak{E}}_n = \sum_m \widehat{\mathfrak{e}}_{nm} \tag{2}$$

and

$$\widehat{\mathfrak{E}}_n(x) = \sum_m \widehat{\mathfrak{e}}_{nm}(x). \tag{3}$$

Now $\mathfrak{e}_{nm}$ being a standard cell, $\widehat{\mathfrak{e}}_{nm}(x)$ has a constant value > 0 for all x contained in the projection of $\mathfrak{e}_{nm}$ on $\mathfrak{q}$. It is thus continuous in $\mathfrak{q}$ except for a discrete set. It thus has an R-integral, and

$$\widehat{\mathfrak{e}}_{nm} = \int_{\mathfrak{q}} \widehat{\mathfrak{e}}_{nm}(x).$$

This in 2) gives

$$\widehat{\mathfrak{E}}_n = \Sigma \int_{\mathfrak{q}} \widehat{\mathfrak{e}}_{nm}(x)$$

$$= \int_{\mathfrak{q}} \Sigma \widehat{\mathfrak{e}}_{nm}(x), \qquad \text{by 413, 2,}$$

$$= \int_{\mathfrak{q}} \widehat{\mathfrak{E}}_n(x), \tag{4}$$

by 3).

On the other hand, $\widehat{\overline{\mathfrak{E}}}(x)$ is a measurable function by 411. Also

$$\overline{\mathfrak{A}} = \widehat{\mathfrak{A}}_e = \lim \widehat{\mathfrak{E}}_n$$
$$= \lim \int_{\mathfrak{q}}^{\bullet} \widehat{\overline{\mathfrak{E}}}_n(x)$$
$$= \int_{\mathfrak{q}} \lim \widehat{\overline{\mathfrak{E}}}_n(x), \qquad \text{by 413, 1.} \tag{5}$$

Now
$$\widehat{\mathfrak{A}}_e(x) = \lim_{n=\infty} \widehat{\mathfrak{E}}_n(x).$$

Thus this in 5) gives 1).

416. *Let $\mathfrak{A}$ lie in the standard cube $\mathfrak{Q}$. Let $\mathfrak{A}_\iota$ be an inner associated set. Then $\widehat{\overline{\mathfrak{A}}}_\iota(x)$ is L-integrable in* $\mathfrak{q}$, *and*

$$\underline{\mathfrak{A}} = \int_{\mathfrak{q}}^{\bullet} \widehat{\overline{\mathfrak{A}}}_\iota(x).$$

For
$$\mathfrak{Q} = \mathfrak{A}_\iota + A_o.$$

Thus
$$\widehat{\overline{\mathfrak{A}}}_\iota(x) = \widehat{\mathfrak{Q}}(x) - \widehat{\overline{A}}_o(x).$$

Hence $\widehat{\overline{\mathfrak{A}}}_\iota(x)$ is L-integrable in $\mathfrak{q}$, and

$$\int_{\mathfrak{q}}^{\bullet} \widehat{\overline{\mathfrak{A}}}_\iota(x) = \int_{\mathfrak{q}} \widehat{\mathfrak{Q}}(x) - \int_{\mathfrak{q}}^{\bullet} \widehat{\overline{A}}_o(x)$$
$$= \widehat{\mathfrak{Q}} - A_o \quad , \quad \text{by 415,}$$
$$= \widehat{\overline{\mathfrak{A}}}_\iota = \underline{\mathfrak{A}} \qquad \text{by 370, 2.}$$

417. *Let measurable $\mathfrak{A}$ lie in the standard cube $\mathfrak{Q}$. Then*

$$\widehat{\mathfrak{A}} = \int_{\mathfrak{q}}^{\bullet} \underline{\mathfrak{A}}(x). \tag{1}$$

For
$$\mathfrak{A}_\iota(x) \leq \mathfrak{A}(x) \leq \mathfrak{A}_e(x).$$

Hence
$$\underline{\underline{\mathfrak{A}}} = \int_{\mathfrak{q}} \widehat{\overline{\mathfrak{A}}}_\iota(x) \leq \underline{\overline{\int}}_{\mathfrak{q}}^{\bullet} \mathfrak{A}(x) \leq \overline{\int}_{\mathfrak{q}}^{\bullet} \widehat{\overline{\mathfrak{A}}}_e(x) = \overline{\overline{\mathfrak{A}}}, \tag{2}$$

using 396, 1, and 415, 416. From 2) we conclude 1) at once.

418. *Let $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$ be measurable. Then $\overline{\underline{\mathfrak{C}}}$ are L-integrable in* $\mathfrak{B}$, *and*

$$\widehat{\mathfrak{A}} = \int_{\mathfrak{B}} \overline{\underline{\mathfrak{C}}}.$$

Here the first $= \overline{\overline{\mathfrak{E}}}$ when a section exists, otherwise it $= 0$, in $\mathfrak{q}$. A similar definition holds for the other function.

3. Let us note that *the sections*

$$\mathfrak{A}_e(x) \quad , \quad \mathfrak{A}_\iota(x),$$

where $\mathfrak{A}_e$, $\mathfrak{A}_\iota$ *are the outer and inner associated sets belonging to* $\mathfrak{A}$, *are always measurable.*

For $\mathfrak{A}_e = Dv\{\mathfrak{E}_n\}$, where each $\mathfrak{E}_n$ is a standard enclosure, each of whose cells $\mathfrak{e}_{nm}$ is rectangular. But the sections $\mathfrak{e}_{nm}(x)$ are also rectangular. Hence

$$\mathfrak{A}_e(x) = Dv\{\mathfrak{e}_{nm}(x)\},$$

being the divisor of measurable sets, is measurable.

415. *Let* $\mathfrak{A}_e$ *be an outer associated set of* $\mathfrak{A}$, *both lying in the standard cube* $\mathfrak{Q}$. *Then* $\widehat{\widehat{\mathfrak{A}}}_e(x)$ *is L-integrable in* $\mathfrak{q}$, *and*

$$\overline{\overline{\mathfrak{A}}} = \int_{\mathfrak{q}} \widehat{\widehat{\mathfrak{A}}}_e(x). \tag{1}$$

For let $\{\mathfrak{E}_n\}$ be a sequence of standard enclosures of $\mathfrak{A}$, and $\mathfrak{E}_n = \{\mathfrak{e}_{nm}\}$. Then

$$\widehat{\widehat{\mathfrak{E}}}_n = \sum_m \widehat{\mathfrak{e}}_{nm} \tag{2}$$

and

$$\widehat{\widehat{\mathfrak{E}}}_n(x) = \sum_m \widehat{\mathfrak{e}}_{nm}(x). \tag{3}$$

Now $\mathfrak{e}_{nm}$ being a standard cell, $\widehat{\mathfrak{e}}_{nm}(x)$ has a constant value > 0 for all x contained in the projection of $\mathfrak{e}_{nm}$ on $\mathfrak{q}$. It is thus continuous in $\mathfrak{q}$ except for a discrete set. It thus has an R-integral, and

$$\widehat{\mathfrak{e}}_{nm} = \int_{\mathfrak{q}} \widehat{\mathfrak{e}}_{nm}(x).$$

This in 2) gives

$$\widehat{\widehat{\mathfrak{E}}}_n = \Sigma \int_{\mathfrak{q}} \widehat{\mathfrak{e}}_{nm}(x)$$

$$= \int_{\mathfrak{q}} \Sigma \widehat{\mathfrak{e}}_{nm}(x), \qquad \text{by 413, 2,}$$

$$= \int_{\mathfrak{q}} \widehat{\widehat{\mathfrak{E}}}_n(x), \tag{4}$$

by 3).

On the other hand, $\widehat{\overline{\mathfrak{E}}}(x)$ is a measurable function by 411. Also

$$\overline{\overline{\mathfrak{A}}} = \widehat{\widehat{\mathfrak{A}}}_e = \lim \widehat{\widehat{\mathfrak{E}}}_n$$

$$= \lim \int_{\mathfrak{q}} \widehat{\widehat{\mathfrak{E}}}_n(x)$$

$$= \int_{\mathfrak{q}} \lim \widehat{\widehat{\mathfrak{E}}}_n(x), \qquad \text{by 413, 1.} \tag{5}$$

Now
$$\widehat{\widehat{\mathfrak{A}}}_e(x) = \lim_{n=\infty} \widehat{\widehat{\mathfrak{E}}}_n(x).$$

Thus this in 5) gives 1).

416. *Let $\mathfrak{A}$ lie in the standard cube $\mathfrak{Q}$. Let $\mathfrak{A}_\iota$ be an inner associated set. Then $\widehat{\widehat{\mathfrak{A}}}_\iota(x)$ is L-integrable in $\mathfrak{q}$, and*

$$\underline{\underline{\mathfrak{A}}} = \int_{\mathfrak{q}} \widehat{\widehat{\mathfrak{A}}}_\iota(x).$$

For
$$\mathfrak{Q} = \mathfrak{A}_\iota + A_e.$$

Thus
$$\widehat{\widehat{\mathfrak{A}}}_\iota(x) = \widehat{\mathfrak{Q}}(x) - \widehat{\widehat{A}}_e(x).$$

Hence $\widehat{\widehat{\mathfrak{A}}}_\iota(x)$ is L-integrable in $\mathfrak{q}$, and

$$\int_{\mathfrak{q}} \widehat{\widehat{\mathfrak{A}}}_\iota(x) = \int_{\mathfrak{q}} \widehat{\mathfrak{Q}}(x) - \int_{\mathfrak{q}} \widehat{\widehat{A}}_e(x)$$

$$= \widehat{\mathfrak{Q}} - A_e \quad , \quad \text{by 415,}$$

$$= \widehat{\widehat{\mathfrak{A}}}_\iota = \underline{\underline{\mathfrak{A}}} \qquad \text{by 370, 2.}$$

417. *Let measurable $\mathfrak{A}$ lie in the standard cube $\mathfrak{Q}$. Then*

$$\widehat{\widehat{\mathfrak{A}}} = \int_{\underline{\mathfrak{q}}} \overline{\overline{\mathfrak{A}}}(x). \tag{1}$$

For
$$\mathfrak{A}_\iota(x) \le \mathfrak{A}(x) \le \mathfrak{A}_e(x).$$

Hence
$$\underline{\underline{\mathfrak{A}}} = \int_{\mathfrak{q}} \widehat{\widehat{\mathfrak{A}}}_\iota(x) \le \overline{\int_{\underline{\mathfrak{q}}}} \overline{\overline{\mathfrak{A}}}(x) \le \overline{\int_{\mathfrak{q}}} \widehat{\widehat{\mathfrak{A}}}_e(x) = \overline{\overline{\mathfrak{A}}}, \tag{2}$$

using 396, 1, and 415, 416. From 2) we conclude 1) at once.

418. *Let $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$ be measurable. Then $\underline{\underline{\overline{\overline{\mathfrak{C}}}}}$ are L-integrable in $\mathfrak{B}$, and*

$$\widehat{\widehat{\mathfrak{A}}} = \int_{\underline{\mathfrak{B}}} \overline{\overline{\mathfrak{C}}}.$$

For by 417,
$$\widehat{\mathfrak{A}} = \int_{\mathfrak{q}} \overline{\overline{\mathfrak{A}}}(x)$$
$$= \int_{\mathfrak{B}} \overline{\overline{\mathfrak{C}}},$$
by 401.

419. *If* $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$ *is measurable, the points of* $\mathfrak{B}$ *at which* $\mathfrak{C}$ *is not measurable form a null set* $\mathfrak{N}$.

For by 418,
$$\widehat{\mathfrak{A}} = \int_{\mathfrak{B}} \overline{\overline{\mathfrak{C}}} = \int_{\mathfrak{B}} \underline{\underline{\mathfrak{C}}}.$$
Hence
$$0 = \int_{\mathfrak{B}} (\overline{\overline{\mathfrak{C}}} - \underline{\underline{\mathfrak{C}}}).$$
Thus
$$\phi = \overline{\overline{\mathfrak{C}}} - \underline{\underline{\mathfrak{C}}}$$
is a null function in $\mathfrak{B}$, and by 402, 2, points where $\phi > 0$ form a null set.

420. *Let* $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$ *be measurable. Let* $\mathfrak{b}$ *denote the points of* $\mathfrak{B}$ *for which the corresponding sections* $\mathfrak{C}$ *are measurable. Then*
$$\widehat{\mathfrak{A}} = \int_{\mathfrak{b}} \widehat{\mathfrak{C}}.$$
For by 419,
$$\mathfrak{B} = \mathfrak{b} + \mathfrak{N},$$
and $\mathfrak{N}$ is a null set. Hence by 418,
$$\widehat{\mathfrak{A}} = \int_{\mathfrak{B}} \overline{\overline{\mathfrak{C}}} = \int_{\mathfrak{b}} \overline{\overline{\mathfrak{C}}} + \int_{\mathfrak{N}} \overline{\overline{\mathfrak{C}}}$$
$$= \int_{\mathfrak{b}} \widehat{\mathfrak{C}}.$$

421. *Let* $f \geq 0$ *in* $\mathfrak{A}$. *If the integrand set* $\mathfrak{J}$, *corresponding to* f *be measurable, then* f *is L-integrable in* $\mathfrak{A}$, *and*
$$\widehat{\mathfrak{J}} = \int_{\mathfrak{A}} f.$$

For the points of $\mathfrak{J}$ lying in an $m+1$ way space $\mathfrak{R}_{m+1}$ may be denoted by
$$x = (y_1 \cdots y_m, z),$$
where $y = (y_1 \cdots y_m)$ ranges over $\mathfrak{R}_m$, in which $\mathfrak{A}$ lies. Thus $\mathfrak{A}$ may be regarded as the projection of $\mathfrak{J}$ on $\mathfrak{R}_m$. To each point y

of $\mathfrak{A}$ corresponds a section $\mathfrak{I}(y)$, which for brevity may be denoted by $\mathfrak{K}$. Thus we may write

$$\mathfrak{I} = \mathfrak{A} \cdot \mathfrak{K}.$$

As $\mathfrak{K}$ is nothing but an ordinate through y of length $f(y)$, we have by 419,

$$\widehat{\mathfrak{I}} = \int_{\mathfrak{A}} \overline{\overline{\mathfrak{K}}} = \int_{\mathfrak{A}} f.$$

422. *Let f be L-integrable over the measurable field $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$. Let $\mathfrak{b}$ denote those points of $\mathfrak{B}$, for which f is L-integrable over the corresponding sections $\mathfrak{C}$. Then*

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{b}} \int_{\mathfrak{C}} f. \tag{1}$$

Moreover $\mathfrak{N} = \mathfrak{B} - \mathfrak{b}$ is a null set.

Let us 1° *suppose* $f \geq 0$. Then by 406, $\mathfrak{I}$ is measurable and

$$\widehat{\mathfrak{I}} = \int_{\mathfrak{A}} f. \tag{2}$$

Let β denote the points of $\mathfrak{B}$ for which $\mathfrak{I}(x)$ is measurable. Then by 420,

$$\widehat{\mathfrak{I}} = \int_{\beta} \widehat{\mathfrak{I}}(x). \tag{3}$$

By 419, the points

$$\mathfrak{P} = \mathfrak{B} - \beta \tag{4}$$

form a null set.

On the other hand, $\mathfrak{I}(x)$ is the integrand set of f, for $\mathfrak{A}(x) = \mathfrak{C}$. Hence by 421, for any x in β,

$$\widehat{\mathfrak{I}}(x) = \int_{\mathfrak{C}} f, \tag{5}$$

and

$$\beta \leq \mathfrak{b}. \tag{6}$$

From 2), 3), 5) we have

$$\int_{\mathfrak{A}} f = \int_{\beta} \int_{\mathfrak{C}} f. \tag{7}$$

From 6) we have

$$\mathfrak{N} = \mathfrak{B} - \mathfrak{b} \leq \mathfrak{B} - \beta = \mathfrak{P},$$

a null set by 4). Let us set

$$\mathfrak{b} = \beta + \mathfrak{n}.$$

Then $\mathfrak{n}$ lying in the null set $\mathfrak{P}$, is a null set. Hence

$$\int_{\beta}\int_{\mathfrak{C}} + \int_{\mathfrak{n}}\int_{\mathfrak{C}} = \int_{\mathfrak{b}}\int_{\mathfrak{C}}.$$

This with 7) gives 1).

Let f be now unrestricted as to sign. We take $C > 0$, such that the auxiliary function

$$g = f + C \geq 0, \qquad \text{in } \mathfrak{A}.$$

Then f, g are simultaneously L-integrable over any section $\mathfrak{C}$. Thus by case 1°

$$\int_{\mathfrak{A}}(f + C) = \int_{\mathfrak{b}}\int_{\mathfrak{C}}(f + C). \tag{8}$$

Now

$$\int_{\mathfrak{A}}(f + C) = \int_{\mathfrak{A}} f + \int_{\mathfrak{A}} C = \int_{\mathfrak{A}} f + C\widehat{\mathfrak{A}}, \tag{9}$$

$$\int_{\mathfrak{C}}(f + C) = \int_{\mathfrak{C}} f + C\overline{\overline{\mathfrak{C}}}. \tag{10}$$

By 418, $\overline{\overline{\mathfrak{C}}}$ is L-integrable in $\mathfrak{B}$, and hence in $\mathfrak{b}$. Thus

$$\int_{\mathfrak{b}}\int_{\mathfrak{C}}(f + C) = \int_{\mathfrak{b}}\int_{\mathfrak{C}} f + C\int_{\mathfrak{b}}\overline{\overline{\mathfrak{C}}}. \tag{11}$$

As $\mathfrak{b}$ differs from $\mathfrak{B}$ by a null set,

$$\underline{\int_{\mathfrak{b}}}\overline{\overline{\mathfrak{C}}} = \int_{\mathfrak{B}}\overline{\overline{\mathfrak{C}}} = \widehat{\mathfrak{A}}, \tag{12}$$

by 418. From 8), 9), 10), 11), 12) we have 1).

423. *If f is L-integrable over the measurable set $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$, then*

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{B}}\overline{\underline{\int_{\mathfrak{C}}}} f. \tag{1}$$

For by 422,

$$\int_{\mathfrak{A}} = \int_{\mathfrak{b}}\int_{\mathfrak{C}}. \tag{2}$$

As $\mathfrak{B} - \mathfrak{b} = \mathfrak{N}$ is a null set,

$$\int_{\mathfrak{N}}\overline{\underline{\int_{\mathfrak{C}}}} f = 0$$

may be added to the right side of 2) without altering its value. Thus

$$\int_{\mathfrak{A}} = \int_{\mathfrak{b}} \int_{\mathfrak{C}} + \int_{\mathfrak{N}} \int_{\bar{\mathfrak{C}}} = \int_{\mathfrak{B}} \int_{\bar{\mathfrak{C}}}.$$

424. 1. (*W. A. Wilson.*) *If $f(x_1 \cdots x_m)$ is L-integrable in measurable $\mathfrak{A}$, f is measurable in $\mathfrak{A}$.*

Let us *first suppose* that $f \geq 0$. We begin by showing that the set of points $\mathfrak{A}_\lambda$ of $\mathfrak{A}$ at which $f \geq \lambda$, is measurable. Then by 408, 3, f is measurable in $\mathfrak{A}$.

Now f being L-integrable in $\mathfrak{A}$, its integrand set $\mathfrak{J}$ is measurable by 406. Let $\mathfrak{J}_\lambda$ be the section of $\mathfrak{J}$ corresponding to $x_{m+1} = \lambda$. Then the projection of $\mathfrak{J}_\lambda$ on $\mathfrak{R}_m$ is $\mathfrak{A}_\lambda$. Since $\mathfrak{J}$ is measurable, the sections $\mathfrak{J}_\lambda$ are measurable, except at most over a null set L of values of λ, by 419. Thus there exists a sequence

$$\lambda_1 < \lambda_2 < \cdots \doteq \lambda$$

none of whose terms lies in L. Hence each $\mathfrak{J}_{\lambda_n}$ is measurable, and hence $\mathfrak{A}_{\lambda_n}$ is also.

As $\mathfrak{A}_{\lambda_{n+1}} \leq \mathfrak{A}_{\lambda_n}$, each point of $\mathfrak{A}_\lambda$ lies in

$$\mathfrak{D} = Dv\{\mathfrak{A}_{\lambda_n}\}, \tag{1}$$

so that

$$\mathfrak{A}_\lambda \leq \mathfrak{D}. \tag{2}$$

On the other hand, each point d of $\mathfrak{D}$ lies in $\mathfrak{A}_\lambda$. For if not, $f(d) < \lambda$.

There thus exists an s such that

$$f(d) < \lambda_s < \lambda. \tag{3}$$

But then d does not lie in $\mathfrak{A}_{\lambda_s}$, for otherwise $f(d) \geq \lambda_s$, which contradicts 3). But not lying in $\mathfrak{A}_{\lambda_s}$, d cannot lie in $\mathfrak{D}$, and this contradicts our hypothesis. Thus

$$\mathfrak{D} \leq \mathfrak{A}_\lambda. \tag{4}$$

From 2), 4) we have

$$\mathfrak{D} = \mathfrak{A}_\lambda.$$

But then from 1), $\mathfrak{A}_\lambda$ is measurable.

Let the sign of f be now unrestricted.

Since f is limited, we may choose the constant C, such that

$$g = f(x) + C \geq 0, \text{ in } \mathfrak{A}.$$

Then g is L-integrable, and hence, by case 1°, g is measurable. Hence f, differing only by a constant from g, is also measurable.

2. *Let $\mathfrak{A}$ be measurable. If f is L-integrable in $\mathfrak{A}$, it is measurable in $\mathfrak{A}$, and conversely.*

This follows from 1 and 409, 1.

3. From 2 and 409, 3, we have at once the theorem :

When the field of integration is measurable, an L-integrable function is integrable in Lebesgue's sense, and conversely; moreover, both have the same value.

Remark. In the theory which has been developed in the foregoing pages, the reader will note that neither the field of integration nor the integrand needs to be measurable. This is not so in Lebesgue's theory. In removing this restriction, we have been able to develop a theory entirely analogous to Riemann's theory of integration, and to extend this to a theory of upper and lower integration. We have thus a perfect counterpart of the theory developed in Chapter XIII of vol. I.

4. *Let $\mathfrak{A}$ be metric or complete. If $f(x_1 \cdots x_m)$ is limited and R-integrable, it is a measurable function in $\mathfrak{A}$.*

For by 381, 2, it is L-integrable. Also since $\mathfrak{A}$ is metric or complete, $\mathfrak{A}$ is measurable. We now apply 1.

IMPROPER L-INTEGRALS

Upper and Lower Integrals

425. 1. We propose now to consider the case that the integrand $f(x_1 \cdots x_m)$ is not limited in the limited field of integration $\mathfrak{A}$. In chapter II we have treated this case for R-integrals. To extend the definitions and theorems there given to L-integrals, we have in general only to replace metric or complete sets by measurable sets; discrete sets by null sets; unmixed sets by separated sets;

visions by separated divisions; sequences of superposed divisions by extremal sequences; etc.

28 we may define an improper L-integral in any of the ys there given, making such changes as just indicated. llowing we shall employ only the 3° Type of definition. plicit we define as follows:

$(x_1 \cdots x_m)$ be defined for each point of the limited set $\mathfrak{A}$. lenote the points of $\mathfrak{A}$ at which

$$-\alpha \leq f(x_1 \cdots x_m) \leq \beta \qquad \alpha, \beta > 0. \tag{1}$$

ts

$$\lim_{\alpha, \beta = \infty} \underline{\int}_{\mathfrak{A}_{\alpha\beta}} f \quad , \quad \lim_{\alpha, \beta = \infty} \overline{\int}_{\mathfrak{A}_{\alpha\beta}} f \tag{2}$$

hey exist, we call the *lower* and *upper* (improper) *L-in*- nd denote them by

$$\underline{\int}_{\mathfrak{A}} f \quad , \quad \overline{\int}_{\mathfrak{A}} f.$$

e the two limits 2) exist and are equal, we denote their value by

$$\int_{\mathfrak{A}} f$$

f is (improperly) *L-integrable in* $\mathfrak{A}$, etc.

order to use the demonstrations of Chapter II without too ouble, we introduce the term *separated function*. A func- such a function when the fields $\mathfrak{A}_{\alpha\beta}$ defined by 1) are l parts of $\mathfrak{A}$.

ve defined measurable functions in 407 in the case that ted in $\mathfrak{A}$. We may extend it to unlimited functions by that the fields $\mathfrak{A}_{\alpha\beta}$ are measurable however large α, β are

oing so, we see that measurable functions are special cases ted functions.

e the field $\mathfrak{A}$ of integration is measurable, $\mathfrak{A}_{\alpha\beta}$ is a meas- rt of $\mathfrak{A}$, if it is a separated part. From this follows the t result:

a separated function in the measurable field $\mathfrak{A}$, *it is L-in- n each* $\mathfrak{A}_{\alpha\beta}$.

From this follows also the theorem:

Let f be a separated function in the measurable field $\mathfrak{A}$. If either the lower or upper integral of f over $\mathfrak{A}$ is convergent, f is L-integrable in $\mathfrak{A}$, and

$$\int_{\mathfrak{A}} f = \lim_{\alpha, \beta = \infty} \int_{\mathfrak{A}_{\alpha\beta}} f.$$

426. To illustrate how the theorems on improper R-integrals give rise to analogous theorems on improper L-integrals, which may be demonstrated along the same lines as used in Chapter II, let us consider the analogue of 38, 2, viz.:

If f is a separated function such that $\overline{\int}_{\mathfrak{A}} f$ converges, so do $\overline{\int}_{\mathfrak{P}} f$.

Let $\{E_n\}$ be an extremal sequence common to both

$$\overline{\int}_{\mathfrak{A}_{\alpha\beta'}}, \quad \overline{\int}_{\mathfrak{A}_{\alpha\beta}} \qquad \beta' > \beta.$$

Let e denote the cells of E_n containing a point of $\mathfrak{P}_\beta$; e' those cells containing a point of $\mathfrak{P}_{\beta'}$; δ those cells containing a point of $\mathfrak{A}_{\alpha\beta}$ but none of $\mathfrak{P}_{\beta'}$. Then

$$\overline{\int}_{\mathfrak{A}_{\alpha\beta'}} = \lim_{n=\infty} \{\Sigma M_e' \cdot e + \Sigma M_{e'}' \cdot e' + \Sigma M_\delta' \cdot \delta\}.$$

In this manner we may continue using the proof of 38, and so establish our theorem.

427. As another illustration let us prove the theorem analogous to 46, viz.:

Let $\mathfrak{A}_1, \mathfrak{A}_2, \cdots \mathfrak{A}_n$ form a separated division of $\mathfrak{A}$. If f is a separated function in $\mathfrak{A}$, then

$$\overline{\int}_{\mathfrak{A}} f = \overline{\int}_{\mathfrak{A}_1} f + \cdots + \overline{\int}_{\mathfrak{A}_n} f,$$

provided the integral on the left exists, or all the integrals on the right exist.

For let $\mathfrak{A}_{s, \alpha\beta}$ denote the points of $\mathfrak{A}_{\alpha\beta}$ in $\mathfrak{A}_s$. Then by 390, 1,

$$\overline{\int}_{\mathfrak{A}_{\alpha\beta}} = \overline{\int}_{\mathfrak{A}_{1, \alpha\beta}} + \cdots + \overline{\int}_{\mathfrak{A}_{n, \alpha\beta}}.$$

In this way we continue with the reasoning of 46.

428. In this way we can proceed with the other theorems; in each case the requisite modification is quite obvious, by a consideration of the demonstration of the corresponding theorem in R-integrals given in Chapter II.

This is also true when we come to treat of *iterated* integrals along the lines of 70–78. We have seen, in 425, 2, that if $\mathfrak{A}$ is measurable, *upper* and *lower* integrals of separated functions do not exist as such; they reduce to L-integrals. We may still have a theory analogous to iterated R-integrals, by extending the notion of iterable fields, using the notion of upper measure. To this end we define:

A limited point set at $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$ is *submeasurable* with respect to $\mathfrak{B}$, when

$$\overline{\overline{\mathfrak{A}}} = \int_{\mathfrak{B}} \overline{\overline{\mathfrak{C}}}.$$

We do not care to urge this point at present, but prefer to pass on at once to the much more interesting case of L-integrals over measurable fields.

L-Integrals

429. These we may define for our purpose as follows:

Let $f(x_1 \cdots x_m)$ be defined over the limited measurable set $\mathfrak{A}$. As usual let $\mathfrak{A}_{\alpha\beta}$ denote the points of $\mathfrak{A}$ at which

$$-\alpha \leq f \leq \beta, \qquad \alpha, \beta \geq 0.$$

Let each $\mathfrak{A}_{\alpha\beta}$ be measurable, and let f have a proper L-integral in each $\mathfrak{A}_{\alpha\beta}$. Then the improper integral of f over $\mathfrak{A}$ is

$$\int_{\mathfrak{A}} f = \lim_{\alpha, \beta \to \infty} \int_{\mathfrak{A}_{\alpha\beta}} f, \tag{1}$$

when this limit exists. We shall also say that the integral on the left of 1) is *convergent*.

On this hypothesis, the reader will note at once that the demonstrations of Chapter II admit ready adaptation; in fact some of the theorems require no demonstration, as they follow easily from results already obtained.

430. Let us group together for reference the following theorems, analogous to those on improper R-integrals.

1. *If f is (improperly) L-integrable in $\mathfrak{A}$, it is in any measurable part of $\mathfrak{A}$.*

2. *If g, h denote as usual the non-negative functions associated with f, then*

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{A}} g - \int_{\mathfrak{A}} h. \tag{1}$$

3. *If $\int_{\mathfrak{A}} f$ is convergent, so is $\int_{\mathfrak{A}} |f|$, and conversely.*

4. *When convergent,*

$$\left| \int_{\mathfrak{A}} f \right| \leq \int_{\mathfrak{A}} f \,. \tag{2}$$

5. *If $\int_{\mathfrak{A}} f$ is convergent, then*

$$\epsilon > 0, \qquad \sigma > 0, \qquad \left| \int_{\mathfrak{B}} f \right| \leq \epsilon,$$

for any measurable $\mathfrak{B} < \mathfrak{A}$, such that $\widehat{\mathfrak{B}} < \sigma$.

6. *Let $\mathfrak{A} = (\mathfrak{A}_1, \mathfrak{A}_2 \cdots \mathfrak{A}_n)$ be a separated division of $\mathfrak{A}$, each $\mathfrak{A}_\iota$ being measurable. Then*

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{A}_1} f + \cdots + \int_{\mathfrak{A}_n} f, \tag{3}$$

provided the integral on the left exists, or all the integrals on the right exist.

7. *Let $\mathfrak{A} = \{\mathfrak{A}_n\}$ be a separated division of $\mathfrak{A}$, into an enumerable infinite set of measurable sets $\mathfrak{A}_n$. Then*

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{A}_1} f + \int_{\mathfrak{A}_2} f + \cdots \tag{4}$$

provided the integral on the left exists.

8. *If $f \leq g$ in $\mathfrak{A}$, except possibly at a null set, then*

$$\int_{\mathfrak{A}} f \leq \int_{\mathfrak{A}} g, \tag{5}$$

when convergent.

431. 1. To show how simple the proofs run in the present case, let us consider, in the first place, the theorem analogous to 38, 2, viz.:

If $\int_{\mathfrak{A}} f$ *converges, so do* $\int_{\mathfrak{P}} f$ *and* $\int_{\mathfrak{R}} f$.

The rather difficult proof of 38, 2 can be replaced by the following simpler one. Since

$$\mathfrak{A}_{\alpha\beta} = \mathfrak{P}_{\beta} + \mathfrak{R}_{\alpha} \tag{1}$$

is a *separated* division of $\mathfrak{A}_{\alpha\beta}$, we have

$$\int_{\mathfrak{A}_{\alpha\beta}} = \int_{\mathfrak{P}_{\beta}} + \int_{\mathfrak{R}_{\alpha}},$$

$$\int_{\mathfrak{A}_{\alpha\beta}} = \int_{\mathfrak{P}_{\beta}} + \int_{\mathfrak{R}_{\alpha}}.$$

Hence
$$\left| \int_{\mathfrak{A}_{\alpha\beta}} - \int_{\mathfrak{A}_{\alpha\beta'}} \right| = \left| \int_{\mathfrak{P}_{\beta}} - \int_{\mathfrak{P}_{\beta'}} \right|.$$

But the left side is $< \epsilon$, for a sufficiently large α, and $\beta, \beta' >$ some β_0. This shows that $\int_{\mathfrak{P}}$ is convergent. Similarly we show the other integral converges.

2. This form of proof could not be used in 38, 2, since 1) in general is not an unmixed division of $\mathfrak{A}_{\alpha\beta}$.

3. In a similar manner we may establish the theorem analogous to 39, viz.:

If $\int_{\mathfrak{P}} f$ *and* $\int_{\mathfrak{R}} f$ *converge, so does* $\int_{\mathfrak{A}} f$.

4. Let us look at the demonstration of the theorem analogous to 43, 1, viz.:

$$\int_{\mathfrak{A}} g = \int_{\mathfrak{P}} f \quad ; \quad \int_{\mathfrak{A}} h = -\int_{\mathfrak{R}} f,$$

provided the integral on either side of these equations converges.

Let us prove the first relation. Let $\mathfrak{B}_\beta$ denote the points of $\mathfrak{A}$ at which $f \leq \beta$. Then

$$\mathfrak{B}_\beta = \mathfrak{N} + \mathfrak{P}_\beta$$

is a separated division of $\mathfrak{B}_\beta$, and hence

$$\int_{\mathfrak{B}_\beta} g = \int_{\mathfrak{N}_\beta} g + \int_{\mathfrak{P}_\beta} g = \int_{\mathfrak{P}_\beta} g = \int_{\mathfrak{P}_\beta} f, \quad \text{etc.}$$

5. It is now obvious that the analogue of 44, 1 is the relation 1) in 430.

6. The analogue of 46 is the relation 3) in 430. Its demonstration is precisely similar to that in 46.

7. We now establish 430, 7. Let

$$\mathfrak{B}_m = (\mathfrak{A}_1, \mathfrak{A}_2 \cdots \mathfrak{A}_m).$$

Then $$\mathfrak{A} = \mathfrak{B}_m + B_m$$

is a separated division of $\mathfrak{A}$, and we may take m so large that $\widehat{B}_m < \sigma$, an arbitrarily small positive number. Hence by 430, 5, we may take m so large that

$$\left| \int_{B_m} f \right| < \epsilon.$$

Thus $$\int_{\mathfrak{A}} f = \int_{\mathfrak{B}_m} f + \int_{B_m} f$$

$$= \int_{\mathfrak{A}_1} f + \cdots + \int_{\mathfrak{A}_m} f + \epsilon' \quad , \quad |\epsilon'| < \epsilon.$$

From this our theorem follows at once.

Iterated Integrals

432. 1. Let us see how the reasoning of Chapter II may be extended to this case. We will of course suppose that the field of integration $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$ is measurable. Then by 419, the points of $\mathfrak{B}$ for which the sections are not measurable form a null set. Since the integral of any function over a null set is zero, we may therefore in our reasoning suppose that every $\mathfrak{C}$ is measurable.

Since $\mathfrak{A}$ is measurable, there exists a sequence of complete components $A_m = B_m C_m$ in $\mathfrak{A}$, such that the measure of $A = \{A_m\}$ is $\widehat{\mathfrak{A}}$.

Since A_m is complete, its projection B_m is complete, by I, 717, 4. The points of B_m for which the corresponding sections C_m are not measurable form a null set ν_m. Hence the union $\{\nu_m\}$ is a null set. Thus we may suppose, without loss of generality in our demonstrations, that $\mathfrak{A}$ is such that every section in each A_m is measurable.

Now from

$$0 = \hat{\bar{\mathfrak{A}}} - \hat{A} = \int_{\mathfrak{B}}^{\cdot} \hat{\bar{\mathfrak{C}}} - \int_{B}^{\cdot} \hat{C} = \int_{\mathfrak{B}}^{\cdot} (\hat{\bar{\mathfrak{C}}} - \hat{C}),$$

we see that those points of $\mathfrak{B}$ where $\hat{\bar{\mathfrak{C}}} > \hat{C}$ form a null set. We may therefore suppose that $\hat{\bar{\mathfrak{C}}} = C$ everywhere. Then $\mathfrak{C} - C$ is a null set at each point; we may thus adjoin them to C. Thus we may suppose that $\mathfrak{C} = C$ at each point of $\mathfrak{B}$, and that $\mathfrak{B} = B$ is the union of an enumerable set of complete sets B_m.

As we shall suppose that

$$\int_{\mathfrak{A}}^{\cdot} f$$

is convergent, let

$$\alpha_1 < \alpha_2 < \cdots \doteq \infty,$$
$$\beta_1 < \beta_2 < \cdots \doteq \infty.$$

Let us look at the sets $\mathfrak{A}_{\alpha_n}$, $\mathfrak{B}_{\beta_n}$, which we shall denote by $\mathfrak{A}_n$. These are measurable by 429. Moreover, the reasoning of 72, 2 shows that without loss of generality we may suppose that $\mathfrak{A}$ is such that $\mathfrak{B}_n = \mathfrak{B}$. We may also suppose that each $\mathfrak{C}_n$ is measurable, as above.

2. Let us finally consider the integrals

$$\int_{\mathfrak{C}}^{\cdot} f. \tag{1}$$

These may not exist at every point of $\mathfrak{B}$, because f does not admit a proper or an improper integral at this point. It will suffice for our purpose to suppose that 1) does not exist at a null set in $\mathfrak{B}$. Then without loss of generality we may suppose in our demonstrations that 1) converges at each point of $\mathfrak{B}$.

On these assumptions let us see how the theorems 73, 74, 75, and 76 are to be modified, in order that the proofs there given may be adapted to the present case.

433. 1. The first of these may be replaced by this:

Let $B_{\sigma, n}$ denote the points of $\mathfrak{B}$ at which $\widehat{\mathfrak{c}}_n > \sigma$. Then

$$\lim_{n=\infty} \overline{\overline{B}}_{\sigma, n} = 0.$$

For by 419,

$$\widehat{\mathfrak{A}} = \int_{\mathfrak{B}} \widehat{\mathfrak{C}},$$

as by hypothesis the sections $\mathfrak{C}$ are measurable. Moreover, by hypothesis

$$\mathfrak{C} = \mathfrak{C}_n + \mathfrak{c}_n$$

is a separated division of $\mathfrak{C}$, each set on the right being measurable. Thus the proof in 73 applies at once.

2. The theorem of 74 becomes:

Let the integrals

$$\int_{\mathfrak{C}} f \quad , \quad f \geq 0$$

be limited in the complete set $\mathfrak{B}$. Let $\mathfrak{E}_n$ denote the points of $\mathfrak{B}$ at which

$$\int_{\mathfrak{c}_n} f \leq \epsilon.$$

Then

$$\lim_{n=\infty} \overline{\overline{\mathfrak{E}}}_n = \widehat{\mathfrak{B}}.$$

The proof is analogous to that in 74. Instead of a cubical division of the space $\mathfrak{R}_p$, we use a standard enclosure. The sets $\mathfrak{B}_n$ are now measurable, and thus

$$\mathfrak{b} = Dv\{\mathfrak{B}_n\}$$

is measurable. Thus $\overline{\overline{\mathfrak{b}}}_n \doteq \widehat{\mathfrak{b}}$. The rest of the proof is as in 74.

3. The theorem of 75 becomes:

Let the integral

$$\int_{\mathfrak{C}} f \quad , \quad f \geq 0$$

be limited in complete $\mathfrak{B}$. Then

$$\lim_{n=\infty} \overline{\int_{\mathfrak{B}}} \int_{\mathfrak{c}_n} f = 0.$$

The proof is entirely similar to that in 75, except that we use extremal sequences, instead of cubical divisions.

4. As a corollary of 3 we have

Let the integral

$$\int_{\mathfrak{C}} f \quad , \quad f \geq 0$$

be limited and L-integrable in $\mathfrak{B}$. *Let* $\mathfrak{B} = \{B_m\}$ *the union of an enumerable set of complete sets. Then*

$$\lim_{n=\infty} \int_{\mathfrak{B}} \int_{\mathfrak{C}_n} f = 0.$$

For if $\mathfrak{B}_m = (B_1, B_2 \cdots B_m)$, and $\mathfrak{B} = \mathfrak{B}_m + \mathfrak{D}_m$, we have

$$\int_{\mathfrak{B}} \int_{\mathfrak{C}_n} = \int_{\mathfrak{B}_m} \int_{\mathfrak{C}_n} + \int_{\mathfrak{D}_m} \int_{\mathfrak{C}_n}.$$

But for m sufficiently large, $\widehat{\mathfrak{D}}_m$ is small at pleasure. Hence

$$\int_{\mathfrak{D}_m} \int_{\mathfrak{C}_n} < \int_{\mathfrak{D}_m} \int_{\mathfrak{C}} < \epsilon.$$

We have now only to apply 3.

434. 1. We are now in position to prove the analogue of 76, viz.:

Let $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$ *be measurable. Let* $\int_{\mathfrak{A}} f$ *be convergent. Let the integrals* $\int_{\mathfrak{C}} f$ *converge in* $\mathfrak{B}$, *except possibly at a null set. Then*

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{B}} \int_{\mathfrak{C}} f, \tag{1}$$

provided the integral on the right is convergent.

We follow along the line of proof in 76, and begin by taking $f \geq 0$ in $\mathfrak{A}$. By 428, we have

$$\int_{\mathfrak{A}_n} f = \int_{\mathfrak{B}} \int_{\mathfrak{C}_n} f;$$

hence

$$\int_{\mathfrak{A}} f = \lim_{n=\infty} \int_{\mathfrak{B}} \int_{\mathfrak{C}_n} f. \tag{2}$$

Now $\epsilon > 0$ being small at pleasure,

$$-\epsilon + \int_{\mathfrak{B}} \int_{\mathfrak{C}} f < \int_{\mathfrak{B}_G} \int_{\mathfrak{C}} f \quad , \quad \text{for } G > \text{some } G_0,$$

$$\leq \int_{\mathfrak{B}_G} \left\{ \int_{\mathfrak{C}_n} + \int_{\mathfrak{c}_n} \right\}$$

$$\leq \int_{\mathfrak{B}} \int_{\mathfrak{C}_n} + \int_{\mathfrak{B}} \int_{\mathfrak{c}_n}.$$

Since we have seen that we may regard $\mathfrak{B}$ as the union of an enumerable set of complete sets, we see that the last term on the right $\doteq 0$, as $n \doteq \infty$, by 433, 4. Thus

$$\int_{\mathfrak{B}} \int_{\mathfrak{C}} \leq \lim \int_{\mathfrak{B}} \int_{\mathfrak{C}_n} = \int_{\mathfrak{A}}, \tag{3}$$

by 2). On the other hand,

$$\int_{\mathfrak{B}} \int_{\mathfrak{C}_n} \leq \int_{\mathfrak{B}} \int_{\mathfrak{C}}.$$

Hence
$$\int_{\mathfrak{A}} = \lim \int_{\mathfrak{B}} \int_{\mathfrak{C}_n} \leq \int_{\mathfrak{B}} \int_{\mathfrak{C}}. \tag{4}$$

From 3) and 4) we have 1), when $f \geq 0$.

The general case is now obviously true. For

$$\mathfrak{A} = \mathfrak{P} + \mathfrak{N},$$

where $f \geq 0$ in $\mathfrak{P}$, and < 0 in $\mathfrak{N}$. Here $\mathfrak{P}$ and $\mathfrak{N}$ are measurable. We have therefore only to use 1) for each of these fields and add the results.

2. The theorem 1 states that if

$$\int_{\mathfrak{A}} f \quad , \quad \int_{\mathfrak{B}} \int_{\mathfrak{C}} f,$$

both converge, they are equal. *Hobson** in a remarkable paper on Lebesgue Integrals has shown that it is only necessary to assume the convergence of the first integral; the convergence of the second follows then as a necessary consequence.

* *Proceedings of the London Mathematical Society*, Ser. 2, vol. 8 (1900), p. 31.

435. We close this chapter by proving a theorem due to *Lebesgue*, which is of fundamental importance in the theory of Fourier's Series.

Let $f(x)$ be properly or improperly L-integrable in the interval $\mathfrak{A} = (a < b)$. *Then*

$$\lim_{\delta=0} J_f = \lim_{\delta=0} \int_a^{\beta} |f(x+\delta) - f(x)|\, dx$$

$$= \lim_{\delta=0} \int_a^{\beta} |\Delta f|\, dx = 0, \qquad a < \beta < \beta + \delta \leq b. \tag{1}$$

For in the first place,

$$J_f \leq \int_a^{\beta} |f(x+\delta)|\, dx + \int_a^{\beta} |f|\, dx \leq 2 \int_a^{b} |f|\, dx. \tag{2}$$

Next we note that

$$|f(x+\delta) - f(x)| - |g(x+\delta) - g(x)|$$
$$\leq |(f(x+\delta) - g(x+\delta)) - (f(x) - g(x))|.$$

Hence

$$\int_a^{\beta} |\Delta f|\, dx - \int_a^{\beta} |\Delta g|\, dx \leq \int_a^{\beta} |\Delta(f-g)|\, dx,$$

or

$$J_f - J_g \leq J_{f-g}. \tag{3}$$

From 2), 3) we have

$$J_f < J_g + 2\int_a^{b} |f-g|\, dx. \tag{4}$$

Let now

$$g = f \qquad \text{for } |f| \leq G,$$
$$= 0 \qquad \text{for } |f| > G.$$

Then by 4),

$$J_f \leq J_g + 2\int_a^{b} |f-g|\, dx$$
$$\leq J_g + \epsilon',$$

where ϵ' is small at pleasure, for G sufficiently large. Thus the theorem is established, if we prove it for a limited function, $|g(x)| < G$.

Let us therefore effect a division of the interval $\Gamma = (-G, G)$, of norm d, by interpolating the points

$$-G < e_1 < e_2 < \cdots < G,$$

causing Γ to fall into the intervals

$$\gamma_1, \gamma_2, \gamma_3 \cdots$$

Let $h_m = c_m$ for those values of x for which $g(x)$ falls in the interval γ_m, and $= 0$ elsewhere in $\mathfrak{A}$. Then

$$J_g \leq \Sigma J_{h_\iota} + 2\int_a^\beta (g - \Sigma h_\iota)dx$$

$$\leq \Sigma J_{h_\iota} + 2\, d\widehat{\mathfrak{A}}$$

$$< \Sigma J_{h_\iota} + \epsilon', \qquad \epsilon' \text{ small at pleasure,}$$

for d sufficiently small.

Thus we have reduced the demonstration of our theorem to a function $h(x)$ which takes on but two values in $\mathfrak{A}$, say 0 and γ.

Let $\mathfrak{E}$ be a $\sigma/4$ enclosure of the points where $h = \gamma$, while $\mathfrak{F}$ may denote a finite number of intervals of $\mathfrak{E}$ such that $\widehat{\mathfrak{F}} - \widehat{\mathfrak{E}} < \sigma/4$.

Let $\phi = \gamma$ in $\mathfrak{E}$, and elsewhere $= 0$; let $\psi = \gamma$ in $\mathfrak{F}$, and elsewhere $= 0$. Thus using 4),

$$J_h \leq J_\phi + \int_a^\beta | h - \phi |$$

$$\leq J_\phi + \frac{\sigma}{2}\cdot\gamma,$$

since $h = \phi$ in (α, β), except at points of measure $< \sigma/4$. Similarly

$$J_\phi \lesseqgtr J_\psi + \frac{\sigma}{2}\gamma.$$

Thus
$$J_h \leq J_\psi + \sigma\gamma < J_\psi + \epsilon,$$

for σ sufficiently small.

Thus the demonstration is reduced to proving it for a ψ which is continuous, except at a finite number of points. But for such a function, it is obviously true.

CHAPTER XIII

FOURIER'S SERIES

Preliminary Remarks

436. 1. Let us suppose that the limited function $f(x)$ can be eveloped into a series of the type

$$f(x) = a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \cdots$$
$$+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots \qquad (1$$

hich is valid in the interval $\mathfrak{A} = (-\pi, \pi)$. If it is also known hat this series can be integrated termwise, the coefficients a_n, b_n an be found at once as follows. By hypothesis

$$\int_{-\pi}^{\pi} f dx = a_0 \int_{-\pi}^{\pi} dx + a_1 \int_{-\pi}^{\pi} \cos x dx + \cdots$$
$$+ b_1 \int_{-\pi}^{\pi} \sin x dx + \cdots$$

As the terms on the right all vanish except the first, we have

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f dx. \qquad (2'$$

Let us now multiply 1) by $\cos nx$ and integrate.

$$\int_{-\pi}^{\pi} f(x) \cos nx dx = a_0 \int_{-\pi}^{\pi} \cos nx dx + a_1 \int_{-\pi}^{\pi} \cos x \cos nx dx + \cdots$$
$$+ b_1 \int_{-\pi}^{\pi} \sin x \cos nx + \cdots$$

Now
$$\int_{-\pi}^{\pi} \cos mx \cos nx dx = 0 \quad , \quad m \neq n,$$
$$\int_{-\pi}^{\pi} \cos^2 nx dx = \pi,$$
$$\int_{-\pi}^{\pi} \sin mx \cos nx = 0.$$

Thus all the terms on the right of the last series vanish except the one containing a_n. Hence

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx dx. \tag{2''}$$

Finally multiplying 1) by $\sin nx$, integrating, and using the relations

$$\int_{-\pi}^{\pi} \sin mx \sin nx dx = 0 \quad , \quad m \neq n,$$

$$\int_{-\pi}^{\pi} \sin^2 nx dx = \pi,$$

we get

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx dx. \tag{2'''}$$

Thus under our present hypothesis,

$$f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(u) du + \frac{1}{\pi}\sum_1^\infty \cos nx \int_{-\pi}^{\pi} f(u) \cos nu du$$
$$+ \frac{1}{\pi}\sum_1^\infty \sin nx \int_{-\pi}^{\pi} f(u) \sin nu du. \tag{3}$$

The series on the right is known as *Fourier's series;* the coefficients 2) are called *Fourier's coefficients* or *constants*. When the relation 3) holds for a set of points $\mathfrak{B}$, we say $f(x)$ can be developed in a Fourier's series in $\mathfrak{B}$, or Fourier's development is valid in $\mathfrak{B}$.

2. Fourier thought that every continuous function in $\mathfrak{A}$ could be developed into a trigonometric series of the type 3). The demonstration he gave is not rigorous. Later *Dirichlet* showed that such a development is possible, provided the continuous function has only a finite number of oscillations in $\mathfrak{A}$. The function still regarded as limited may also have a finite number of *discontinuities of the first kind*, *i.e.* where

$$f(a+0) \quad , \quad f(a-0) \tag{4}$$

exist, but one at least is $\neq f(a)$.

At such a point a, Fourier's series converges to

$$\tfrac{1}{2}\{f(a+0) + f(a-0)\}.$$

Jordan has extended Dirichlet's results to functions having limited variation in $\mathfrak{A}$. Thus Fourier's development is valid in certain cases when f has an infinite number of oscillations or points of discontinuity. Fourier's development is also valid in certain cases when f is not limited in $\mathfrak{A}$, as we shall see in the following sections.

We have supposed that $f(x)$ is given in the interval $\mathfrak{A}=(-\pi, \pi)$. This restriction was made only for convenience. For if $f(x)$ is given in the interval $\mathfrak{J}=(a<b)$, we have only to change the variable by means of the relation

$$u=\frac{\pi(2x-a-b)}{b-a}.$$

Then when x ranges over $\mathfrak{J}$, u will range over $\mathfrak{A}$.

Suppose f is an *even function* in $\mathfrak{A}$; its development in Fourier's series will contain only cosine terms. For

$$f(x)=\sum_0^\infty(a_n\cos nx+b_n\sin nx),$$

$$f(-x)=\sum_0^\infty(a_n\cos nx-b_n\sin nx).$$

Adding and remembering that $f(x)=f(-x)$ in $\mathfrak{A}$, we get

$$f(x)=\tfrac{1}{2}\sum_0^\infty a_n\cos nx, \qquad f\ \textit{even}.$$

Similarly if f is *odd*, its development in Fourier's series will contain only sine terms;

$$f(x)=\tfrac{1}{2}\sum_1^\infty b_n\sin nx, \qquad f\ \textit{odd}.$$

Let us note that if $f(x)$ is given only in $\mathfrak{B}=(0, \pi)$, and has limited variation in $\mathfrak{B}$, we may develop f either as a sine or a cosine series in $\mathfrak{B}$. For let

$$\begin{aligned} g(x)&=f(x) \quad , \quad x \text{ in } \mathfrak{B}\\ &=f(-x) \quad , \quad x \text{ in } (-\pi, 0). \end{aligned}$$

Then g is an even function in $\mathfrak{A}$ and has limited variation. Using Jordan's result, we see g can be developed in a cosine series valid in $\mathfrak{A}$. Hence f can be developed in a cosine series valid in $\mathfrak{B}$.

In a similar manner, let

$$h(x) = f(x) \quad , \quad x \text{ in } \mathfrak{B}$$
$$= -f(-x) \quad , \quad -\pi \leq x < 0.$$

Then h is an odd function in $\mathfrak{A}$, and Fourier's development contains only sine terms.

Unless $f(0) = 0$, the Fourier series will not converge to $f(0)$ but to 0, on account of the discontinuity at $x = 0$. The same is true for $x = \pi$.

If f can be developed in Fourier's series valid in $\mathfrak{A} = (-\pi, \pi)$, the series 3) will converge for all x, since its terms admit the period 2π. Thus 3) will represent $f(x)$ in $\mathfrak{A}$, but will not represent it unless f also admits the period 2π. The series 3) defines a periodic function admitting 2π as a period.

EXAMPLES

437. We give now some examples. They may be verified by the reader under the assumption made in 436. Their justification will be given later

Example 1. $f(x) = x \quad , \quad$ for $-\pi \leq x \leq \pi$.
Then

$$x = 2\left\{\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right\}.$$

If we set $x = \frac{\pi}{2}$, we get *Leibnitz's formula*,

$$\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Example 2. $f(x) = x \quad , \quad 0 \leq x \leq \pi$

$$= -x \quad , \quad -\pi \leq x \leq 0.$$

Then

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left\{\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right\}.$$

If we set $x = 0$, we get

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

Example 3.
$$f(x) = 1 \quad , \quad 0 < x < \pi$$
$$= 0 \quad , \quad x = 0, \pm \pi$$
$$= -1 \quad , \quad -\pi < x < 0.$$

Then
$$f(x) = \frac{4}{\pi}\left\{\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right\}.$$

Example 4.
$$f(x) = x \quad , \quad 0 \leq x \leq \frac{\pi}{2}$$
$$= \pi - x \quad , \quad \frac{\pi}{2} \leq x < \pi.$$

By defining f as an odd function, it can be developed in a sine eries, *valid in* $(0, \pi)$. We find
$$f(x) = \frac{4}{\pi}\left\{\frac{\sin x}{1} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \cdots\right\}.$$

Example 5.
$$f(x) = 1 \quad , \quad 0 \leq x \leq \frac{\pi}{2}$$
$$= -1 \quad , \quad \frac{\pi}{2} < x < \pi.$$

By defining f as an even function, we get a development in osines,
$$f(x) = \frac{4}{\pi}\left\{\frac{\cos x}{1} - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} - \cdots\right\},$$
alid in $(0, \pi)$.

Example 6.
$$f(x) = \tfrac{1}{2}(\pi - x) \quad , \quad 0 < x \leq \pi.$$

By defining f as an odd function we get a development in ines,
$$f(x) = \sin x + \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x + \cdots$$
alid in $(-\pi, \pi)$.

Example 7. Let
$$f(x) = \frac{\pi}{3} \quad , \quad 0 < x < \frac{\pi}{3}$$
$$= 0 \quad , \quad \frac{\pi}{3} < x < \frac{2\pi}{3}$$
$$= -\frac{\pi}{3} \quad , \quad \frac{2\pi}{3} < x < \pi.$$

Developing f as a sine series, we get

$$f(x) = \sin 2x + \frac{\sin 4x}{2} + \frac{\sin 8x}{4} + \cdots$$

valid in $(0, \pi)$.

Example 8. $f(x) = e^x$, in $(-\pi, \pi)$.

We find

$$f(x) = \frac{2 \sinh \pi}{\pi} \left\{ \frac{1}{2} - \frac{1}{1+1^2} \cos x + \frac{1}{1+2^2} \cos 2x - \frac{1}{1+3^2} \cos 3x + \cdots \right.$$
$$\left. + \frac{1}{1+1^2} \sin x - \frac{2}{1+2^2} \sin 2x + \frac{3}{1+3^2} \sin 3x - \cdots \right\}$$

valid for $-\pi < x < \pi$.

Example 9. We find

$$\cos \mu x = \frac{2\mu}{\pi} \sin \pi\mu \left\{ \frac{1}{2\mu^2} - \frac{\cos x}{\mu^2 - 1} + \frac{\cos 2x}{\mu^2 - 2^2} - \frac{\cos 3x}{\mu^2 - 3^2} + \cdots \right\}$$

valid for

$$-\pi \leq x \leq \pi \quad , \quad \mu^2 \neq 1, 2^2, 3^2, \cdots$$

Let us set $x = \pi$, and replace μ by x; we get

$$\frac{\pi}{2x} \cot \pi x = \frac{1}{2x^2} + \frac{1}{x^2 - 1^2} + \frac{1}{x^2 - 2^2} + \frac{1}{x^2 - 3^2} + \cdots$$

a decomposition of $\cot \pi x$ into partial fractions, a result already found in 216.

Example 10. We find

$$\sin x = \frac{2}{\pi} \left\{ 1 - \frac{2 \cos 2x}{1 \cdot 3} - \frac{2 \cos 4x}{3 \cdot 5} - \frac{2 \cos 6x}{5 \cdot 7} \cdots \right\},$$

valid for $0 \leq x \leq \pi$.

Summation of Fourier's Series

438. In order to justify the development of $f(x)$ in Fourier's series F, we will actually sum the F series and show that it converges to $f(x)$ in certain cases. To this end let us suppose that $f(x)$ is given in the interval $\mathfrak{A} = (-\pi, \pi)$, and let us extend f by giving it the period 2π. Moreover, at the points of discontinuity of the first kind, let us suppose

$$f(x) = \tfrac{1}{2} \{ f(x+0) + f(x-0) \}.$$

hen the function

$$\phi(u) = f(x+2u) + f(x-2v) - 2f(x)$$

continuous at $u=0$, and has the value 0, at points of continuity, l at points of discontinuity of 1° kind of f. Finally let us suppose that f is (properly or improperly) L-integrable in $\mathfrak{A}$; this t condition being necessary, in order to make the Fourier coeficients a_n, b_n have a sense.

Let

$$F = F(x) = \tfrac{1}{2}a_0 + a_1 \cos x + a_2 \cos 2x + \cdots$$
$$+ b_1 \sin x + b_2 \sin 2x + \cdots \qquad (1$$
$$= \tfrac{1}{2}a_0 + \sum_1^\infty (a_n \cos nx + b_n \sin nx),$$

ere we will now write

$$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x) \cos nx dx, \qquad (2'$$

$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x) \sin nx dx. \qquad (2''$$

Since $f(x)$ is periodic, the coefficients a_n, b_n have the same value vever c is chosen. If we make $c = -\pi$, these integrals reduce hose given in 436.

We may write

$$F = \frac{1}{\pi}\int_c^{c+2\pi} f(t)dt\ \{\tfrac{1}{2} + \sum_1^\infty(\cos nx \cos nt + \sin nx \sin nt)\}$$

$$= \frac{1}{\pi}\int_c^{c+2\pi}\{\tfrac{1}{2} + \sum_1^\infty \cos n(t-x)\} f(t)dt. \qquad (3$$

us

$$F_n = \frac{1}{\pi}\int_c^{c+2\pi} P_n \cdot f(t)dt,$$

ere

$$P_n = \tfrac{1}{2} + \sum_1^n \cos m(t-x). \qquad (4$$

Provided

$$\sin \tfrac{1}{2}(t-x) \neq 0, \qquad (5$$

may write

$$= \frac{1}{2\sin\frac{1}{2}(t-x)}\{\sin\tfrac{1}{2}(t-x) + \sum_1^m 2\sin\tfrac{1}{2}(t-x)\cos m(t-x)$$

$$= \frac{1}{2\sin\frac{1}{2}(t-x)}\Big[\sin\tfrac{1}{2}(t-x)$$
$$+ \sum_1^n\Big\{\sin\frac{2m+1}{2}(t-x) - \sin\frac{2m-1}{2}(t-x)\Big\}\Big].$$

Thus
$$P_n = \frac{\sin \frac{1}{2}(2n+1)(t-x)}{2 \sin \frac{1}{2}(t-x)}, \tag{6}$$

if 5) holds. Let us see what happens when 5) does not hold. In this case $\frac{1}{2}(t-x)$ is a multiple of π. As both t and x lie in $(c, c+2\pi)$, this is only possible for three singular values:

$$t = x \quad ; \quad t = c, \quad x = c + 2\pi \quad ; \quad t = c + 2\pi, \quad x = c.$$

For these singular values 4) gives

$$P_n = \frac{2n+1}{2}. \tag{7}$$

As P_n is a continuous function of t, x, the expression on the right of 6) must converge to the value 7) as x, t converge to these singular values. We will therefore assign to the expression on the right of 6) the value 7), for the above singular values. Then in all cases

$$F_n = \frac{1}{\pi}\int_c^{c+2\pi} \frac{\sin \frac{1}{2}(2n+1)(t-x)}{2 \sin \frac{1}{2}(t-x)} f(t)dt.$$

Let us set
$$2n + 1 = \nu \quad , \quad t - x = u.$$

Then
$$F_n = \frac{1}{\pi}\int_{\frac{1}{2}(c-x)}^{\frac{1}{2}(c-x)+\pi} f(x+2u)\frac{\sin \nu u}{\sin u}du.$$

Let us choose c so that
$$c - x = -\pi,$$

then
$$\pi F_n = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \int_{-\frac{\pi}{2}}^{0} + \int_0^{\frac{\pi}{2}}.$$

Replacing u by $-u$ in the first integral on the right, it becomes

$$\int_0^{\frac{\pi}{2}} f(x-2u)\frac{\sin \nu u}{\sin u}du.$$

Thus we get

$$F_n = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \{f(x+2u) + f(x-2u)\} \frac{\sin \nu u}{\sin u}du. \tag{8}$$

Let us now introduce the term $-2f(x)$ under the sign of integration in order to replace the brace by $\phi(u)$. To this end let us

ve x an arbitrary but fixed value and consider the Fourier's ies for the function

$$g(t)=f(x), \text{ a constant.}$$

If we denote the Fourier series corresponding to the g function

$$G=\tfrac{1}{2}g_0+g_1\cos t+g_2\cos 2t+\cdots$$
$$+h_1\sin t+h_2\sin 2t+\cdots$$

have

$$g_0=\frac{1}{\pi}\int_c^{c+2\pi}f(x)dt=2f(x),$$

$$g_n=\frac{f(x)}{\pi}\int_c^{c+2\pi}\cos ntdt=0,$$

$$h_n=\frac{f(x)}{\pi}\int_c^{c+2\pi}\sin ntdt=0.$$

Thus the sum of the first $n+1$ terms of the Fourier series longing to $g(t)$ reduces to

$$G_n=f(x). \tag{9}$$

But this sum is also given by 8), if we replace

$$f(x+2u)+f(x-2u)$$
$$g(x+2u)+g(x-2u)=2f(x),$$

nce g is a constant. We get thus

$$G_n=\frac{1}{\pi}\int_0^{\frac{\pi}{2}}2f(t)\frac{\sin nu}{\sin u}du. \tag{10}$$

Let us therefore subtract $f(x)$ from both sides of 8), using 9),). We get

$$(x)-f(x)=\frac{1}{\pi}\int_0^{\frac{\pi}{2}}\{f(x+2u)+f(x-2u)-2f(x)\}\frac{\sin \nu u}{\sin u}du.$$

Setting

$$D_n(x)=\pi\{F_n(x)-f(x)\}, \tag{11}$$

have

$$D_n(x)=\int_0^{\frac{\pi}{2}}\phi(u)\frac{\sin \nu u}{\sin u}du. \tag{12}$$

We have thus the theorem:

For the Fourier Series to converge to $f(x)$ at the point x, it is cessary and sufficient that $D_n(x)\doteq 0$, as $n\doteq\infty$.

Validity of Fourier's Development *

439. The integral on the right side of 438, 12), on which the validity of Fourier's development at the point x depends, is a special case of the integral

$$J_n = \int_{\mathfrak{B}} g(u) \sin nu du \quad , \quad \mathfrak{B} = (a < b). \qquad (1$$

In fact J_n goes over into D_n, if we set

$$g = \frac{\phi}{\sin u} \quad , \quad a = 0, \quad b = \frac{\pi}{2}.$$

To evaluate J_n let us break $\mathfrak{B}$ up into the intervals

$$\mathfrak{B}_0 = \left(a, a + \frac{\pi}{n}\right) \quad , \quad \mathfrak{B}_1 = \left(a + \frac{\pi}{n}, a + \frac{2\pi}{n}\right) \cdots \mathfrak{B}_r = \left(a + r\frac{\pi}{n}, b\right).$$

These intervals are equal except the last, which is shorter than the others unless $b - a$ is a multiple of π/n. We have thus

$$J_n = \int_{\mathfrak{B}_0} + \int_{\mathfrak{B}_1} + \cdots + \int_{\mathfrak{B}_r}.$$

If we set

$$v = u + \frac{\pi}{n},$$

we see that while v ranges over $\mathfrak{B}_{2s}$, u ranges over $\mathfrak{B}_{2s-1}$. This substitution enables us to replace the integrals over $\mathfrak{B}_{2s}$ by those over $\mathfrak{B}_{2s-1}$, since

$$\int_{\mathfrak{B}_{2s}} g(v) \sin nv dv = -\int_{\mathfrak{B}_{2s-1}} g\left(u + \frac{\pi}{n}\right) \sin nu du.$$

Hence grouping the integrals in pairs, we get

$$J_n = \int_{\mathfrak{B}_0} g(u) \sin nu du + \sum_s \int_{\mathfrak{B}_{2s-1}} \left\{ g(u) - g\left(u + \frac{\pi}{n}\right)\right\} \sin nu du$$
$$+ \int_{\mathfrak{B}'} g(u) \sin nu du,$$

* The presentation given in 439–448 is due in the main to *Lebesgue*. Cf. his classic paper, *Mathematische Annalen*, vol. 61 (1905), p. 251. Also his *Leçons sur les Séries Trigonométriques*, Paris, 1906.

where $\mathfrak{B}'$ is $\mathfrak{B}_r$ or $\mathfrak{B}_{r-1}+\mathfrak{B}_r$, depending on the parity of r. Now

$$\left|\int_{\mathfrak{B}_0}\right| \leq \int_{\mathfrak{B}_0}|g|, \tag{2}$$

$$\left|\sum\int_{\mathfrak{B}_{2s-1}}\left\{g(u)-g\left(u+\frac{\pi}{n}\right)\right\}\sin nu\,du\right|$$

$$<\sum\int_{\mathfrak{B}_{2s-1}}\left|g\left(u+\frac{\pi}{n}\right)-g(u)\right|du$$

$$\leq \int_{a+\frac{\pi}{n}}^{b-\frac{\pi}{n}}\left|g\left(u+\frac{\pi}{n}\right)-g(u)\right|du. \tag{3}$$

$$\left|\int_{\mathfrak{B}'}\right| \leq \int_{\mathfrak{B}'}|g|. \tag{4}$$

Thus $J_n \doteq 0$, if the three integrals 2), 3), 4) $\doteq 0$. Moreover, if these three integrals are *uniformly evanescent* with respect to some point set $\mathfrak{C} \leq \mathfrak{B}$, J_n is also uniformly evanescent in $\mathfrak{C}$. In particular we note the theorem

$J_n \doteq 0$, if g is L-integrable in $\mathfrak{B}$.

We are now in a position to draw some important conclusions with respect to Fourier's series.

440. 1. *Let $f(x)$ be L-integrable in $(c, c+2\pi)$. Then the Fourier constants $a_n, b_n \doteq 0$, as $n \doteq \infty$.*

For
$$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx$$

is a special case of the J_n integral. As f is L-integrable, we need only apply the theorem at the close of the last article. Similar reasoning applies to b_n.

2. *For a given value of x in $\mathfrak{A} = (-\pi, \pi)$ let*

$$\psi(u) = \frac{\phi(u)}{\sin u}, \tag{1}$$

be L-integrable in $\mathfrak{B} = \left(0, \frac{\pi}{2}\right)$. Then Fourier's development is valid at the point x.

For by 438, Fourier's series $\doteq f(x)$ at the point x, if $D_n(x) \doteq 0$. But D_n is a special case of J_n for which the g function is integrable. We thus need only apply 439.

3. *For a given x in $\mathfrak{A} = (-\pi, \pi)$, let*

$$\chi(u) = \frac{\phi(u)}{u} \tag{2}$$

be L-integrable in $\mathfrak{B} = \left(0, \frac{\pi}{2}\right)$. Then Fourier's development is valid at the point x.

For let $\delta > 0$, then

$$\int_0^\delta |\psi| du = \int_0^\delta \left|\frac{\phi(u)}{\sin u}\right| \leq \int_0^\delta \frac{|\phi(u)|(1+\eta)}{u} du$$

$$\leq (1+\eta) \int_0^\delta |\chi(u)| du$$

$$\doteq 0 \quad , \quad \text{as } \delta \doteq 0 \quad , \quad \text{by hypothesis.}$$

4. *For a given x in $\mathfrak{A} = (-\pi, \pi)$, let*

$$\omega(u) = \frac{f(x+u) - f(x)}{u} \tag{3}$$

be L-integrable in $\mathfrak{A}$. Then Fourier's development is valid at the point x.

For

$$\chi(u) = \frac{f(x+2u) - f(x)}{u} + \frac{f(x-2u) - f(x)}{u}$$

$$= 2[\omega(2u) + \omega(-2u)].$$

Thus χ is L-integrable in $\left(0, \frac{\pi}{2}\right)$, as it is the difference of two integrable functions.

441. (*Lebesgue*). *For a given x in $\mathfrak{A} = (-\pi, \pi)$ let*

$$1° \quad \lim_{n=\infty} n \int_0^{\frac{\pi}{n}} |\phi(u)| du = 0 ;$$

$$2° \quad \lim_{\delta=0} \int_\delta^\eta |\psi(u+\delta) - \psi(u)| du = 0$$

for some η such that

$$0 < \delta < \eta \leq \frac{\pi}{2}.$$

Then Fourier's development is valid at the point x.

For as we have seen,

$$|D_n| \leq \int_0^{\frac{\pi}{\nu}} \left|\frac{\phi(u)}{\sin u}\sin \nu u\right| du + \int_{\frac{\pi}{\nu}}^{\frac{\pi}{2}} \left|\psi\left(u+\frac{\pi}{n}\right) - \psi(u)\right| du$$

$$+ \int_{\beta_n}^{\frac{\pi}{2}} \left|\frac{\phi(u)}{\sin u}\sin \nu u\right| du \leq D' + D'' + D''',$$

where β_n is a certain number which $\doteq \frac{\pi}{2}$, as $n \doteq \infty$.

Let us first consider D'. Since $0 \leq u \leq \frac{\pi}{\nu}$, we have $0 \leq \nu u \leq \pi$. Hence

$$\frac{\sin \nu u}{\sin u} = \frac{\nu u - \frac{\nu^3 u^3}{6} + \sigma \frac{\nu^4 u^4}{24}}{u - \frac{u^3}{6} + \tau \frac{u^4}{24}}, \qquad 0 < \sigma,\ \tau \leq 1$$

$$= \nu \frac{1 - \frac{\nu^2 u^2}{6}\left(1 - \frac{\sigma \nu u}{4}\right)}{1 - \frac{u^2}{6}\left(1 - \frac{\tau u}{4}\right)} = \nu \frac{1 - s\frac{u^2}{6}}{1 - t\frac{u^2}{6}}$$

$$\leq \nu, \text{ provided } s \geq t.$$

But this is indeed so. For

$$1 - \frac{\nu\sigma u}{4} \geq 1 - \frac{\pi}{4}.$$

Hence
$$s \geq \nu^2\left(1 - \frac{\pi}{4}\right) > 1 > t \quad , \quad \text{if } \nu \geq 5.$$

Thus
$$D' < \nu \int_0^{\frac{\pi}{\nu}} |\phi|\, du \doteq 0, \text{ by hypothesis.}$$

We now turn to D''. We have

$$D'' = \int_{\frac{\pi}{\nu}}^{\frac{\pi}{2}} = \int_{\frac{\pi}{\nu}}^{\eta} + \int_{\eta}^{\frac{\pi}{2}} \quad , \quad \frac{\pi}{\nu} < \eta < \frac{\pi}{2}.$$

Now f being L-integrable,

$$\left|\psi\left(u+\frac{\pi}{n}\right)-\psi(u)\right|$$

is L-integrable in $\left(\eta, \frac{\pi}{2}\right)$. Thus

$$\lim_{n=\infty}\int_{\eta}^{\frac{\pi}{2}}=0.$$

But by condition 2°, $$\lim_{\nu=\infty}\int_{\frac{\pi}{\nu}}^{\eta}=0.$$

Thus $$\lim_{\delta=0} D''=0.$$

Finally we consider D'''. But the integrand is an integrable function in $\left(\beta, \frac{\pi}{2}\right)$. Thus it $\doteq 0$ as $n \doteq \infty$.

442. 1. *The validity of Fourier's development at the point x depends only on the nature of f in a vicinity of x, of norm δ as small as we please.*

For the conditions of the theorem in 441 depend only on the value of f in such a vicinity.

2. Let us call a point x at which the function

$$\phi(u)=f(x+2u)+f(x-2u)-2f(x)$$

is continuous at $u=0$, and has the value 0, a *regular point.*

In 438, we saw that if x is a point of discontinuity of the first kind for $f(x)$, then x is a regular point.

3. *Fourier's development is valid at a regular point x, provided for some η*

$$\lim_{\delta=0}\int_{\delta}^{\eta}|\psi(u+\delta)-\psi(u)|\,du=0\ ,\quad 0<\delta<\eta\leq\frac{\pi}{2}.$$

For at a regular point x, $\phi(u)$ is continuous at $u=0$, and $=0$ for $u=0$. Now

$$\lim_{h=0}\frac{1}{h}\int_{0}^{h}|\phi(u)|\,du=|\phi(0)|=0.$$

Thus
$$n\int_0^{\frac{\pi}{n}} |\phi(u)|\, du = \pi \cdot \frac{1}{\frac{\pi}{n}} \cdot \int_0^{\frac{\pi}{n}} |\phi|\, du$$
$$\doteq \pi\,|\phi(0)| = 0.$$

Hence condition 1° of 441 is satisfied.

Limited Variation

443. 1. Before going farther we must introduce a few notions lative to the variation of a function $f(x)$ defined over an interval $= (a < b)$. Let us effect a division D of $\mathfrak{A}$ into subintervals, interpolating a finite number of points $a_1 < a_2 < \cdots$ The sum
$$V_D = \Sigma\,|\,f(a_\iota) - f(a_{\iota+1})\,| \qquad (1$$
called *the variation of f in $\mathfrak{A}$ for the division D.* If
$$\text{Max } V_D \qquad (2$$
finite with respect to the class of all finite divisions of $\mathfrak{A}$, we say *has finite variation in* $\mathfrak{A}$. When 2) is finite, we denote its value by
$$\text{Var} f, \quad \text{or} \quad V_f, \quad \text{or} \quad V$$
d call it the *variation of f in* $\mathfrak{A}$.

We shall show in 5 that finite variation means the same thing limited variation introduced in I, 509. We use the term finite riation in sections 1 to 4 only for clearness.

2. A most important property of functions having finite vari- ion is brought out by the following geometric consideration.

Let us take two monotone increasing curves A, B such that one them crosses the other a finite or infinite number of times. If (x), $g(x)$ are the continuous functions having these curves as aphs, it is obvious that
$$d(x) = f(x) - g(x)$$
a continuous function which changes its sign, when the curves , B cross each other. Thus we can construct functions in infinite riety, which oscillate infinitely often in a given interval, and hich are the difference of two monotone increasing functions.

For simplicity we have taken the curves A, B continuous. A moment's reflection will show that this is not necessary.

Since $d(x)$ is the difference of two monotone increasing functions, its variation is obviously finite. Jordan has proved the following *fundamental theorem.*

3. *If $f(x)$ has finite variation in the interval $\mathfrak{A} = (a < b)$, there exists an infinity of limited monotone increasing functions $g(x)$, $h(x)$ such that*

$$f = g - h. \tag{1}$$

For let D be a finite division of $\mathfrak{A}$. Let

$$P_D = \text{sum of terms } \{f(a_{m+1}) - f(a_m)\} \text{ which are } \geq 0,$$
$$-N_D = \ldots\ldots\ldots\ldots\ldots\ldots < 0.$$

Then

$$V_D = \Sigma \,|\, f(a_{m+1}) - f(a_m) \,|\, = P_D + N_D. \tag{2}$$

Also

$$\{f(a_1) - f(a)\} + \{f(a_2) - f(a_1)\} + \cdots + \{f(b) - f(a_n)\} = P_D - N_D.$$

On the left the sum is telescopic, hence

$$f(b) - f(a) = P_D - N_D. \tag{3}$$

From 2), 3) we have

$$V_D = 2\,P_D + f(a) - f(b) = 2\,N_D + f(b) - f(a). \tag{4}$$

Let now

$$\text{Max}\, P_D = P \quad , \quad \text{Max}\, N_D = N$$

with respect to the class of finite divisions D.

We call them the *positive* and *negative variation* of $f(x)$ in $\mathfrak{A}$. Then 4) shows that

$$V = 2\,P + f(a) - f(b) \quad , \quad V = 2\,N + f(b) - f(a). \tag{5}$$

Adding these, we get

$$V = P + N. \tag{6}$$

From 5) we have

$$f(b) - f(a) = P - N. \tag{7}$$

Instead of the interval $\mathfrak{A} = (a < b)$, let us take the interval $(a < x)$, where x lies in $\mathfrak{A}$. Replacing b by x in 7), we have

$$f(x) = f(a) + P(x) - N(x). \tag{8}$$

Obviously $P(x)$, $N(x)$ are monotone increasing functions. et $\mu(x)$ be a monotone increasing function in $\mathfrak{A}$. If we set

$$g(x) = f(a) + P(x) + \mu(x)$$
$$h(x) = N(x) + \mu(x), \qquad (9$$

e get 1) from 8) at once.

4. From 8) we have

$$|f(x)| \leq |f(a)| + P(x) + N(x)$$
$$\leq |f(a)| + V(x). \qquad (10$$

5. We can now show that *when $f(x)$ has finite variation in the terval $\mathfrak{A} = (a < b)$ it has limited variation and conversely.*

For if *f has finite variation* in $\mathfrak{A}$ we can set

$$f(x) = \phi(x) - \psi(x),$$

here ϕ, ψ are monotone increasing in $\mathfrak{A}$. Then if $\mathfrak{A}$ is divided o the intervals $\delta_1, \delta_2 \cdots$ we have

$$\mathrm{Osc}\, f \leq \mathrm{Osc}\, \phi + \mathrm{Osc}\, \psi \quad , \quad \text{in } \delta_\iota.$$

But

$$\mathrm{Osc}\, \phi = \Delta\phi \quad , \quad \mathrm{Osc}\, \psi = \Delta\psi \quad , \quad \text{in } \delta_\iota$$

nce these functions are monotone. Hence summing over all the ervals δ_ι,

$$\Sigma\, \mathrm{Osc}\, f \leq \Sigma\Delta\phi + \Sigma\Delta\psi$$
$$\leq \{\phi(b) - \phi(a)\} + \{\psi(b) - \psi(a)\}$$
$$\leq \text{some } M, \text{ for any division.}$$

Hence f has limited variation.

If *f has limited variation* in $\mathfrak{A}$,

$$|\Delta f| \leq \mathrm{Osc}\, f \quad , \quad \text{in } \delta_\iota.$$

Hence

$$\Sigma\, |\Delta f| \leq \Sigma\, \mathrm{Osc}\, f \leq \text{some } M.$$

Hence f has finite variation.

6. *If $f(x)$ has limited variation in the interval $\mathfrak{A}$, its points of ntinuity form a pantactic set in $\mathfrak{A}$.*

This follows from 5, and I, 508.

7. *Let $a < b < c$; then if f has finite variation in (a, c),*

$$V_{a,b}f + V_{b,c}f = V_{a,c}f, \tag{11}$$

where $V_{a,b}$ means the variation of f in the interval (a, b), etc.

For
$$V_{ac}f = \text{Max } V_D f$$

with respect to the class of all finite divisions D of (a, c). The divisions D fall into two classes:

1° those divisions E containing the point b,

2° the divisions F which do not.

Let Δ be a division obtained by interpolating one or more points in the interval. Obviously

$$V_\Delta f \geq V_D f.$$

Let now G be obtained from a division F by adding the point b. Then
$$V_G f \geq V_F f.$$

Hence
$$\underset{E}{\text{Max}}\ V_E \geq \underset{F}{\text{Max}}\ V_F.$$

Hence to find $V_{a,c}f$, we may consider only the class E. Let now E_1 be a division of (a, b), and E_2 a division of (b, c). Then $E_1 + E_2$ is a division of class E. Conversely each division of class E gives a division of (a, b), (b, c). Now

$$V_E f = V_{E_1} f + V_{E_2} f.$$

From this 11) follows at once.

444. We establish now a few simple relations concerning the variation of two functions in an interval $\mathfrak{A} = (a < b)$.

1.
$$V(f + c) = Vf. \tag{1}$$

For
$$\Sigma \,|\, (f_{\iota+1} + c) - (f_\iota + c) \,|= \Sigma \,|\, f_{\iota+1} - f_\iota \,|,$$

where for brevity we set
$$f_\iota = f(a_\iota).$$

2.
$$V(cf) = |\, c \,|\, Vf. \tag{2}$$

For
$$\Sigma \,|\, cf_{\iota+1} - cf_\iota \,| = |\, c \,|\, \Sigma \,|\, f_{\iota+1} - f_\iota \,|.$$

. *Let f, g be monotone increasing functions in $\mathfrak{A}$. Then*

$$V(f+g) = Vf + Vg. \tag{3}$$

or $\Sigma \, | (f_{\iota+1} + g_{\iota+1}) - (f_\iota + g_\iota) | = \Sigma \, | (f_{\iota+1} - f_\iota) + (g_{\iota+1} - g_\iota) |$

$$= \Sigma \, | f_{\iota+1} - f_\iota | + \Sigma \, | g_{\iota+1} - g_\iota |.$$

. *For any two functions f, g having limited variation,*

$$V(f+g) \leq Vf + Vg. \tag{4}$$

. *Let f, f_1 have limited variation in $\mathfrak{A} = (a, b)$.*

et $\qquad \alpha = | f(a) | \quad , \quad \alpha_1 = | f_1(a) |.$

hen $\qquad V(ff_1) \leq (\alpha + Vf)(\alpha_1 + Vf_1). \tag{5}$

or by 443, 8) we have

$$f = P - N + A \quad , \quad f_1 = P_1 - N_1 + A_1,$$

re $\qquad A = f(a) \quad , \quad A_1 = f_1(a).$

hus

$$= PP_1 - PN_1 + PA_1 - NP_1 + NN_1 - NA_1 + AP_1 - AN_1 + AA_1.$$

ence by 2, 4,

$$\begin{aligned} Vff_1 &\leq VPP_1 + VPN_1 + VPA_1 + \cdots \\ &\leq V(PP_1 + PN_1 + \cdots) \quad , \quad \text{by } 3 \\ &\leq PP_1 + PN_1 + P\alpha_1 + \cdots \\ &\leq (P + N + \alpha)(P_1 + N_1 + \alpha_1). \end{aligned}$$

ut $\qquad Vf = P + N \quad$, hence, etc.

45. *Fourier's development is valid at the regular point x, if there ts a $0 < \zeta \leq \frac{\pi}{2}$, such that in $(0, \zeta)$ the variation $V(u)$ of $\psi(u)$ ny (u, ζ) is limited, and such that $u V(u) \doteq 0$, $u \doteq 0$.*

y 442, we have only to show that

$$\Psi = \int_\delta^\eta | \psi(u + \delta) - \psi(u) | \, du \qquad 0 < \delta < \eta \leq \frac{\pi}{2}$$

vanescent with δ.

Let us *first* suppose that $\psi(u)$ is monotone in some $(0, \zeta)$, say monotone increasing. Similar reasoning will apply, if it is monotone decreasing. Then, taking $0 < \eta + \delta < \zeta$,

$$\Psi = \int_\delta^\eta \{\psi(u+\delta) - \psi(u)\}du = \int_\delta^\eta \psi(u+\delta)du - \int_\delta^\eta \psi(u)du.$$

In the second integral from the end, set $v = u + \delta$.

Then $$\int_\delta^\eta \psi(u+\delta)du = \int_{2\delta}^{\eta+\delta} \psi(v)dv.$$

Hence, $$\Psi = \int_{2\delta}^{\eta+\delta} \psi(u)du - \int_\delta^\eta \psi(u)du$$

$$= \int_{2\delta}^{\eta+\delta} - \int_\delta^{2\delta} - \int_{2\delta}^{\eta}.$$

Thus $$|\Psi(u)| \leq \int_\delta^{2\delta} |\psi|\,du + \int_\eta^{\eta+\delta} |\psi|\,du = \Psi_1 + \Psi_2.$$

We will consider the integrals on the right separately. Let

$$\phi_m = \text{Max}\,|\phi|, \qquad \text{in } (\delta, 2\delta).$$

Then $$\Psi_1 \leq \int_\delta^{2\delta} \frac{|\phi|}{\sin u}du \leq \phi_m \int_\delta^{2\delta} \frac{du}{\sin u}.$$

Now $$\sin u = u - \sigma' u^3 \quad , \quad 0 < \sigma' < \tfrac{1}{2}.$$

Hence, $$\frac{1}{\sin u} = \frac{1}{u} + \sigma u \quad , \quad |\sigma| < \text{some } M.$$

Thus, $$\Psi_1 \leq \phi_m \left\{ \int_\delta^{2\delta} \frac{du}{u} + M \int_\delta^{2\delta} u\,du \right\}$$

$$< \phi_m \{\log 2 + M'\delta^2\}$$

$$\doteq 0 \quad , \quad \text{as } \delta \doteq 0 \quad , \quad \text{since } \phi(u) \doteq 0,$$

as x is a regular point.

We turn now to Ψ_2. In $(\eta, \eta + \delta)$, δ, η sufficiently small,

$$\sin u > u - \tfrac{1}{6}u^3 > \eta(1 - \eta^2).$$

Thus, if $\phi_\mu = \text{Max} \mid \phi \mid$ in $(\eta, \eta + \delta)$,

$$\Psi_2 \leq \frac{\phi_\mu}{\eta(1-\eta)} \int_\eta^{\eta+\delta} du = \frac{\phi_\mu \delta}{\eta(1-\eta)} \doteq 0$$

with δ.

Thus, when ψ is monotone in some $(0, \zeta)$, Fourier's development is valid. But obviously when ψ is monotone, the condition that $uV(u) \doteq 0$ is satisfied. Our theorem is thus established in this case.

Let us now consider the case that the variation $V(u)$ of ψ is limited in (u, ζ).

From 443, 10), we have

$$\mid \psi(u) \mid \leq \mid \psi(\zeta) \mid + V(u).$$

As before we have

$$\mid \Psi \mid < \int_\delta^{2\delta} \mid \psi \mid du + \int_\eta^{\eta+\delta} \mid \psi \mid du = \Psi_1 + \Psi_2.$$

By hypothesis there exists for each $\epsilon > 0$, a $\delta_0 > 0$, such that

$$uV(u) \leq \epsilon \quad , \quad \text{for any } 0 < u \leq \delta_0.$$

Hence,

$$V(u) \leq \frac{\epsilon}{u} \quad , \quad 0 < u \leq \delta_0.$$

Thus,

$$\Psi_1 \leq \int_\delta^{2\delta} \mid \psi(\zeta) \mid du + \int_\delta^{2\delta} V(u) du$$

$$\leq \mid \psi(\zeta) \mid \delta + \epsilon \int_\delta^{2\delta} \frac{du}{u}$$

$$= \mid \psi(\zeta) \mid \delta + \epsilon \log 2.$$

Let us turn now to Ψ_2. Since $V(u)$ is the sum of two limited monotone decreasing functions P, N in (u, ζ), it is integrable.

Thus,

$$\Psi_2 \leq \mid \psi(\zeta) \mid \int_\eta^{\eta+\delta} du + \int_\eta^{\eta+\delta} V(u) du \leq \delta \{ \mid \psi(\zeta) \mid + V(\eta) \}$$

is evanescent with δ.

Fourier's development is valid at the regular point x, if ... *limited variation in some interval* $(0 < \zeta)$, $\zeta < \frac{\pi}{2}$.

... $0 < u < \gamma < \zeta$, then

$$V_{u\zeta}\psi = V_{u\gamma}\psi + V_{\gamma\zeta}\psi.$$

$$\psi = \phi(u) \cdot \frac{1}{\sin u}.$$

$$V_{u\gamma}\psi \leq \{ V_{u\gamma}\phi + |\phi(\gamma)| \} \left\{ V_{u\gamma}\frac{1}{\sin u} + \frac{1}{\sin \gamma} \right\}.$$

... n u being monotone,

$$V_{u\gamma}\frac{1}{\sin u} = \frac{1}{\sin u} - \frac{1}{\sin \gamma}.$$

$$V_{u\gamma}\psi \leq \frac{V_{u\gamma}\phi + |\phi(\gamma)|}{\sin u} = V_1.$$

...arly,
$$V_{\gamma\zeta}\psi \leq \frac{V_{\gamma\zeta}\phi + |\phi(\zeta)|}{\sin \gamma} = V_2.$$

$$0 < \frac{u}{\sin u} < M \quad , \quad \text{in } (0^*, \zeta).$$

... theorem now follows by 445. For we may take γ so small

$$V_{0\gamma}\phi < \frac{\epsilon}{4M} \quad , \quad |\phi(\gamma)| < \frac{\epsilon}{4M}. \tag{1}$$

...s for any $u < \gamma$,

$$uV_1 < \frac{\epsilon}{2}.$$

... the other hand, $\mathfrak{M}$ being sufficiently large, and γ chosen as ... and then fixed,

$$V_2 < \mathfrak{M}.$$

...us
$$uV_2 < \frac{\epsilon}{2},$$

... $<$ some δ'. Hence
$$uV_{u\zeta}\psi < \epsilon,$$

... $< u <$ some δ.

... *(Jordan.) Fourier's development is valid at the regular point* ... $f(x)$ *has limited variation in some domain of x.*

For ...

has lim...

3. ... *if f is* ...

447. ...

If X ...

For ...

where ...

Obvi...
provide...

Let ...

Then ...

Now ...

Z...

Thus ...

For $\phi(u) = \{f(x+2u) - f(u)\} + \{f(x-2u) - f(u)\}$

has limited variation also.

3. *Fourier's development is valid at every point of $\mathfrak{A} = (0, 2\pi)$, if f is limited and has only a finite number of oscillations in $\mathfrak{A}$.*

Other Criteria

447. *Let* $X = \int_\delta^\eta |\chi(u+\delta) - \chi(u)|\, du \quad , \quad \chi(u) = \frac{\phi(u)}{u},$

$$\Psi = \int_\delta^\eta |\psi(u+\delta) - \psi(u)|\, du \quad , \quad 0 < \delta < \eta \leq \frac{\pi}{2}.$$

If $X \doteq 0$ as $\delta \doteq 0$, so does Ψ, and conversely.

For
$$\chi(u+\delta) - \chi(u) = \psi(u+\delta)\frac{\sin(u+\delta)}{u+\delta} - \psi(u)\frac{\sin u}{u}$$
$$= \{\psi(u+\delta) - \psi(u)\}\frac{\sin(u+\delta)}{u+\delta} + \rho,$$

where
$$\rho = \psi(u)\left\{\frac{\sin(u+\delta)}{u+\delta} - \frac{\sin u}{u}\right\}.$$

Obviously X and Ψ are simultaneously evanescent with δ, provided
$$R = \int_\delta^\eta |\rho| \doteq 0 \quad , \quad \text{as } \delta \doteq 0.$$

Let
$$Z(u) = \frac{\sin u}{u}.$$

Then
$$\rho = \psi(u)\{Z(u+\delta) - Z(u)\}$$
$$= \delta\psi(u)Z'(v) \quad , \quad u < v < u + \delta.$$

Now
$$Z'(v) = \frac{v\cos v - \sin v}{v^2} = \frac{v\left(1 - \frac{v^2}{2} + \cdots\right) - v\left(1 - \frac{v^2}{6} + \cdots\right)}{v^2}$$
$$= -\tfrac{1}{3}v + \tfrac{1}{30}v^3 + \cdots$$

Thus
$$|Z'(v)| < Mv < M \cdot 2u.$$

Hence

$$|\rho| < \frac{2u\delta\,|\phi|\,M}{\sin u} < 2\,\delta\,|\phi|\,\mathfrak{M}.$$

As

$$|\phi| \leq |f(x+2u)| + |f(x-2u)| + 2\,|f(x)|,$$

$$R < 2\,\delta\mathfrak{M}\int_\delta^\eta |\phi| \doteq 0 \quad , \quad \text{with } \delta.$$

448. (*Lipschitz-Dini.*) *At the regular point x, Fourier's development is valid, if for each $\epsilon > 0$, there exists a $\delta_0 > 0$, such that for each $0 < \delta < \delta_0$,*

$$|\phi(u+\delta) - \phi(u)| < \frac{\epsilon}{|\log\delta|}, \qquad \textit{for any } u \textit{ in } (\delta, \delta_0).$$

For

$$|\chi(u+\delta) - \chi(u)| = \left|\frac{\phi(u+\delta) - \phi(u)}{u+\delta} + \left\{\frac{1}{u+\delta} - \frac{1}{u}\right\}\phi(u)\right|$$

$$< \frac{|\phi(u+\delta) - \phi(u)|}{u} + \delta\frac{|\phi(u)|}{u^2}.$$

Now x being a regular point, there exists an η' such that

$$|\phi(u)| < \epsilon, \qquad \text{for } u \text{ in any } (\delta, \eta').$$

Thus taking

$$\eta > \delta_0, \eta',$$

$$\mathrm{X} = \int_\delta^\eta |\chi(u+\delta) - \chi(u)|\,du < \frac{\epsilon}{|\log\delta|}\int_\delta^\eta \frac{du}{u} + \epsilon\delta\int_\delta^\eta \frac{du}{u^2}$$

$$< \epsilon\frac{\log\eta - \log\delta}{|\log\delta|} + \epsilon\delta\left(\frac{1}{\delta} - \frac{1}{\eta}\right)$$

$$< 2\,\epsilon, \qquad \text{for any } \delta < \eta.$$

Thus

$$\mathrm{X} \doteq 0, \text{ as } \delta \doteq 0.$$

Uniqueness of Fourier's Development

449. Suppose $f(x)$ can be developed in Fourier's series

$$f(x) = \tfrac{1}{2}a_0 + \sum_1^\infty (a_n\cos nx + b_n\sin nx), \tag{1}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx \quad , \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx, \tag{2}$$

lid in $\mathfrak{A} = (-\pi, \pi)$. We ask can $f(x)$ be developed in a simi- series

$$f(x) = \tfrac{1}{2} a_0' + \sum_1^\infty (a_n' \cos nx + b_n' \sin nx), \tag{3}$$

o valid in $\mathfrak{A}$, where the coefficients are not Fourier's coefficients, least not all of them.

Suppose this were true. Subtracting 1), 3) we get

$$0 = \tfrac{1}{2}(a_0 - a_0') + \Sigma\{(a_n - a_n') \cos nx + (b_n - b_n') \sin nx\} = 0,$$

$$c_0 + \Sigma\{c_n \cos nx + d_n \sin nx\} = 0, \qquad \text{in } \mathfrak{A}. \tag{4}$$

us it would be possible for a trigonometric series of the type to vanish without all the coefficients c_m, d_m vanishing.

For a power series

$$p_0 + p_1 x + p_2 x^2 + \cdots \tag{5}$$

vanish in an interval about the origin, however small, we know t all the coefficients p_m in 5) must $= 0$.

We propose to show now that a similar theorem holds for a gonometric series. In fact we shall prove the *fundamental*

Theorem 1. Suppose it is known that the series 4) *converges to* 0 *all the points of* $\mathfrak{A} = (-\pi, \pi)$, *except at a reducible set* $\mathfrak{R}$. *en the coefficients* c_m, d_m *are all* 0, *and the series* 4) $\doteq 0$ *at all the ints of* $\mathfrak{A}$.

From this we deduce at once as corollaries:

Theorem 2. Let $\mathfrak{R}$ *be a reducible set in* $\mathfrak{A}$. *Let the series*

$$\alpha_0 + \sum_1^\infty \{\alpha_n \cos nx + \beta_n \sin nx\} \tag{6}$$

verge in $\mathfrak{A}$, *except possibly at the points* $\mathfrak{R}$. *Then* 6) *defines a nction* $F(x)$ *in* $\mathfrak{A} - \mathfrak{R}$.

If the series

$$\alpha_0' + \Sigma\{\alpha_n' \cos nx + \beta_n' \sin nx\}$$

verges to $F(x)$ *in* $\mathfrak{A} - \mathfrak{R}$, *its coefficients are respectively equal to se in* 6).

Theorem 3. If $f(x)$ *admits a development in Fourier's series for set* $\mathfrak{A} - \mathfrak{R}$, *any other development of* $f(x)$ *of the type* 6), *valid in* $-\mathfrak{R}$ *is necessarily Fourier's series, i.e. the coefficients* α_m, β_m *have values given in* 2).

In order to establish the fundamental theorem, we shall make use of some results due to *Riemann*, *G. Cantor*, *Harnack* and *Schwarz* as extended by later writers. Before doing this let us prove the easy

Theorem 4. If $f(x)$ admits a development in Fourier's series which is uniformly convergent in $\mathfrak{A} = (-\pi, \pi)$, it admits no other development of the type 3), *which is also uniformly convergent in $\mathfrak{A}$.*

For then the corresponding series 4) is uniformly convergent in $\mathfrak{A}$, and may be integrated termwise. Thus making use of the method employed in 436, we see that all the coefficients in 4) vanish.

450. 1. Before attempting to prove the fundamental theorem which states that the coefficients a_n, b_n are 0, we will first show that the coefficients of any trigonometric series which converges in $\mathfrak{A}$, except possibly at a point set of a certain type, must be such that they $\doteq 0$, as $n \doteq \infty$. We have already seen, in 440, 1, that this is indeed so in the case of Fourier's series, whether it converges or not. It is not the case with every trigonometric series as the following example shows, viz.:

$$\sum_1^\infty \sin n!\, x. \tag{1}$$

When $x = \frac{m\pi}{r!}$ all the terms, beginning with the $r!^{\text{th}}$, vanish, and hence 1) is convergent at such points. Thus 1) is convergent at a pantactic set of points. In this series the coefficients a_n of the *cosine terms* are all 0, while the coefficients of the sine terms b_n, are 0 or 1. Thus b_n does not $\doteq 0$, as $n \doteq \infty$.

2. Before enunciating the theorem on the convergence of the coefficients of a trigonometric series to 0, we need the notion of *divergence of a series* due to *Harnack*.

Let
$$A = a_1 + a_2 + \cdots \tag{2}$$
be a series of real terms. Let g_n, G_n be the minimum and maximum of all the terms

$$A_{n+1} \quad , \quad A_{n+2} \quad , \quad \cdots$$

where as usual A_n is the sum of the first n terms of 2). Obviously

$$g_n \leq g_{n+1} \quad , \quad G_n \geq G_{n+1}.$$

hus the two sequences $\{g_n\}$, $\{G_n\}$ are monotone, and if limited, eir terms converge to fixed values. Let us say

$$g_n \doteq g \quad , \quad G_n \doteq G.$$

he difference

$$\mathfrak{d} = G - g$$

called the *divergence* of the series 2).

3. *For the series* 2) *to converge it is necessary and sufficient that divergence* $\mathfrak{d} = 0$.

For if A is convergent,

$$-\epsilon + A \leq A_{n+p} \leq A + \epsilon \quad , \quad p = 1, 2 \ldots$$

us

$$-\epsilon + A \leq g_n \leq G_n \leq A + \epsilon.$$

us the limits G, g exist, and

$$G - g \leq 2\epsilon \quad ; \quad \text{or } G = g,$$

$\epsilon > 0$ is small at pleasure.

Suppose now $\mathfrak{d} = 0$. Then by hypothesis, G, g exist and are ual. There exists, therefore, an n, such that

$$g - \epsilon \leq g_n < G_n < G + \epsilon,$$

$$G_n - g_n \leq 2\epsilon.$$

us

$$|A_{n+p} - A_n| \leq 2\epsilon \quad , \quad p = 1, 2 \ldots$$

l A is convergent.

451. *Let the series*

$$\sum_0^\infty (a_n \cos nx + b_n \sin nx)$$

such that for each $\delta > 0$, *there exists a subinterval of*

$$\mathfrak{A} = (-\pi, \pi)$$

each point of which its divergence $\mathfrak{d} < \delta$. *Then* $a_n, b_n \doteq 0$, *as* $= \infty$.

For, as in 450, there exists for each x an m_x, such that

$$|a_n \cos nx + b_n \sin nx| < \frac{\delta}{8} \quad , \quad n > m_x \qquad (1$$

for any point x in some interval $\mathfrak{B}$ of $\mathfrak{A}$. Thus if b is an inner point of $\mathfrak{B}$, $x = b + \beta$ will lie in $\mathfrak{B}$, if β lies in some interval $\boldsymbol{B} = (p, q)$. Now

$$a_n \cos n(b+\beta) + b_n \sin n(b+\beta)$$
$$= (a_n \cos nb + b_n \sin nb) \cos n\beta - (a_n \sin nb - b_n \cos nb) \sin n\beta.$$

$$a_n \cos n(b-\beta) + b_n \sin n(b-\beta)$$
$$= (a_n \cos nb + b_n \sin nb) \cos n\beta + (a_n \sin nb - b_n \cos nb) \sin n\beta.$$

Adding and subtracting these equations, and using 1) we have

$$|(a_n \cos nb + b_n \sin nb) \cos n\beta| < \frac{\delta}{4},$$

$$|(a_n \sin nb - b_n \cos nb) \sin n\beta| < \frac{\delta}{4},$$

for all $n > m_x$. Let us multiply the first of these inequalities by $\cos nb \sin n\beta$, and the second by $\sin nb \cos n\beta$, and add. We get

$$|a_n \sin n\beta_1| < \delta \quad , \quad \beta_1 = 2\beta \quad , \quad n > m_x. \tag{2}$$

Again if we multiply the first inequality by $\sin nb \sin n\beta$, and the second by $\cos nb \cos n\beta$, and subtract, we get

$$|b_n \sin n\beta_1| < \delta \quad , \quad n > m_x. \tag{3}$$

From 2), 3), we can infer that for any $\epsilon > 0$

$$|a_n| < \epsilon \quad , \quad |b_n| < \epsilon \quad , \quad n > \text{some } m, \tag{4}$$

or what is the same, that $a_n, b_n \doteq 0$.

For suppose that the first inequality of 4) did not hold. Then there exists a sequence

$$n_1 < n_2 < \cdots \doteq \infty \tag{5}$$

such that on setting

$$|a_{n_r}| = \delta + \delta'_{n_r} \quad , \quad \epsilon - \delta = \delta'$$

we will have

$$\delta_{n_r} \geq \delta'. \tag{6}$$

If this be so, we can show that there exists a sequence

$$\nu_1 < \nu_2 < \cdots \doteq \infty$$

in 5), such that for some β' in B,

$$|a_{\nu_r} \sin \nu_r \beta'| \geq \delta, \tag{7}$$

h contradicts 2). To this end we note that $\gamma_0 > 0$ may be
en so small that for any r and any $|\gamma| \leq \gamma_0$,

$$|a_{\nu_r}| \cos\gamma \geq (\delta + \delta') \cos\gamma_0 > \delta. \tag{8}$$

et us take the integer ν_1 so that

$$\nu_1 > \frac{\pi + 2\gamma_0}{q - p}. \tag{9}$$

hen
$$\frac{2}{\pi}(\nu_1(q-p) - 2\gamma_0) \geq 2.$$

hus at least *one* odd integer lies in the interval determined by
two numbers

$$\frac{2}{\pi}(p\nu_1 + \gamma_0) \quad , \quad \frac{2}{\pi}(q\nu_1 - \gamma_0).$$

et m_1 be such an integer. Then

$$\frac{2}{\pi}(p\nu_1 + \gamma_0) \leq m_1 \leq \frac{2}{\pi}(q\nu_1 - \gamma_0). \tag{10}$$

we set

$$p_1 = \frac{1}{\nu_1}\left(m_1\frac{\pi}{2} - \gamma_0\right) \quad , \quad q_1 = \frac{1}{\nu_1}\left(m_1\frac{\pi}{2} + \gamma_0\right) \tag{11}$$

ee that the interval $B_1 = (p_1, q_1)$ lies in B. The length of
$2\gamma_0/\nu_1$. Then for any β in B_1,

$$\nu_1\beta = m_1\frac{\pi}{2} + \gamma_1 \quad , \quad |\gamma_1| \leq \gamma_0.$$

hus by 8),
$$|a_{\nu_1} \sin \nu_1\beta| = |a_{\nu_1}| \cos\gamma_1 > \delta. \tag{12}$$

at we may reason on B_1 as we have on B. We determine ν_2
), replacing p, q by p_1, q_1. We determine the odd integer m_2
0), replacing p, q, ν_1 by p_1, q_1, ν_2. The relation 11) deter-
s the new interval $B_2 = (p_2, q_2)$, on replacing m_1, ν_1 by m_2, ν_2.
length of B_2 is $2\gamma_0/\nu_2$, and B_2 lies in B_1. For this relation
, and for any β in B_2 we have, similar to 12),

$$|a_{\nu_2} \sin \nu_2\beta| > \delta.$$

this way we may continue indefinitely. The intervals
$> B_2 > \cdots \doteq$ to a point β', and obviously for this β', the rela-

tion 7) holds for any x. In a similar manner we see that if b_n does not $\doteq 0$, the relation 3) cannot hold.

452. As corollaries of the last theorem we have:

1. *Let the series*

$$\sum_0^\infty (a_n \cos nx + b_n \sin nx) \tag{1}$$

be such that for each $\delta > 0$, *the points in* $\mathfrak{A} = (-\pi, \pi)$ *at which the divergence of* 1) *is* $\geq \delta$, *form an apantactic set in* $\mathfrak{A}$. *Then* $a_n, b_n \doteq 0$, *as* $n \doteq \infty$.

2. *Let the series* 1) *converge in* $\mathfrak{A}$, *except possibly at the points of a reducible set* $\mathfrak{R}$. *Then* $a_n, b_n \doteq 0$.

For $\mathfrak{R}$ being reducible [318, 6], there exists in $\mathfrak{A}$ an interval $\mathfrak{B}$ in which 1) converges at every point. We now apply 451.

453. *Let*
$$F(x) = \sum_0^\infty (a_n \cos nx + b_n \sin nx)$$
at the points of $\mathfrak{A} = (-\pi, \pi)$, *where the series is convergent. At the other points of* $\mathfrak{A}$, *let* $F(x)$ *have an arbitrarily assigned value, lying between the two limits of indetermination* g, G *of the series. If* F *is* R*-integrable in* $\mathfrak{A}$, *the coefficients* $a_n, b_n \doteq 0$.

For there exists a division of $\mathfrak{A}$, such that the sum of those intervals in which Osc $F \geq \omega$ is $< \sigma$. There is therefore an interval $\mathfrak{I}$ in which Osc $F < \omega$. If $\mathfrak{K}$ is an inner interval of $\mathfrak{I}$, the divergence of the above series is $< \omega$ at each point of $\mathfrak{K}$. We now apply 451.

454. *Riemann's Theorem.*

Let $F(x) = \frac{1}{2} a_0 + \sum_1^\infty (a_n \cos nx + b_n \sin nx) = \Sigma A_n$ *converge at each point of* $\mathfrak{A} = (-\pi, \pi)$, *except possibly at the points of a reducible set* $\mathfrak{R}$. *The series obtained by integrating this series termwise, we denote by*

$$G(x) = \frac{1}{4} a_0 x^2 - \sum_1^\infty \frac{1}{n^2}(a_n \cos nx + b_n \sin nx) = \frac{1}{2} A_0 x^2 - \sum_1^\infty \frac{A_n}{n^2}.$$

Then G *is continuous in* $\mathfrak{A}$.

$$\Phi(u) = G(x+2u) + G(x-2u) - 2\,G(x). \qquad (1$$

at each point of $\mathfrak{B} = \mathfrak{A} - \mathfrak{R}$,

$$\lim_{u=0} \frac{\Phi(u)}{4u^2} = F(x)\,; \qquad (2$$

t each point of $\mathfrak{A}$,

$$\lim_{u=0} \frac{\Phi(u)}{u} = 0. \qquad (3$$

, in the first place, since $\mathfrak{R}$ is a reducible set, $a_n, b_n \doteq 0$. The G is therefore uniformly convergent in $\mathfrak{A}$, and is thus a nuous function.

us now compute Φ. We have

$$a_n \cos n(x+2u) + a_n \cos n(x-2u) - 2\,a_n \cos nx$$
$$= 2\,a_n \cos nx\,(\cos 2\,nu - 1)$$
$$= -4\,a_n \cos nx \sin^2 nu.$$

o

$$b_n \sin n(x+2u) + b_n \sin n(x-2u) - 2\,b_n \sin nx$$
$$= 2\,b_n \sin nx(\cos 2\,nu - 1)$$
$$= -4\,b_n \sin nx \sin^2 nu.$$

s

$$\frac{\Phi(u)}{4u^2} = \sum_0^\infty A_n \left(\frac{\sin nu}{nu}\right)^2,$$

agree to give the coefficient of A_0 the value 1. Let us an arbitrary but fixed value in $\mathfrak{B}$. Then for each $\epsilon > 0$, exists an m such that

$$_0 + A_1 + \cdots + A_{n-1} = F(x) + \epsilon_n \quad , \quad |\epsilon_n| < \epsilon, \quad n \geq m.$$

s

$$A_n = \epsilon_{n+1} - \epsilon_n.$$

ce

$$\frac{\Phi}{4u^2} = F(x) + \epsilon_1 + \sum_1^\infty (\epsilon_{n+1} - \epsilon_n)\left(\frac{\sin nu}{nu}\right)^2$$

$$= F(x) + \sum_1^\infty \epsilon_n \left\{\left[\frac{\sin (n-1)u}{(n-1)u}\right]^2 - \left[\frac{\sin nu}{nu}\right]^2\right\} \qquad (4$$

$$= F(x) + S$$

$$u < \frac{\pi}{m} \quad , \quad \text{so that} \quad m < \frac{\pi}{u};$$

and break S into three parts

$$S_1 = \sum_1^m \quad , \quad S_2 = \sum_{m+1}^{\kappa} \quad , \quad S_3 = \sum_{\kappa+1}^{\infty},$$

where κ is the greatest integer $< \pi/u$, and then consider each sum separately, as $u \doteq 0$.

Obviously
$$\lim_{u=0} S_1 = 0.$$

As to the second sum, the number of its terms increases indefinitely as $u \doteq 0$.

For any u,

$$\begin{aligned} | S_2 | &< \epsilon \sum_{m+1}^{\kappa} \{ \cdots \} \\ &< \epsilon \left\{ \left[\frac{\sin mu}{mu} \right]^2 - \left[\frac{\sin \kappa u}{\kappa u} \right]^2 \right\} \\ &< \epsilon \left[\frac{\sin mu}{mu} \right]^2 < \epsilon, \end{aligned}$$

since each term in the brace is positive. In fact

$$\frac{\sin v}{v}$$

is a decreasing function of v as v ranges from 0 to π, and

$$nu \leq \kappa u \leq \pi \quad , \quad n = m, m+1, \cdots \kappa.$$

Finally we consider S_3. We may write the general term as follows:

$$\epsilon_n \left\{ \left[\frac{\sin (n-1)u}{(n-1)u} \right]^2 - \left[\frac{\sin (n-1)u}{nu} \right]^2 \right\}$$

$$+ \epsilon_n \left\{ \left[\frac{\sin (n-1)u}{nu} \right]^2 - \left[\frac{\sin nu}{nu} \right]^2 \right\}.$$

Now
$$\frac{\sin^2 (n-1)u - \sin^2 nu}{n^2u^2} = \frac{-\sin (2n-1)u \sin u}{n^2u^2} | \leq | \frac{1}{n^2u}.$$

...us
$$|S_3| < \frac{\epsilon}{u^2}\sum_{\kappa+1}^{\infty}\left\{\frac{1}{(n-1)^2}-\frac{1}{n^2}\right\}+\frac{\epsilon}{u}\sum_{\kappa+1}^{\infty}\frac{1}{n^2}$$
$$\leq \frac{\epsilon}{\kappa^2 u^2}+\frac{\epsilon}{\kappa u},$$
$$\sum_{\kappa+1}^{\infty}\frac{1}{n^2} < \int_{\kappa}^{\infty}\frac{dx}{x^2}=\frac{1}{\kappa}.$$
...t
$$\kappa \geq \frac{\pi}{u}-1 \quad , \quad \text{or } \kappa u > \pi - u.$$
...us
$$|S_3| < \epsilon\left\{\frac{1}{(\pi-u)^2}+\frac{1}{\pi-u}\right\}.$$
...ence
$$S = S_1 + S_2 + S_3 \doteq 0, \text{ as } u \doteq 0,$$
...h proves the limit 2), on using 4).

...o prove the limit 3), we have
$$\frac{\Phi(u)}{4u} = \sum_0^{\infty} uA_n\left(\frac{\sin nu}{nu}\right)^2 = T.$$
...et us give u a definite value and break T into three sums.
$$T_1 = \sum_1^m,$$
...re m is chosen so that
$$|A_n| < \epsilon \quad , \quad n > m;$$
$$T_2 = \sum_{m+1}^{\lambda},$$
...re λ is the greatest integer such that
$$\lambda u \leq 1;$$
$$T_3 = \sum_{\lambda+1}^{\infty}.$$
...bviously for some M,
$$|T_1| \leq uM.$$
...lso
$$|T_2| \leq \epsilon u\lambda \leq \epsilon,$$
...
$$\left(\frac{\sin nu}{nu}\right)^2 < 1.$$

As to the last sum,

$$|T_3| \leq \frac{\epsilon}{u}\sum_{\lambda+1}^{\infty}\frac{1}{n^2} \leq \epsilon\lambda\cdot\frac{1}{\lambda} \quad , \quad \text{since } \frac{1}{u} \leq \lambda,$$
$$< \epsilon.$$

Thus
$$T \doteq 0 \quad , \quad \text{as } u \doteq 0.$$

455. *Schwarz-Lüroth Theorem.*

In $\mathfrak{A} = (a < b)$ let the continuous function $f(x)$ be such that

$$S(x, u) = \frac{f(x+u) + f(x-u) - 2f(x)}{u^2} \doteq 0, \text{ as } u \doteq 0, \tag{1}$$

except possibly at an enumerable set $\mathfrak{E}$ in $\mathfrak{A}$. At the points $\mathfrak{E}$, let

$$uS(x, u) \doteq 0 \qquad \text{as } u \doteq 0. \tag{2}$$

Then f is a linear function in $\mathfrak{A}$.

Let us *first suppose* with Schwarz that $\mathfrak{E} = 0$. We introduce the auxiliary function,

$$g(x) = \eta L(x) - \tfrac{1}{2}c(x-a)(x-b),$$

where
$$L(x) = f(x) - f(a) - \frac{x-a}{b-a}\{f(b) - f(a)\},$$

$\eta = \pm 1$, and c is an arbitrary constant.

The function $g(x)$ is continuous in $\mathfrak{A}$, and $g(a) = g(b) = 0$. Moreover
$$\frac{g(x+u) + g(x-u) - 2g(x)}{u^2} \doteq c, \qquad \text{as } u \doteq 0.$$

Thus for all $0 < u <$ some δ,

$$G = g(x+u) + g(x-u) - 2g(x) > 0. \tag{3}$$

From this follows that $g(x) \leq 0$ in $\mathfrak{A}$. For if $g(x) > 0$, at any point in $\mathfrak{A}$, it takes on its maximum value at some point ξ within $\mathfrak{A}$. Thus
$$g(\xi+u) - g(\xi) \leq 0 \quad , \quad g(\xi-u) - g(\xi) \leq 0,$$

for $0 < u < \delta$, δ being sufficiently small. Adding these two inequalities gives $G \leq 0$, which contradicts 3). Thus $g \leq 0$ in $\mathfrak{A}$.

Let us now suppose $L \neq 0$ for some x in $\mathfrak{A}$. We take c so small that
$$\operatorname{sgn} g = \operatorname{sgn} \eta L = \eta \operatorname{sgn} L.$$

nt η is at pleasure ± 1, hence the supposition that $L \neq 0$ is admissible. Hence $L = 0$ in $\mathfrak{A}$, or

$$f(x) = f(a) - \frac{x-a}{b-a}\{f(b) - f(a)\} \tag{4}$$

deed a linear function.

t us now suppose with Lüroth that $\mathfrak{E} > 0$. We introduce the iary continuous function.

$$h(x) = L(x) + e(x-a)^2 \quad , \quad e > 0.$$

us

$$h(a) = 0 \quad , \quad h(b) = e(b-a)^2.$$

ppose at some inner point ξ of $\mathfrak{A}$

$$L(\xi) > 0. \tag{5}$$

is leads to a contradiction, as we proceed to show. For then

$$h(\xi) - h(b) = L(\xi) + e\{(\xi - a)^2 - (b-a)^2\} > 0,$$

ded

$$C = \frac{L(\xi)}{(b-a)^2 - (\xi - a)^2} > e.$$

e shall take e so that this inequality is satisfied, *i.e.* e lies in nterval $\mathfrak{E} = (0^*, C^*)$. Thus

$$h(\xi) > h(b) > h(a).$$

nce $h(x)$ takes on its maximum value at some inner point e Hence for $\delta > 0$ sufficiently small,

$$(e+u) - h(e) < 0 \quad , \quad h(e-u) - h(e) < 0 \quad , \quad 0 < u \leq \delta. \tag{6}$$

$$H(e, u) = \frac{h(e+u) + h(e-u) - 2h(e)}{u^2} \leq 0. \tag{7}$$

v if e is a point of $\mathfrak{A} - \mathfrak{E}$,

$$\lim_{u=0} H(e, u) = 2e > 0.$$

this contradicts 7), which requires that

$$\lim_{u=0} H(e, u) \leq 0.$$

Hence e is a point of $\mathfrak{E}$. Hence by 2),

$$\frac{h(e+u)-h(e)}{u}+\frac{h(e-u)-h(e)}{u} \doteq 0 \quad , \quad \text{as } u \doteq 0.$$

By 6), both terms have the same sign. Hence each term $\doteq 0$. Thus for $u > 0$

$$0 = \lim_{u=0} \frac{h(e \pm u)-h(e)}{\pm u} = \lim \frac{f(e \pm u)-f(e)}{\pm u} - \frac{f(b)-f(a)}{b-a} + 2\,c(e-a).$$

Hence
$$f'(e) = \frac{f(b)-f(a)}{b-a} + 2\,c(e-a). \tag{8}$$

Thus to each c in the interval $\mathfrak{C}$, corresponds an e in $\mathfrak{E}$, at which point the derivative of $f(x)$ exists and has the value given on the right of 8). On the other hand, two different c's, say c and c', in $\mathfrak{C}$ cannot correspond to the same e in $\mathfrak{E}$.

For then 8) shows that

$$c(e-a) = c'(e-a),$$

or as

$$e > a, \qquad c' = c.$$

Thus there is a uniform correspondence between $\mathfrak{C}$ whose cardinal number is $\mathfrak{c}$, and $\mathfrak{E}$ whose cardinal number is $\mathfrak{e}$, which is absurd. Thus the supposition 5) is impossible. In a similar manner, the assumption that $L < 0$ at some point in $\mathfrak{A}$, leads to a contradiction. Hence $L = 0$ in $\mathfrak{A}$, and 4) again holds, which proves the theorem.

456. *Cantor's Theorem. Let*

$$\tfrac{1}{2} a_0 + \sum_1^\infty (a_n \cos nx + b_n \sin nx) \tag{1}$$

converge to 0 in $\mathfrak{A} = (-\pi, \pi)$, except possibly at a reducible set $\mathfrak{R}$, where nothing is asserted regarding its convergence. Then it converges to 0 at every point in $\mathfrak{A}$, and all its coefficients

$$a_0, a_1, a_2 \cdots b_1, b_2, b_3 \cdots = 0.$$

For by 452, 2, $a_n, b_n \doteq 0$. Then Riemann's function

$$f(x) = \tfrac{1}{4} a_0 x^2 - \sum \frac{1}{n^2}(a_n \cos nx + b_n \sin nx)$$

satisfies the conditions of the Schwarz-Lüroth theorem, 455, since $\mathfrak{R}$ is enumerable. Thus $f(x)$ is a linear function of x in $\mathfrak{A}$, and has the form $\alpha + \beta x$. Hence

$$\alpha + \beta x - \tfrac{1}{4} a_0 x^2 = - \sum_1^\infty \frac{1}{n^2}(a_n \cos nx + b_n \sin nx). \tag{2}$$

The right side admits the period 2π, and is therefore periodic. Its period ω must be 0. For if $\omega > 0$, the left side has this period, which is absurd. Hence $\omega = 0$, and the left side reduces to a constant, which gives $\beta = 0$, $a_0 = 0$. But in $\mathfrak{A} - \mathfrak{R}$, the right side of 1) has the sum 0. Hence $\alpha = 0$. Thus the right side of 2) vanishes in $\mathfrak{A}$. As it converges uniformly in $\mathfrak{A}$, we may determine its coefficients as in 436. This gives

$$a_n = 0 \quad , \quad b_n = 0 \quad , \quad n = 1, 2 \cdots$$

CHAPTER XIV

DISCONTINUOUS FUNCTIONS

Properties of Continuous Functions

457. 1. In Chapter VII of Volume I we have discussed some of the elementary properties of continuous and discontinuous functions. In the present chapter further developments will be given, paying particular attention to discontinuous functions. Here the results of Baire * are of foremost importance. Lebesgue † has shown how some of these may be obtained by simpler considerations, and we have accordingly adopted them.

2. Let us begin by observing that the definition of a continuous function given in I, 339, may be extended to sets having isolated points, if we use I, 339, 2 as definition.

Let therefore $f(x_1 \cdots x_m)$ be defined over $\mathfrak{A}$, being either limited or unlimited. Let a be any point of $\mathfrak{A}$. If for each $\epsilon > 0$, there exists a $\delta > 0$, such that

$$|f(x) - f(a)| < \epsilon, \qquad \text{for any } x \text{ in } V_\delta(a),$$

we say f *is continuous at* a.

By the definition it follows at once that f is continuous at each isolated point of $\mathfrak{A}$. Moreover, when a is a proper limiting point of $\mathfrak{A}$, the definition here given coincides with that given in I, 339. If f is continuous at each point of $\mathfrak{A}$, we say it is *continuous in* $\mathfrak{A}$. The definition of discontinuity given in I, 347, shall still hold, except that we must regard isolated points as points of continuity.

* "*Sur les Fonctions de Variables réelles,*" *Annali di Mat.*, Ser. 3, vol. 3 (1899).

Also his *Leçons sur les Fonctions Discontinues.* Paris, 1905.

† *Bulletin de la Société Mathématique de France*, vol. 32 (1904), p. 229.

3. The reader will observe that the theorems I, 350 to 354 inclusive, are valid not only for limited perfect domains, but also for limited complete sets.

458. 1. *If $f(x_1 \cdots x_m)$ is continuous in the limited set $\mathfrak{A}$, and its values are known at the points of $\mathfrak{B} < \mathfrak{A}$, then f is known at all points of $\mathfrak{B}'$ lying in $\mathfrak{A}$.*

For let $b_1, b_2, b_3 \cdots$ be points of $\mathfrak{B}$, whose limiting point b lies in $\mathfrak{A}$. Then

$$\lim_{n=\infty} f(b_n) = f(b).$$

2. *If f is known for a dense set $\mathfrak{B}$ in $\mathfrak{A}$, and is continuous in $\mathfrak{A}$, f is known throughout $\mathfrak{A}$.*

For

$$\mathfrak{B}' \geq \mathfrak{A}.$$

3. *If $f(x_1 \cdots x_m)$ is continuous in the complete set $\mathfrak{A}$, the points $\mathfrak{B}$ in $\mathfrak{A}$ where $f = c$, a constant, form a complete set. If $\mathfrak{A}$ is an interval, there is a first and a last point of $\mathfrak{B}$.*

For $f = c$ at $x = \alpha_1, \alpha_2 \cdots$ which $\doteq \alpha$; we have therefore

$$f(\alpha) = \lim_{n=\infty} f(\alpha_n) = c.$$

459. *The points of continuity $\mathfrak{C}$ of $f(x_1 \cdots x_m)$ in $\mathfrak{A}$ lie in a deleted enclosure $\mathfrak{K}$. If $\mathfrak{A}$ is complete, $\mathfrak{K} = \mathfrak{C}$.*

For let $\epsilon_1 > \epsilon_2 > \cdots \doteq 0$. For each ϵ_n, and for each point of continuity c in $\mathfrak{A}$, there exists a cube $\mathfrak{Q}$ whose center is c, such that

$$\text{Osc}\, f < \epsilon_n, \qquad \text{in } \mathfrak{Q}.$$

Thus the points of continuity of f lie in an enumerable non-overlapping set of complete metric cells, in each of which $\text{Osc}\, f < \epsilon_n$. Let $\mathfrak{Q}_n$ be the inner points of this enclosure. Then each point of the *deleted enclosure*

$$\mathfrak{K} = Dv\,\{\mathfrak{Q}_n\}$$

which lies in $\mathfrak{A}$ is a point of continuity of f. For such a point c lies within each $\mathfrak{Q}_n$.

Hence

$$\text{Osc}\, f < \epsilon, \qquad \text{in } V_\delta(c),$$

for $\delta > 0$ sufficiently small and n sufficiently great.

Oscillation

460. Let
$$\omega_\delta = \operatorname{Osc} f(x_1 \cdots x_m) \qquad \text{in } V_\delta(a).$$

This is a monotone decreasing function of δ. Hence if ω_δ is finite, for some $\delta > 0$,
$$\omega = \lim_{\delta=0} \omega_\delta$$
exists. We call ω *the oscillation of f at $x = a$*, and write
$$\omega = \operatorname*{Osc}_{x=a} f.$$

Should $\omega_\delta = +\infty$, however small $\delta > 0$ is taken, we say $\omega = +\infty$. When $\omega = 0$, f is continuous at $x = a$, if a is a point in the domain of definition of f. When $\omega > 0$, f is discontinuous at this point. It is a *measure of the discontinuity of f at $x = a$*; we write
$$\omega = \operatorname*{Disc}_{x=a} f(x_1 \cdots x_m).$$

461. 1. *Let*
$$d = \operatorname{Disc} f(x_1 \cdots x_m) \quad , \quad e = \operatorname{Disc} g(x_1 \cdots x_m),$$
at $x = a$. Then
$$| d - e | \leq \operatorname*{Disc}_{x=a} (f \pm g) \leq d + e.$$

For in $V_\delta(a)$,
$$| \operatorname{Osc} f - \operatorname{Osc} g | \leq \operatorname{Osc} (f \pm g) \leq \operatorname{Osc} f + \operatorname{Osc} g.$$

2. *If f is continuous at $x = a$, while* $\operatorname{Disc} g = d$, *then*
$$\operatorname*{Disc}_{x=a} (f + g) = d.$$

For f being continuous at a, $\qquad \operatorname{Disc} f = 0.$

Hence
$$\operatorname{Disc} g \leq \operatorname{Disc} (f + g) \leq \operatorname{Disc} g = d.$$

3. *If c is a constant,*
$$\operatorname{Disc} (cf) = | c | \operatorname{Disc} f \quad , \quad \textit{at } x = a.$$

For
$$\operatorname{Osc} (cf) = | c | \operatorname{Osc} f \quad , \quad \text{in any } V_\delta(a).$$

4. When the limits
$$f(x - 0) \quad , \quad f(x + 0)$$

exist and at least one of them is different from $f(x)$, the point x is a discontinuity of the first kind, as we have already said. When at least one of the above limits does not exist, the point x is a *point of discontinuity of the second kind.*

462. 1. *The points of infinite discontinuity $\mathfrak{J}$ of f, defined over a limited set $\mathfrak{A}$, form a complete set.*

For let $\iota_1, \iota_2 \cdots$ be points of $\mathfrak{J}$, having k as limiting point. Then in any $V(k)$ there are an infinity of the points ι_n and hence in any $V(k)$, $\text{Osc}\, f = +\infty$. The point k does not of course need to lie in $\mathfrak{A}$.

2. We cannot say, however, that the points of discontinuity of a function form a complete set as is shown by the following

Example. In $\mathfrak{A} = (0, 1)$, let $f(x) = x$ when x is irrational, and $= 0$ when x is rational. Then each point of $\mathfrak{A}$ is a point of discontinuity except the point $x = 0$. Hence the points of discontinuity of f do not form a complete set.

3. *Let f be limited or unlimited in the limited complete set $\mathfrak{A}$. The points $\mathfrak{K}$ of $\mathfrak{A}$ at which $\text{Osc}\, f \geq k$ form a complete set.*

For let $a_1, a_2 \cdots$ be points of $\mathfrak{K}$ which $\doteq a$. However small $\delta > 0$ is taken, there are an infinity of the a_n lying in $V_\delta(a)$. But at any one of these points, $\text{Osc}\, f \geq k$. Hence $\text{Osc}\, f \geq k$ in $V_\delta(a)$, and thus a lies in $\mathfrak{K}$.

4. *Let $f(x_1 \cdots x_m)$ be limited and R-integrable in the limited set $\mathfrak{A}$. The points $\mathfrak{K}$ at which $\text{Osc}\, f \geq k$ form a discrete set.*

For let D be a rectangular division of space. Let us suppose $\overline{\mathfrak{K}}_D >$ some constant $c > 0$, however D is chosen. In each cell δ of D,

$$\text{Osc}\, f \geq k.$$

Hence the sum of the cells in which the oscillation is $\geq k$ cannot be made small at pleasure, since this sum is $\overline{\mathfrak{K}}_D$. But this contradicts I, 700, 5.

5. *Let $f(x_1 \cdots x_m)$ be limited in the complete set $\mathfrak{A}$. If the points $\mathfrak{K}$ in $\mathfrak{A}$ at which $\text{Osc}\, f \geq k$ form a discrete set, for each k, then f is R-integrable in $\mathfrak{A}$.*

For about each point of $\mathfrak{A} - \mathfrak{K}$ as center, we can describe a cube $\mathfrak{C}$ of varying size, such that $\mathrm{Osc}\, f < 2k$ in $\mathfrak{C}$. Let D be a cubical division of space of norm d. We may take d so small that $\overline{\mathfrak{K}}_D = \Sigma d_\iota$ is as small as we please. The points of $\mathfrak{A}$ lie now within the cubes $\mathfrak{C}$ and the set formed of the cubes d_ι. By Borel's theorem there are a finite number of cubes, say

$$\eta_1 \quad , \quad \eta_2 \cdots$$

such that all the points of $\mathfrak{A}$ lie within these η's. If we prolong the faces of these η's, we effect a rectangular division such that the sum of those cells in which the oscillation is $> 2k$ is as small as we choose, since this sum is obviously $\leq \overline{\mathfrak{K}}_D$. Hence by I, 700, 5, f is R-integrable.

6. *Let $f(x_1 \cdots x_m)$ be limited in $\mathfrak{A}$; let its points of discontinuity in $\mathfrak{A}$ be $\mathfrak{D}$. If f is R-integrable, $\mathfrak{D}$ is a null set. If $\mathfrak{A}$ is complete and $\mathfrak{D}$ is a null set, f is R-integrable.*

Let f be R-integrable. Then $\mathfrak{D}$ is a null set. For let $\epsilon_1 > \epsilon_2 > \cdots \doteq 0$. Let $\mathfrak{D}_n$ denote the points at which $\mathrm{Osc}\, f \geq \epsilon_n$. Then $\mathfrak{D} = \{\mathfrak{D}_n\}$. But since f is R-integrable, each $\mathfrak{D}_n$ is discrete by 4. Hence $\mathfrak{D}$ is a null set.

Let $\mathfrak{A}$ be complete and $\mathfrak{D}$ a null set. Then each $\mathfrak{D}_n$ is complete by 3. Hence by 365, $\widehat{\mathfrak{D}}_n = \overline{\mathfrak{D}}_n$. As $\widehat{\mathfrak{D}} = 0$, we see $\mathfrak{D}_n$ is discrete. Hence by 5, f is R-integrable.

If $\mathfrak{A}$ is not complete, f does not need to be R-integrable when $\mathfrak{D}$ is a null set.

Example. Let $\mathfrak{A}_1 = \left\{\dfrac{m}{2^n}\right\} \quad , \quad n = 1, 2 \cdots ; \ m < 2^n.$

$$\mathfrak{A}_2 = \left\{\frac{r}{3^s}\right\} \quad , \quad s = 1, 2 \cdots ; \ r < 3^s.$$

Let $\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{A}_2$.

Let
$$f(x) = \frac{1}{2^n} \quad , \quad \text{at } x = \frac{m}{2^n}$$
$$= 1 \qquad \text{in } \mathfrak{A}_2.$$

Then each point of $\mathfrak{A}$ is a point of discontinuity, and $\mathfrak{A} = \mathfrak{D}$. But $\mathfrak{A}_1$, $\mathfrak{A}_2$ are null sets, hence $\mathfrak{A}$ is a null set.

On the other hand,

$$\overline{\int}_{\mathfrak{A}} f = 1 \quad , \quad \underline{\int}_{\mathfrak{A}} f = 0,$$

and f is not R-integrable in $\mathfrak{A}$.

Pointwise and Total Discontinuity

463. Let $f(x_1 \cdots x_m)$ be defined over $\mathfrak{A}$. If each point of $\mathfrak{A}$ is a point of discontinuity, we say f is *totally discontinuous* in $\mathfrak{A}$.

We say f is *pointwise discontinuous* in $\mathfrak{A}$, if f is not continuous in $\mathfrak{A} = \{a\}$, but has in any $V(a)$ a point of continuity. If f is continuous or pointwise discontinuous, we may say it is *at most pointwise* discontinuous.

Example 1. A function $f(x_1 \cdots x_m)$ having only a finite number of points of discontinuity in $\mathfrak{A}$ is pointwise discontinuous in $\mathfrak{A}$.

Example 2. Let

$$\begin{aligned} f(x) &= 0 \quad , \quad \text{for irrational } x \text{ in } \mathfrak{A} = (0, 1) \\ &= \frac{1}{n} \quad , \quad \text{for } x = \frac{m}{n} \\ &= 1 \quad , \quad \text{for } x = 0, 1. \end{aligned}$$

Obviously f is continuous at each irrational x, and discontinuous at the other points of $\mathfrak{A}$. Hence f is pointwise discontinuous in $\mathfrak{A}$.

Example 3. Let $\mathfrak{D}$ be a Harnack set in the unit interval $\mathfrak{A} = (0, 1)$. In the associate set of intervals, end points included, let $f(x) = 1$. At the other points of $\mathfrak{A}$, let $f = 0$. As $\mathfrak{D}$ is apantactic in $\mathfrak{A}$, f is pointwise discontinuous.

Example 4. In Ex. 3, let $\mathfrak{D} = \mathfrak{E} + \mathfrak{F}$, where $\mathfrak{E}$ is the set of end points of the associate set of intervals. Let $f = 1/n$ at the end points of these intervals belonging to the n^{th} stage. Let $f = 0$ in $\mathfrak{F}$. Here f is defined only over $\mathfrak{D}$. The points $\mathfrak{F}$ are points of continuity in $\mathfrak{D}$. Hence f is pointwise discontinuous in $\mathfrak{D}$.

Example 5. Let $f(x)$ be Dirichlet's function, *i.e.* $f = 0$, for the irrational points $\mathfrak{J}$ in $\mathfrak{A} = (0, 1)$, and $= 1$ for the rational points.

As each point of $\mathfrak{A}$ is a point of discontinuity, f is totally discontinuous in $\mathfrak{A}$. Let us remove the rational points in $\mathfrak{A}$; the deleted domain is $\mathfrak{J}$. In this domain f is continuous. Thus on removing certain points, a discontinuous function becomes a continuous function in the remaining point set.

This is not always the case. For if in Ex. 4 we remove the points $\mathfrak{F}$, retaining only the points $\mathfrak{E}$, we get a function which is *totally* discontinuous in $\mathfrak{E}$, whereas before f was only pointwise discontinuous.

464. 1. *If $f(x_1 \cdots x_m)$ is totally discontinuous in the infinite complete set $\mathfrak{A}$, then the points $\mathfrak{d}_\omega$ where*

$$\text{Disc} f \geq \omega \quad , \quad \omega > 0,$$

form an infinite set, if ω is taken sufficiently small.

For suppose $\mathfrak{d}_\omega$ were finite however small ω is taken. Let $\omega_1 > \omega_2 > \cdots \doteq 0$. Let D_1, D_2, $\cdots$ be a sequence of superposed cubical divisions of space of norms $d_n \doteq 0$. We shall only consider cells containing points of $\mathfrak{A}$. Then if d_1 is taken sufficiently small, D_1 contains a cell δ_1, containing an infinite number of points of $\mathfrak{A}$, but no point at which $\text{Disc} f \geq \omega_1$. If d_2 is taken sufficiently small, D_2 contains a cell $\delta_2 < \delta_1$, containing no point at which $\text{Disc} f \geq \omega_2$. In this way we get a sequence of cells,

$$\delta_1 > \delta_2 > \cdots$$

which $\doteq$ a point p. As $\mathfrak{A}$ is complete, p lies in $\mathfrak{A}$. But f is obviously continuous at p. Hence f is not totally discontinuous in $\mathfrak{A}$.

2. If $\mathfrak{A}$ is not complete, $\mathfrak{d}_\omega$ does not need to be infinite for any $\omega > 0$.

Example. Let $\mathfrak{A} = \left\{\frac{m}{2^n}\right\}$, $n = 1, 2, \cdots$ and m odd and $< 2^n$. At $\frac{m}{2^n}$, let $f = \frac{1}{2^n}$. Then each point of $\mathfrak{A}$ is a point of discontinuity. But $\mathfrak{d}_\omega$ is finite, however small $\omega > 0$ is taken.

3. We cannot say f is not pointwise discontinuous in complete $\mathfrak{A}$, when $\mathfrak{d}_\omega$ is infinite.

Example. At the points $\left\{\frac{1}{n}\right\} = \mathfrak{N}$, let $f = 0$; at the other points of $\mathfrak{A} = (0, 1)$, let $f = 1$.

Obviously f is pointwise discontinuous in $\mathfrak{A}$. But $\mathfrak{d}_\omega$ is an infinite set for $\omega \leq 1$, as in this case it is formed of $\mathfrak{N}$, and the point 0.

Examples of Discontinuous Functions

465. In volume I, 330 *seq.* and 348 *seq.*, we have given examples of discontinuous functions. We shall now consider a few more.

Example 1. Riemann's Function.

Let (x) be the difference between x and the nearest integer; and when x has the form $n + \frac{1}{2}$, let $(x) = 0$. Obviously (x) has the period 1.

It can be represented by Fourier's series thus:

$$(x) = \frac{1}{\pi}\left\{\frac{\sin 2\,\pi x}{1} - \frac{\sin 2 \cdot 2\,\pi x}{2} + \frac{\sin 3 \cdot 2\,\pi x}{3} - \cdots\right\}. \tag{1}$$

Riemann's function is now

$$F(x) = \sum_1^\infty \frac{(nx)}{n^2}. \tag{2}$$

This series is obviously uniformly convergent in $\mathfrak{A} = (-\infty, \infty)$.

Since (x) has the period 1 and is continuous within $(-\frac{1}{2}, \frac{1}{2})$, we see that (nx) has the period $\frac{1}{n}$, and is continuous within $\left(-\frac{1}{2\,n}, \frac{1}{2\,n}\right)$. The points of discontinuity of (nx) are thus

$$\mathfrak{E}_n = \left\{\frac{1}{2\,n} + \frac{s}{n}\right\} \quad , \quad s = 0, \pm 1, \pm 2, \cdots$$

Let $\mathfrak{E} = \{\mathfrak{E}_n\}$. Then at any x not in $\mathfrak{E}$, each term of 2) is a continuous function of x. Hence $F(x)$ is continuous at this point.

On the other hand, F is discontinuous at any point e of $\mathfrak{E}$. For F being uniformly convergent,

$$R\lim_{x=e} F(x) = \Sigma R \lim_{x=e} \frac{(nx)}{n^2} \tag{3}$$

$$L\lim_{x=e} F(x) = \Sigma L \lim_{x=e} \frac{(nx)}{n^2}. \tag{4}$$

We show now that 3) has the value

$$F(e)-\frac{\pi^2}{16\,n^2},\qquad \text{for } e=\frac{2\,s+1}{2\,n},\qquad e \text{ irreducible.} \tag{5}$$

and 4) the value

$$F(e)+\frac{\pi^2}{16\,n^2}. \tag{6}$$

Hence

$$\operatorname*{Disc}_{x=e} F(x)=\frac{\pi^2}{8\,n^2}. \tag{7}$$

To this end let us see when two of the numbers

$$\frac{1}{2\,m}+\frac{r}{m},\quad \text{and}\quad \frac{1}{2\,n}+\frac{s}{n}\qquad m\neq n$$

are equal. If equal, we have

$$\frac{2\,r+1}{m}=\frac{2\,s+1}{n}. \tag{8}$$

Thus if we take $2\,s+1$ relatively prime to n, no two of the numbers in $\mathfrak{E}_n$ are equal. Let us do this for each n. Then no two of the numbers in $\mathfrak{E}$ are equal.

Let now $x=e=\dfrac{1}{2\,n}+\dfrac{s}{n}\cdot$ Then (mx) is continuous at this point, unless 8) holds; *i.e.* unless m is a multiple of n, say $m=ln$. In this case, 8) gives

$$2\,r+1=l(2\,s+1).$$

Thus l must be odd; $l=1, 3, 5\cdots$ In this case $(mx)=0$ at e, while $R\lim\limits_{x=e}(mx)=-\frac{1}{2}$. When m is not an odd multiple of n, obviously

$$R\lim_{x=e}(mx)=(me).$$

Thus when $m=ln$, l odd,

$$R\lim_{x=e}\frac{(mx)}{m^2}=-\frac{1}{2}\,\frac{1}{l^2n^2}=\frac{(mx)}{m^2}-\frac{1}{2}\,\frac{1}{n^2}\cdot\frac{1}{l^2}.$$

When m is not a multiple of n,

$$R\lim_{x=e}\frac{(mx)}{m^2}=\frac{(mx)}{m^2}.$$

o

$$R \lim_{x=e} F(x) = F(e) - \frac{1}{2\, n^2}\left\{\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right\}$$

$$= F(e) - \frac{\pi^2}{16\, n^2}, \qquad \text{by 218.}$$

is establishes 5). Similarly we prove 6). Thus $F(x)$ is ntinuous at each point of $\mathfrak{E}$. As

$$|F(x)| \leq \sum \frac{1}{n^2}$$

limited. As the points $\mathfrak{E}$ form an enumerable set, F is egrable in any finite interval.

. *Example 2.* Let $f(x) = 0$ at the points of a Cantor set $n \cdot a_1 a_2 \cdots$; $m = 0$, or a positive or negative integer, and the 0 or 2. Let $f(x) = 1$ elsewhere. Since $f(x)$ admits the d 1, $f(3\,nx)$ admits the period $\frac{1}{3\,n}\cdot$ Let C_1 be the points of ich fall in $\mathfrak{A} = (0, 1)$. Let D_1 be the corresponding set of vals. Let $C_2 = C_1 + \Gamma_1$, where Γ_1 is obtained by putting a t in each interval of D_1. Let D_2 be the intervals correspond- o C_2. Let $C_3 = C_2 + \Gamma_2$ where Γ_2 is obtained by putting a C_2 each interval of D_2, etc.

e zeros of $f(3\,nx)$ are obviously the points of C_n. Let

$$F = \sum \frac{1}{n^2} f(3\,nx) = \Sigma f_n(x).$$

zeros of F are the points of $\mathfrak{C} = \{C_n\}$. Since each C_n is a null is also a null set. Let $A = \mathfrak{A} - \mathfrak{C}$. The points A, $\mathfrak{C}$ are pantactic in $\mathfrak{A}$. Obviously F converges uniformly in $\mathfrak{A}$, $0 \leq f(3\,nx) \leq 1$. Since $f_n(x)$ is continuous at each point a F is continuous at a, and

$$F(a) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = H.$$

We show now that F is discontinuous at each point of $\mathfrak{E}$. For let e_m be an end point of one of the intervals of D_{m+1} but not of D_m. Then

$$f_1(e_m) = \frac{1}{1^2} \quad , \quad \cdots f_m(e_m) = \frac{1}{m^2},$$

$$f_{m+p}(e_m) = 0 \quad , \quad p = 1, 2 \cdots$$

Hence $$F(e_m) = H_m = \frac{1}{1^2} + \cdots + \frac{1}{m^2}.$$

As the points A are pantactic in $\mathfrak{A}$, there exists a sequence in A which $\doteq e$. For this sequence $F \doteq H$. Hence

$$\operatorname*{Disc}_{x=e_m} F = H - H_m = \bar{H}_m.$$

Similarly, if η_m is not an end point of the intervals D_{m+1}, but a limiting point of such end points,

$$\operatorname*{Disc}_{x=\eta_m} = \bar{H}_m.$$

The function F is R-integrable in $\mathfrak{A}$ since its points of discontinuity $\mathfrak{E}$ form a null set.

467. *Let $\mathfrak{E} = \{e_{\iota_1 \cdots \iota_s}\}$ be an enumerable set of points lying in the limited or unlimited set $\mathfrak{A}$, which lies in $\mathfrak{R}_m$. For any x in $\mathfrak{A}$ and any e_ι in $\mathfrak{E}$, let $x - e_\iota$ lie in $\mathfrak{B}$. Let $g(x_1 \cdots x_m)$ be limited in $\mathfrak{B}$ and continuous, except at $x = 0$, where*

$$\operatorname{Disc} g(x) = \mathfrak{d}.$$

Let $C = \Sigma c_{\iota_1 \cdots \iota_s}$ converge absolutely. Then

$$F(x_1 \cdots x_m) = \Sigma c_\iota g(x - e_\iota)$$

is continuous in $A = \mathfrak{A} - \mathfrak{E}$, and at $x = e_\iota$,

$$\operatorname{Disc} F(x) = c_\iota \mathfrak{d}.$$

For when x ranges over $\mathfrak{A}$, $x - e_\iota$ remains in $\mathfrak{B}$, and g is limited in $\mathfrak{B}$. Hence F is uniformly and absolutely convergent in $\mathfrak{A}$.

Now each $g(x - e_\iota)$ is continuous in A; hence F is continuous in A by 147, 2.

er hand, F is discontinuous at $x = e_\kappa$. For

$$F(x) = c_\kappa g(x - e_\kappa) + H(x),$$

he series F after removing the term on the right of tion. But H, as has just been shown, is continuous

ple 1. Let $\mathfrak{E} = \{e_n\}$ denote the rational numbers.

$$g(x) = \sin \frac{\pi}{x} \quad , \quad x \neq 0$$

$$= 0 \quad , \quad x = 0.$$

$$F(x) = \sum \frac{1}{n_\mu} g(x - e_n) \quad , \quad \mu > 1$$

except at the points $\mathfrak{E}$. At $x = e_n$,

$$\text{Disc } F = \frac{2}{n^\mu}.$$

Let $\mathfrak{E} = \}e_n\}$ denote the rational numbers.

$$g(x) = \lim_{n=\infty} \frac{nx}{1 + nx} = 1 \quad , \quad x \neq 0$$

$$= 0 \quad , \quad x = 0,$$

sidered in I, 331.

$$F(x) = \sum \frac{1}{n!} g(x - e_n)$$

except at the rational points, and at $x = e_m$,

$$\text{Disc } F(x) = \frac{1}{m!}.$$

foregoing $g(x)$ is limited. This restriction may be ny cases, as the reader will see from the following as an example.

$\cdots_{\iota_s}\}$ *be an enumerable apantactic set in* $\mathfrak{A}$. *Let* $\mathfrak{E} =$ *any* x *in* $\mathfrak{A}$, *and any* e_ι *in* E, *let* $x - e_\iota$ *lie within a* $(x_1 \cdots x_m)$ *be continuous in* $\mathfrak{B}$ *except at* $x = 0$, *where* $= 0$. *Let* $\Sigma c_{\iota_1 \ldots \iota_s}$ *be a positive term convergent series.*

Then $$F(x_1 \cdots x_m) = \Sigma e_\iota g(x - e_\iota)$$

is continuous in $A = \mathfrak{A} - \mathfrak{E}$. *On the other hand, each point of* $\mathfrak{E}$ *is a point of infinite discontinuity.*

For any given point $x = a$ of A lies at a distance > 0 from $\mathfrak{E}$. Thus $$\text{Min}\,(x - e_\iota) > 0,$$

as x ranges over some $V_\eta(a)$, and e_ι over E.

Hence $$|g(x - e_\iota)| < \text{some } M,$$

for x in $V_\eta(a)$, and e_ι in E. Thus F is uniformly convergent at $x = a$. As each $g(x - e_\iota)$ is continuous at $x = a$, F is continuous at a.

Let next $x = e_\kappa$. Then there exists a sequence
$$x', x'' \cdots \doteq e_\kappa \tag{1}$$
whose points lie in A. Thus the term $g(x - e_\kappa) \doteq +\infty$ as x ranges over 1). Hence a fortiori $F = +\infty$. Thus each point of E is a point of infinite discontinuity.

Finally any limit point of E is a point of infinite discontinuity, by 462, 1.

470. *Example.* Let $g(x) = \dfrac{1}{x}$, $a_n = -\dfrac{1}{a^n}$, $a > 1$.
$$e_n = \frac{1}{a^n n!}.$$

Then $$F(x) = \Sigma e_n g(x - a_n)$$
$$= \sum \frac{1}{n!} \frac{1}{1 + a^n x}$$

is a continuous function, except at the points
$$0, -\frac{1}{a}, -\frac{1}{a^2}, -\frac{1}{a^3} \cdots$$

which are points of infinite discontinuity.

471. Let us show how to construct functions by limiting processes, whose points of discontinuity are any given complete limited apantactic set $\mathfrak{E}$ in an m-way space $\mathfrak{R}_m$.

1. Let us for simplicity take $m = 2$, and call the coördinates of a point x, y.

Let Q denote the square whose center is the origin, and one of whose vertices is the point $(1, 0)$.

The edge of Q is given by the points x, y satisfying

$$|x| + |y| = 1. \tag{1}$$

Thus

$$Q(x, y) = \lim_{n=\infty} \frac{1}{1 + (|x| + |y|)^n} = \begin{cases} \frac{1}{2}, \text{ on the edge} \\ 1, \text{ inside} \\ 0, \text{ outside} \end{cases} \tag{2}$$

of the square Q. Hence

$$L(x, y) = \tfrac{1}{2}\left[1 - \lim_{n=\infty} \frac{n\{1 - |x| - |y|\}}{1 + n\{1 - |x| - |y|\}}\right] = \begin{cases} \frac{1}{2}, \text{ on the edge,} \\ 0, \text{ off the edge.} \end{cases} \tag{3}$$

Thus

$$G(x, y) = Q(x, y) + L(x, y) = \begin{cases} 1, \text{ in } Q, \\ 0, \text{ without } Q. \end{cases}$$

2. We next show how to construct a function g which shall $= 0$ on one or more of the edges of Q. Let us call these sides e_1, e_2, e_3, e_4, as we go around the edge of Q beginning with the first quadrant. If $G = 0$ on e_ι, let us denote it by G_ι; if $G = 0$ on e_ι, e_κ let us denote it by $G_{\iota\kappa}$, etc. We begin by constructing G_1. We observe that

$$1 - \lim_{n=\infty} \frac{n|t|}{1 + n|t|} = \begin{cases} 1, \text{ for } t = 0, \\ 0, \text{ for } t \neq 0. \end{cases}$$

Now the equation of a right line l may be given the form

$$x \cos\alpha + y \sin\alpha = p$$

where $0 \leq \alpha < 2\pi$, $p \geq 0$. Hence

$$Z(x, y) = 1 - \lim_{n=\infty} \frac{n|x \cos\alpha + y \sin\alpha - p|}{1 + n|x \cos\alpha + y \sin\alpha - p|} = \begin{cases} 1, \text{ on } l, \\ 0, \text{ off } l. \end{cases}$$

If now we make l coincide with e_1, we see that

$$E_1(x, y) = 2\,Z(x, y)\,L(x, y) = \begin{cases} 1, \text{ on } e_1, \\ 0, \text{ off } e_1. \end{cases}$$

Hence

$$G_1(x, y) = G(x, y) - E_1(x, y) = \begin{cases} 1, \text{ in } Q \text{ except on } e_1, \\ 0, \text{ on } e_1 \text{ and without } Q. \end{cases}$$

In the same way,

$$G_{\iota j} = G - (E_\iota + E_j),$$

$$G_{\iota j\kappa} = G - (E_\iota + E_j + E_\kappa),$$

$$G_{1234} = G - (E_1 + E_2 + E_3 + E_4).$$

By introducing a constant factor we can replace Q by a square Q_e whose sides are in the ratio $e: 1$ to those of Q.

Thus

$$Q(x, y) = \lim_{n=\infty} \frac{1}{1 + \left(\frac{\lfloor x \rfloor}{e} + \frac{\lfloor y \rfloor}{e}\right)^n} = \begin{cases} \frac{1}{2}, \text{ on the edge of } Q_e, \\ 1, \text{ inside}, \\ 0, \text{ outside}. \end{cases}$$

We can replace the square Q by a similar square whose center is a, b on replacing $|x|$, $|y|$ by $|x - a|$, $|y - b|$.

We have thus this result: by a limiting process, we can construct a function $g(x, y)$ having the value 1 inside Q, and on any of its edges, and $= 0$ outside Q, and on the remaining edges. Q has any point a, b as center, its edges have any length, and its sides are tipped at an angle of 45° to the axes.

We may take them parallel to the axes, if we wish, by replacing $|x|$, $|y|$ in our fundamental relation 1) by

$$|x - y| \quad , \quad |x + y|.$$

Finally let us remark that we may pass to m-way space, by replacing 1) by

$$|x_1| + |x_2| + \cdots + |x_m| = 1.$$

3. Let now $\mathfrak{Q} = \{\mathfrak{q}_n\}$ be a border set [328], of non-overlapping squares belonging to the complete apantactic set $\mathfrak{C}$, such that $\mathfrak{Q} + \mathfrak{C} = \mathfrak{R}$ the whole plane. We mark these squares in the plane and note which sides $\mathfrak{q}_n$ has in common with the preceding $\mathfrak{q}$'s. We take the $g_n(xy)$ function so that it is $= 1$ in $\mathfrak{q}_n$, except on these sides, and there 0. Then

$$G(x, y) = \Sigma g_n(xy)$$

has for each point only one term $\neq 0$, if x, y lies in $\mathfrak{Q}$, and no term $\neq 0$ if it lies in $\mathfrak{C}$.

Hence

$$G(xy) = \begin{cases} 1, \text{ for each point of } \mathfrak{Q}, \\ 0, \text{ for each point of } \mathfrak{C}. \end{cases}$$

Since $\mathfrak{C}$ is apantactic, each point of $\mathfrak{C}$ is a point of discontinuity of the 2° kind; each point of $\mathfrak{D}$ is a point of continuity.

4. Let $f(xy)$ be a function defined over $\mathfrak{A}$ which contains the complete apantactic set $\mathfrak{C}$.

Then
$$F(xy) = \Sigma f(xy)\, g_n(xy) = \begin{cases} f(xy), \text{ in } \mathfrak{A} - \mathfrak{C}, \\ 0, \text{ in } \mathfrak{C}. \end{cases}$$

472. 1. Let $\mathfrak{A} = (0, 1)$, $\mathfrak{B}_n =$ the points $\dfrac{2m+1}{2^n}$ in $\mathfrak{A}$.

Then $\mathfrak{B}_n$, $\mathfrak{B}_\nu$ have no points in common.

Let $f_n(x) = 1$ in $\mathfrak{B}_n$, and $= 0$ in $B_n = \mathfrak{A} - \mathfrak{B}_n$.

Let $\mathfrak{B} = \{\mathfrak{B}_n\}$. Then
$$F(x) = \Sigma f_n(x) = \begin{cases} 1, \text{ in } \mathfrak{B}, \\ 0, \text{ in } B = \mathfrak{A} - \mathfrak{B}. \end{cases}$$

The function F is totally discontinuous in $\mathfrak{B}$, oscillating between 0 and 1. The series F does not converge uniformly in any subinterval of $\mathfrak{A}$.

2. Keeping the notation in 1, let
$$G(x) = \sum \frac{1}{n} f_n(x).$$

At each point of $\mathfrak{B}_n$, $G = \dfrac{1}{n}$, while $G = 0$ in B.

The function G is discontinuous at the points of $\mathfrak{B}$, but continuous at the points B. The series G converges uniformly in $\mathfrak{A}$, yet an infinity of terms are discontinuous in any interval in $\mathfrak{A}$.

473. Let the limited set $\mathfrak{A}$ be the union of an enumerable set of complete sets $\{\mathfrak{A}_n\}$. We show how to construct a function f, which is discontinuous at the points of $\mathfrak{A}$, but continuous elsewhere in an m-way space.

Let us suppose first that $\mathfrak{A}$ consists of but one set and is complete. A point all of whose coördinates are rational, let us call rational, the other points of space we will call non-rational. If $\mathfrak{A}$ has an inner rational point, let $f = 1$ at this point, on the frontier of $\mathfrak{A}$ let $f = 1$ also; at all other points of space let $f = 0$. Then each point a of $\mathfrak{A}$ is a point of discontinuity. For if x is a fron-

tier or an inner rational point of $\mathfrak{A}$, $f(x)=1$, while in any $V(x)$ there are points where $f=0$. If x is not in $\mathfrak{A}$, all the points of some $D(x)$ are also not in $\mathfrak{A}$. At these points $f=0$. Hence f is continuous at such points.

We turn now to the general case. We have

$$\mathfrak{A}=A_1+A_2+A_3+\cdots$$

where $A_1=\mathfrak{A}_1$, $A_2=$ points of $\mathfrak{A}_2$ not in $\mathfrak{A}_1$, etc. Let $f_1=1$ at the rational inner points of A_1, and at the frontier points of $\mathfrak{A}_1$; at all other points let $f_1=0$. Let $f_2=\frac{1}{2}$ at the rational inner points of A_2, and at the frontier points of A_2 not in A_1; at all other points let $f_2=0$. At similar points of A_3 let $f_3=\frac{1}{3}$, and elsewhere $=0$, etc.

Consider now $$F=\Sigma f_n(x_1\cdots x_m).$$

Let $x=a$ be a point of $\mathfrak{A}$. If it is an inner point of some A_s, it is obviously a point of discontinuity of F. If not, it is a proper frontier point of one of the A's. Then in any $D(a)$ there are points of space not in $\mathfrak{A}$, or there are points of an infinite number of the A's. In either case a is a point of discontinuity. Similarly we see F is continuous at a point not in $\mathfrak{A}$.

2. We can obviously generalize the preceding problem by supposing $\mathfrak{A}$ to lie in a complete set $\mathfrak{B}$, such that each frontier point of $\mathfrak{A}$ is a limit point of $A=\mathfrak{B}-\mathfrak{A}$.

For we have only to replace our m-way space by $\mathfrak{B}$.

Functions of Class 1

474. 1. Baire has introduced an important classification of functions as follows:

Let $f(x_1\cdots x_m)$ be defined over $\mathfrak{A}$; f and $\mathfrak{A}$ limited or unlimited. If f is continuous in $\mathfrak{A}$, we say its class is 0 in $\mathfrak{A}$, and write

$$\text{Class } f=0 \quad , \quad \text{or Cl } f=0 \quad , \quad \text{Mod } \mathfrak{A}.$$

If
$$f=\lim_{n=\infty} f_n(x_1\cdots x_m),$$

each f_n being of class 0 in $\mathfrak{A}$, we say its class is 1, if f does not lie in class 0, mod $\mathfrak{A}$.

e series $F(x) = \Sigma f_n(x)$

$\mathfrak{A}$, each term f_n being continuous in $\mathfrak{A}$. Since

$$F(x) = \lim_{n=\infty} F_n(x),$$

of class 0, or class 1, according as F is continuous, or us in $\mathfrak{A}$. A similar remark holds for infinite prod-

$$G(x) = \Pi g_n(x).$$

rivatives of a function $f(x)$ give rise to functions of For let $f(x)$ have a unilateral differential coeffi- t each point of $\mathfrak{A}$. Both f and $\mathfrak{A}$ may be unlimited. deas, suppose the right-hand differential coefficient $h_1 > h_2 > \cdots \doteq 0$. Then

$$q_n(x) = \frac{f(x+h_n) - f(x)}{h_n}, \qquad x + h_n \text{ in } \mathfrak{A},$$

us function of x in $\mathfrak{A}$. But

$$q(x) = \lim_{n=\infty} q_n(x)$$

n x in $\mathfrak{A}$ by hypothesis.

remark applies to the partial derivatives

$$\frac{\partial f}{\partial x_1}, \cdots \frac{\partial f}{\partial x_m}$$

$f(x_1 \cdots x_n)$.

$$f(x) = \lim_{n=\infty} f_n(x_1 \cdots x_m),$$

of class 1 in $\mathfrak{A}$. Then we say, $\mathrm{Cl}\, f = 2$ if f does not er class. In this way we may continue. It is of sary to show that such functions actually exist.

mple 1.

$$f(x) = \lim_{n=\infty} \frac{nx}{1+nx} = \begin{cases} 1, \text{ for } x > 0, \\ 0, \text{ for } x = 0. \end{cases}$$

tion was considered in I, 331. In any interval containing the origin $x = 0$, $\mathrm{Cl}\, f = 1$; in any inter- $x > 0$, not containing the origin, $\mathrm{Cl}\, f = 0$.

Example 2.

Let
$$f(x) = \lim_{n=\infty} \frac{nx}{e^{nx^2}} = 0, \quad \text{in } \mathfrak{A} = (-\infty, \infty).$$

The class of $f(x)$ is 0 in $\mathfrak{A}$. Although each f_n is limited in $\mathfrak{A}$, the graphs of f_n have peaks near $x = 0$ which $\doteq \infty$, as $n \doteq \infty$.

Example 3. If we combine the two functions in Ex. 1, 2, we get
$$f(x) = \lim_{n=\infty} \left\{ \frac{1}{1+nx} + \frac{1}{e^{nx^2}} \right\} nx = \begin{cases} 1, \text{ for } x \neq 0, \\ 0, \text{ for } x = 0. \end{cases}$$

Hence $\mathrm{Cl}\, f(x) = 1$ for any set $\mathfrak{B}$ embracing the origin; $= 0$ for any other set.

Example 4.

Let
$$f(x) = \lim_{n=\infty} x e^{\frac{1}{x+\frac{1}{n}}}, \quad \text{in } \mathfrak{A} = (0, 1).$$

Then
$$f(x) = 0 \quad , \quad \text{for } x = 0$$
$$= x e^{\frac{1}{x}} \quad , \quad \text{for } x > 0.$$

We see thus that f is continuous in $(0^*, 1)$, and has a point of infinite discontinuity at $x = 0$.

Hence
$$\text{Class } f(x) = 1, \quad \text{in } \mathfrak{A}$$
$$= 0, \quad \text{in } (0^*, 1).$$

Example 5.

Let
$$f(x) = \lim_{n=\infty} \frac{1}{x + \frac{1}{n}} \qquad \text{in } \mathfrak{A} = (0, \infty).$$

Then
$$f(x) = \frac{1}{x} \quad , \quad \text{for } x > 0$$
$$= +\infty \quad , \quad \text{for } x = 0.$$

Here
$$\lim_{n=\infty} f_n(x)$$

does not exist at $x = 0$. We cannot therefore speak of the class of $f(x)$ in $\mathfrak{A}$ since it is not defined at the point $x = 0$. It is defined in $\mathfrak{B} = (0^*, \infty)$, and its class is obviously 0, mod $\mathfrak{B}$.

Example 6.

Let
$$f(x) = \sin \frac{1}{x} \quad , \quad \text{for } x \neq 0$$
$$= \text{a constant } c \quad , \quad \text{for } x = 0.$$

We show that $\mathrm{Cl}\, f = 1$ in $\mathfrak{A} = (-0, \infty)$. For let
$$f_n(x) = c\left(1 - \frac{nx}{1+nx}\right) + \frac{nx}{1+nx} \sin \left[\frac{1}{x + \frac{1}{n}}\right]$$
$$= g_n(x) + h_n(x).$$

Now by Ex. 1,
$$\lim g_n(x) = \begin{cases} 0, \text{ for } x > 0, \\ c, \text{ for } x = 0; \end{cases}$$
hile
$$\lim h_n(x) = \begin{cases} \sin \frac{1}{x}, \text{ for } x > 0, \\ 0, \text{ for } x = 0. \end{cases}$$

As each f_n is continuous in $\mathfrak{A}$, and
$$\lim f_n(x) = f(x) \text{ in } \mathfrak{A},$$
e see its class is ≤ 1. As f is discontinuous at $x = 0$, its class not 0 in $\mathfrak{A}$.

Example 7. Let
$$f(x) = \lim_{n=\infty} \frac{1}{n} \cdot \sin \frac{1}{x}.$$

Here the functions $f_n(x)$ under the limit sign are not defined r $x = 0$. Thus f is not defined at this point. We cannot there-re speak of the class of f with respect to any set embracing the int $x = 0$. For any set $\mathfrak{B}$ not containing this point, $\mathrm{Cl}\, f = 0$, nce $f(x) = 0$ in $\mathfrak{B}$.

Let us set
$$\phi(x) = \sin \frac{1}{x} \quad , \quad \text{for } x \neq 0$$
$$= \text{a constant } c \quad , \quad \text{for } x = 0.$$

Let
$$g(x) = \lim_{n=\infty} \frac{1}{n} \phi(x) = \lim_{n=\infty} \phi_n(x).$$

Here g is a continuous function in $\mathfrak{A} = (-\infty, \infty)$. Its class is thus 0 in $\mathfrak{A}$. On the other hand, the functions ϕ_n are each of class 1 in $\mathfrak{A}$.

Example 3.

$$\Gamma(x) = \frac{1}{x}\prod_1^{\infty} \frac{\left(1+\frac{1}{n}\right)^x}{1+\frac{x}{n}}$$

is defined at all the points of $(-\infty, \infty)$ except $0, -1, -2, \ldots$ These latter are points of infinite discontinuity. In its domain of definition, Γ is a continuous function. Hence $\mathrm{Cl}\,\Gamma(x) = 0$ with respect to this domain.

476. 1. If $\mathfrak{A}$, limited or unlimited, is the union of an enumerable set of complete sets, we say $\mathfrak{A}$ is *hypercomplete*.

Example 1. The points S^* within a unit sphere S, form a hypercomplete set. For let Σ_r have the same center as S, and radius $r<1$. Obviously each Σ_r is complete, while $\{\Sigma_r\} = S^*$, r ranging over $r_1 < r_2 < \cdots \doteq 1$.

Example 2. An enumerable set of points $a_1, a_2 \cdots$ form a hypercomplete set. For each a_n may be regarded as a complete set, embracing but a single point.

2. *If $\mathfrak{A}_1, \mathfrak{A}_2 \cdots$ are limited hypercomplete sets, so is their union $\{\mathfrak{A}_n\} = \mathfrak{A}$.*

For each $\mathfrak{A}_m$ is the union of an enumerable set of complete sets $\mathfrak{A}_{m,n}$. Thus $\mathfrak{A} = \{\mathfrak{A}_{m,n}\}$ $m, n = 1, 2 \cdots$ is hypercomplete.

Let $\mathfrak{A}$ be complete. If $\mathfrak{B}$ is a complete part of $\mathfrak{A}$, $A = \mathfrak{A} - \mathfrak{B}$ is hypercomplete.

For let $\mathfrak{Q} = \{\mathfrak{q}_n\}$ be a border set of $\mathfrak{B}$, as in 328. The points A_n of A in each $\mathfrak{q}_n$ are complete, since $\mathfrak{A}$ is complete. Thus $A = \{A_n\}$, and A is hypercomplete.

Let $\mathfrak{A} = \{\mathfrak{A}_n\}$ be hypercomplete, each $\mathfrak{A}_n$ being complete. If $\mathfrak{B}$ is a complete part of $\mathfrak{A}$, $A = \mathfrak{A} - \mathfrak{B}$ is hypercomplete.

For let A_n denote the points of $\mathfrak{A}_n$ not in $\mathfrak{B}$. Then as above, A_n is hypercomplete. As $A = \{A_n\}$, A is also hypercomplete.

477. 1. $\mathfrak{E}_\epsilon$ *Sets.* If the limited or unlimited set $\mathfrak{A}$ is the union of an enumerable set of limited complete sets, in each of which Osc $f < \epsilon$, we shall say $\mathfrak{A}$ *is an* $\mathfrak{E}_\epsilon$ *set.* If, however small $\epsilon > 0$ is taken, $\mathfrak{A}$ is an $\mathfrak{E}_\epsilon$ set, we shall say $\mathfrak{A}$ is an $\mathfrak{E}_\epsilon$ set, $\epsilon \doteq 0$, which we may also express by $\mathfrak{E}_{\epsilon\doteq 0}$.

2. *Let* $f(x_1 \cdots x_m)$ *be continuous in the limited complete set* $\mathfrak{A}$. *Then* $\mathfrak{A}$ *is an* $\mathfrak{E}_\epsilon$ *set*, $\epsilon \doteq 0$.

For let $\epsilon > 0$ be taken small at pleasure and fixed. By I, 353, there exists a cubical division of space D, such that if $\mathfrak{A}_n$ denote the points of $\mathfrak{A}$ in one of the cells of D, Osc $f < \epsilon$ in $\mathfrak{A}_n$. As $\mathfrak{A}_n$ is complete, since $\mathfrak{A}$ is, $\mathfrak{A}$ is an $\mathfrak{E}_\epsilon$ set.

3. *An enumerable set of points* $\mathfrak{A} = \{a_n\}$ *is an* $\mathfrak{E}_{\epsilon\doteq 0}$ *set.*

For each a_n may be regarded as a complete set, embracing but a *single* point. But in a set embracing but one point, Osc $f = 0$.

4. *The union of an enumerable set of* $\mathfrak{E}_\epsilon$ *sets* $\mathfrak{A} = \{\mathfrak{A}_m\}$ *is an* $\mathfrak{E}_\epsilon$ *set.*

For each $\mathfrak{A}_m$ is the union of an enumerable set of limited sets $\mathfrak{A}_m = \{\mathfrak{A}_{m,n}\}$, $n = 1, 2, \cdots$ and Osc $f < \epsilon$ in each $\mathfrak{A}_{mn}$.

Thus
$$\mathfrak{A} = \{\mathfrak{A}_{mn}\} \quad , \quad m, n = 1, 2, \cdots$$

But an enumerable set of enumerable sets is an enumerable set. Hence $\mathfrak{A}$ is an $\mathfrak{E}_\epsilon$ set.

5. *Let* $f(x_1 \cdots x_m)$ *be continuous in the complete set* $\mathfrak{A}$, *except at the points* $\mathfrak{D} = d_1, d_2 \cdots d_s$. *Then* $\mathfrak{A}$ *is an* $\mathfrak{E}_{\epsilon\doteq 0}$ *set.*

For let $\epsilon > 0$ be taken small at pleasure and fixed. About each point of $\mathfrak{D}$ we describe a sphere of radius ρ. Let $\mathfrak{A}_\rho$ denote the points of $\mathfrak{A}$ not *within* one of these spheres. Obviously $\mathfrak{A}_\rho$ is complete. Let ρ range over $r_1 > r_2 > \cdots \doteq 0$. If we set $\mathfrak{A} = A + \mathfrak{D}$, obviously $A = \{\mathfrak{A}_{r_n}\}$. As f is continuous in $\mathfrak{A}_{r_n}$, it is an $\mathfrak{E}_\epsilon$ set. Hence $\mathfrak{A}$, being the union of A and $\mathfrak{D}$, is an $\mathfrak{E}_\epsilon$ set.

478. 1. *Let* $\mathfrak{A}$ *be an* $\mathfrak{E}_\epsilon$ *set. The points* $\mathfrak{D}$ *of* $\mathfrak{A}$ *common to the limited complete set* $\mathfrak{B}$ *form an* $\mathfrak{E}_\epsilon$ *set.*

For $\mathfrak{A}$ is the union of the complete sets $\mathfrak{A}_n$, in each of which Osc $f < \epsilon$. But the points of $\mathfrak{A}_n$ in $\mathfrak{B}$ form a complete set A_n, and of course Osc $f < \epsilon$ in A_n. As $\mathfrak{D} = \{A_n\}$, it is an $\mathfrak{E}_\epsilon$ set.

2. *Let* $\mathfrak{A}$ *be a limited* $\mathfrak{E}_e$ *set. Let* $\mathfrak{B}$ *be a complete part of* $\mathfrak{A}$. *Then* $A = \mathfrak{A} - \mathfrak{B}$ *is an* $\mathfrak{E}_e$ *set.*

For $\mathfrak{A}$ is the union of the complete sets $\mathfrak{A}_n$, in each of which $\text{Osc} f < \epsilon$. The points of $\mathfrak{A}_n$ not in $\mathfrak{B}$ form a set A_n, such that $\text{Osc} f < \epsilon$ in A_n also. But $A = \{A_n\}$, and each A_n being hypercomplete, is an $\mathfrak{E}_e$ set.

3. Let $f(x_1 \cdots x_m)$ be defined over $\mathfrak{A}$, either f or $\mathfrak{A}$ being limited or unlimited. The points of $\mathfrak{A}$ at which

$$\alpha \leq f \leq \beta \tag{1}$$

may be denoted by

$$(\alpha \leq f \leq \beta). \tag{2}$$

If in 1) one of the equality signs is missing, it will of course be dropped in 2).

479. 1. *Let* $f_1, f_2, \cdots$ *be continuous in the limited complete set* $\mathfrak{A}$. *If at each point of* $\mathfrak{A}$, $\lim_{n=\infty} f_n$ *exists,* $\mathfrak{A}$ *is an* $\mathfrak{E}_{e\doteq 0}$ *set and so is any complete* $\mathfrak{B} < \mathfrak{A}$.

For let $\lim_{n=\infty} f_n(x_1 \cdots x_m) = f(x_1 \cdots x_m)$ in $\mathfrak{A}$. Let us effect a division of norm $\epsilon/2$ of the interval $(-\infty, \infty)$ by interpolating the points $\cdots m_{-2}, m_{-1}, m_0 = 0, m_1, m_2 \cdots$

Let $\mathfrak{A}_\iota = (m_\iota < f < m_{\iota+2})$, then $\mathfrak{A} = \{\mathfrak{A}_\iota\}$.

Next let
$$\mathfrak{D}_{n,p} = \underset{q \geq p}{Dv}\left\{m_\iota + \frac{1}{n} \leq f_q \leq m_{\iota+2} - \frac{1}{n}\right\}.$$

Then
$$\mathfrak{A}_\iota = \{\mathfrak{D}_{n,p}\} \quad , \quad n, p = 1, 2 \cdots \tag{1}$$

For let a be a point of $\mathfrak{A}_\iota$, and say $f(a) = \alpha$. Then

$$m_\iota < \alpha < m_{\iota+2}.$$

But
$$\alpha - \epsilon < f_q(a) < \alpha + \epsilon \quad , \quad q > \text{some } p,$$

and we may take ϵ and n so that

$$m_\iota + \frac{1}{n} \leq f_q(a) \leq m_{\iota+2} - \frac{1}{n}.$$

Hence a is in $\mathfrak{D}_{n,p}$.

Conversely, let a be a point of $\{\mathfrak{D}_{n,p}\}$. Then a lies in some $\mathfrak{D}_{n,p}$. Hence,

$$m_\iota + \frac{1}{n} \leq f_q(a) \leq m_{\iota+2} - \frac{1}{n} \quad , \quad q \geq p.$$

But as $f_n(a) \doteq f(a)$, we have

$$|f(a) - f_q(a)| < \epsilon \quad , \quad q > \text{some } p'.$$

Hence if ϵ is sufficiently small,

$$m_\iota < f(a) < m_{\iota+2},$$

d thus a is in $\mathfrak{A}_\iota$.

Thus 1) is established. But $\mathfrak{D}_{np}$ is a divisor of complete sets, d is therefore complete. Thus $\mathfrak{A}$ is the union of an enumerable of complete sets $\{\mathfrak{B}_\iota\}$, in each of which Osc $f < \epsilon$, ϵ small at asure.

Let now $\mathfrak{B}$ be any complete part of $\mathfrak{A}$. Let $\mathfrak{a}_\iota = Dv\,\{\mathfrak{B}, \mathfrak{B}_\iota\}$. en $\mathfrak{a}_\iota$ is complete, and Osc $f < \epsilon$, in $\mathfrak{a}_\iota$. Moreover, $\mathfrak{B} = \{\mathfrak{a}_\iota\}$.

Hence $\mathfrak{B}$ is an $\mathfrak{E}_{\epsilon \doteq 0}$ set.

2. *If* Class $f \leq 1$ *in limited complete* $\mathfrak{A}$, *f limited or unlimited, is an* $\mathfrak{E}_\epsilon$ *set.*

This is an obvious result from 1.

3. *Let $f(x_1 \cdots x_m)$ be a totally discontinuous function in the non-umerable set* $\mathfrak{A}$. *Then* Class f *is not* 0 *or* 1 *in* $\mathfrak{A}$, *if* $\mathfrak{d} =$ Disc f *at ch point is* $\leq k > 0$.

For in any subset $\mathfrak{B}$ of $\mathfrak{A}$ containing the point x, Osc $f \geq k$. ence Osc f is not $\leq \epsilon$, in any part of $\mathfrak{A}$, if $\epsilon < k$. Thus $\mathfrak{A}$ cannot an $\mathfrak{E}_\epsilon$ set.

4. *If* Class $f(x_1 \cdots x_m) \leq 1$ *in the limited complete set* $\mathfrak{A}$, *the set* $= (a < f < b)$ *is a hypercomplete set, a, b being arbitrary numbers.*

For we have only to take $a = m_\iota$, $b = m_{\iota+2}$. Then $\mathfrak{B} = \mathfrak{A}_\iota$, which, in 1, is hypercomplete.

480. (*Lebesgue.*) *Let the limited or unlimited function* $f(x_1 \cdots x_m)$ *defined over the limited set* $\mathfrak{A}$. *If* $\mathfrak{A}$ *may be regarded as an* $_{\doteq 0}$ *set with respect to f, the class of f is* ≤ 1.

For let $\omega_1 > \omega_2 > \cdots \doteq 0$. By hypothesis $\mathfrak{A}$ is the union of a quence of complete sets

$$\mathfrak{A}_{11} \quad , \quad \mathfrak{A}_{12} \quad , \quad \mathfrak{A}_{13} \cdots \qquad (S_1$$

each of which Osc $f \leq \omega_1$. $\mathfrak{A}$ is also the union of a sequence complete sets

$$\mathfrak{B}_{11} \quad , \quad \mathfrak{B}_{12} \quad , \quad \mathfrak{B}_{13} \cdots \qquad (1$$

in each of which $\operatorname{Osc} f \leq \omega_2$. If we superpose the division 1) of $\mathfrak{A}$ on the division S_1, each $\mathfrak{A}_{1\kappa}$ will fall into an enumerable set of complete sets, and together they will form an enumerable sequence

$$\mathfrak{A}_{21} \quad , \quad \mathfrak{A}_{22} \quad , \quad \mathfrak{A}_{23} \cdots \tag*{$(S_2$}$$

in each of which $\operatorname{Osc} f \leq \omega_2$. Continuing in this way we see that $\mathfrak{A}$ is the union of the complete sets

$$\mathfrak{A}_{n1} \quad , \quad \mathfrak{A}_{n2} \quad , \quad \mathfrak{A}_{n3} \cdots \tag*{$(S_n$}$$

such that in each set of S_n, $\operatorname{Osc} f < \omega_n$, and such that each set lies in some set of the preceding sequence S_{n-1}.

With each $\mathfrak{A}_{n,s}$ we associate a constant C_{ns}, such that

$$|f(x) - C_{ns}| \leq \omega_n \quad , \quad \text{in } \mathfrak{A}_{ns}, \tag*{(2}$$

and call C_{ns} the corresponding *field constant.*

We show now how to define a sequence of continuous functions $f_1, f_2 \cdots$ which $\doteq f$. To this end we effect a sequence of superimposed divisions of space $D_1, D_2 \cdots$ of norms $\doteq 0$. The vertices of the cubes of D_n we call the *lattice points* L_n. The cells of D_n containing a given lattice point l of L_n form a cube $\mathfrak{Q}$. Let $\mathfrak{A}_{1\iota_1}$ be the first set of S_1 containing a point of $\mathfrak{Q}$. Let $\mathfrak{A}_{2\iota_2}$ be the first set of S_2 containing a point of $\mathfrak{Q}$ lying in $\mathfrak{A}_{1\iota_1}$. Continuing in this way we get

$$\mathfrak{A}_{1\iota_1} \geq \mathfrak{A}_{2\iota_2} \geq \cdots \geq \mathfrak{A}_{n\iota_n}.$$

To $\mathfrak{A}_{n\iota_n}$ belongs the field constant $C_{n\iota_n}$; this we associate with the lattice point l and call it the corresponding *lattice constant.*

Let now $\mathfrak{C}$ be a cell of D_n containing a point of $\mathfrak{A}$. It has 2^n vertices or lattice points. Let P_s denote any product of s different factors $x_{r_1}, x_{r_2}, \cdots x_{r_s}$. We consider the polynomial

$$\phi = AP_n + \Sigma BP_{n-1} + \Sigma CP_{n-2} + \cdots + \Sigma KP_1 + L,$$

the summation in each case extending over all the distinct products of that type. The number of terms in ϕ is, by I, 96,

$$\binom{n}{n} + \binom{n}{1} + \binom{n}{2} + \cdots + 1 = 2^n.$$

We can thus determine the 2^n coefficients of ϕ so that the values of ϕ at the lattice points of $\mathfrak{C}$ are the corresponding lattice constants. Thus ϕ is a continuous function in $\mathfrak{C}$, whose greatest and least values are the greatest and least lattice constants belonging to $\mathfrak{C}$. Each cube $\mathfrak{C}$ containing a point of $\mathfrak{A}$ has associated with it a ϕ function.

We now define $f_n(x_1 \cdots x_m)$ by stating that its value in any cube $\mathfrak{C}$ of D_n, containing a point of $\mathfrak{A}$, is that of the corresponding ϕ function. Since ϕ is linear in each variable, two ϕ's belonging to adjacent cubes have the same values along their common points.

We show now that $f_n(x) \doteq f(x)$ at any point x of $\mathfrak{A}$, or that

$$\epsilon > 0, \quad \nu, \quad |f(x) - f_n(x)| < \epsilon \quad , \quad n > \nu. \tag{3}$$

Let $\omega_e < \epsilon/8$. Let $\mathfrak{A}_{1\iota_1}$ be the first set in S_1 containing the point x, $\mathfrak{A}_{2\iota_2}$ the first set of S_2 lying in $\mathfrak{A}_{1\iota_1}$ and containing x. Continuing we get

$$\mathfrak{A}_{1\iota_1} \geq \mathfrak{A}_{2\iota_2} \geq \mathfrak{A}_{3\iota_3} \geq \cdots \geq \mathfrak{A}_{e\iota_e}.$$

Let $\mathfrak{P}_e$ be the union of the sets in S_1 preceding $\mathfrak{A}_{\iota\iota_1}$; of the sets in S_2 preceding $\mathfrak{A}_{2\iota_2}$ and lying in $\mathfrak{A}_{1\iota_1}$, and so on, finally the sets of S_e preceding $\mathfrak{A}_{e\iota_e}$, and lying in $\mathfrak{A}_{e-1, \iota_e - 1}$. Their number being finite, $\delta = \text{Dist}\,(\mathfrak{A}_{e\iota_e}, \mathfrak{P}_e)$ is obviously > 0. We may therefore take $\nu > e$ so large that cubes of D_ν about the point x lie wholly in $D_\eta(x)$, $\eta < \delta$.

Consider now $f_n(x)$, $n > \nu$, and *let us suppose first* that x is not a lattice point of D_n. Let it lie within the cell $\mathfrak{C}$ of D_n. Then $f_n(x)$ is a mean of the values of

$$f_n(l) = C_{n, j_n},$$

where l is any one of the 2^n vertices of $\mathfrak{C}$, and C_{nj_n} is the corresponding lattice constant, which we know is associated with the set $\mathfrak{A}_{nj_n}$.

We observe now that each of the

$$\mathfrak{A}_{nj_n} \leq \mathfrak{A}_{e, \iota_e}. \tag{4}$$

For each set in S_n is a part of some set in any of the preceding sequences. Now $\mathfrak{A}_{nj_n}$ cannot be a part of $\mathfrak{A}_{1k}$, $k < \iota_1$, for none of

these points lie in $D_\eta(x)$. Hence $\mathfrak{A}_{nj_n}$ is a part of $\mathfrak{A}_{1\iota_1}$. For t
same reason it is a part of $\mathfrak{A}_{2\iota_2}$, etc., which establishes 4).

Let now x' be a point of $\mathfrak{A}_{nj_n}$. Then

$$| C_{nj_n} - C_{e\iota_e} | \leq | C_{nj_n} - f(x') | + | f(x') - C_{e\iota_e} |$$

$$\leq \omega_n + \omega_e < \frac{\epsilon}{4} \quad , \quad \text{by } 2).$$

From this follows, since $f_n(x)$ is a mean of these C_{nj_n}, that

$$| f_n(x) - C_{nj_n} | < \frac{\epsilon}{2}.$$

But now

$$| f(x) - f_n(x) | \leq | f(x) - C_{nj_n} | + | C_{nj_n} - f_n(x) |.$$

As x lies in $\mathfrak{A}_{e\iota_e}$,

$$| f(x) - C_{nj_n} | < | f(x) - C_{e\iota_e} | + | C_{e\iota_e} - C_{nj_n} |$$

$$\leq \omega_e + \frac{\epsilon}{4} < \frac{\epsilon}{2},$$

by 2), 5). From 6), 8) we have 3) for the present case.

The case that x is a lattice point for some division and hen
for all following, has really been established by the foregoin
reasoning.

481. 1. *Let f be defined over the limited set $\mathfrak{A}$. If for arbitrar
a, b, the sets $\mathfrak{B} = (a < f < b)$ are hypercomplete, then* Class $f \leq$ 1.

For let us effect a division of norm $\epsilon/2$ of $(-\infty, \infty)$ as i
479, 1. Then $\mathfrak{A} = \{\mathfrak{A}_\iota\}$, where as before $\mathfrak{A}_\iota = (m_\iota < f < m_{\iota+2})$.
But as Osc $f < \epsilon$ in $\mathfrak{A}_\iota$, and as each $\mathfrak{A}_\iota$ is hypercomplete b
hypothesis, our theorem is a corollary of 480.

2. *For $f(x_1 \cdots x_m)$ to be of class ≤ 1 in the limited complete s
$\mathfrak{A}$, it is necessary and sufficient that the sets $(a < f < b)$ are hyper
complete, a, b being arbitrary.*

This follows from 1 and 479, 2.

3. *Let limited $\mathfrak{A}$ be the union of an enumerable set of complete se
$\{\mathfrak{A}_n\}$, such that* Cl $f \leq 1$ *in each $\mathfrak{A}_n$, then* Cl $f \leq 1$ *in $\mathfrak{A}$.*

For by 479, 1, $\mathfrak{A}_n$ is the union of an enumerable set of complete ts in each of which $\text{Osc} f \leq \epsilon$. Thus $\mathfrak{A}$ is also such a set, *i.e.* an $_\epsilon$ set. We now apply 480, 1.

4. *If* $\text{Class} f \leq 1$ *in the limited complete set* $\mathfrak{A}$, *its class is* ≤ 1, *any complete part* $\mathfrak{B}$ *of* $\mathfrak{A}$.

This follows from 479, 1 and 480, 1.

482. 1. *Let* $f(x_1 \cdots x_m)$ *be defined over the complete set* $\mathfrak{A}$, *and ive only an enumerable set* $\mathfrak{E}$ *of points of discontinuity in* $\mathfrak{A}$. *hen* $\text{Class} f = 1$ *in* $\mathfrak{A}$.

For the points E of $\mathfrak{A}$ at which $\text{Osc} f \geq \epsilon/2$ form a complete ırt of $\mathfrak{A}$, by 462, 3. But E, being a part of $\mathfrak{E}$, is enumerable d is hence an $\mathfrak{E}_e$ set by 477, 3. Let us turn to $\mathfrak{B} = \mathfrak{A} - E$. For ch of its points b, there exists a $\delta > 0$, such that $\text{Osc} f < \epsilon$ in e set $\mathfrak{b}$ of points of $\mathfrak{B}$ lying in $D_\delta(b)$. As $\mathfrak{A}$ is complete, so is $\mathfrak{b}$. s E is complete, there is an enumerable set of these $\mathfrak{b}$, call them , $\mathfrak{b}_2 \cdots$, such that $\mathfrak{B} = \{\mathfrak{b}_s\}$. As $\mathfrak{A} = \mathfrak{B} + E$, it is the union of enumerable set of complete sets, in each of which $\text{Osc} f < \epsilon$. his is true however small $\epsilon > 0$ is taken. We apply now 480, 1.

2. We can now construct functions of class 2.

Example. Let $f_n(x_1 \cdots x_m) = 1$ at the rational points in the it cube $\mathfrak{Q}$, whose coördinates have denominators $\leq n$. Else- ere let $f_n = 0$. Since f_n has only a finite number of discontinu- es in $\mathfrak{Q}$, $\text{Cl} f_n = 1$ in $\mathfrak{Q}$. Let now

$$f(x_1 \cdots x_m) = \lim_{n=\infty} f_n.$$

At a non-rational point, each $f_n = 0$, $\therefore f = 0$. At a rational int, $f_n = 1$ for all $n >$ some s. Hence at such a point $f = 1$. ıus each point of $\mathfrak{Q}$ is a point of discontinuity and $\text{Disc} f = 1$. ence $\text{Cl} f$ is not 1. As f is the limit of functions of class 1, its ıss is 2.

483. *Let* $f(x_1 \cdots x_m)$ *be continuous with respect to each* x_ι, *at each int of a limited set* $\mathfrak{A}$, *each of whose points is an inner point. hen* $\text{Class} f \leq 1$.

For let $\mathfrak{A}$ lie within a cube $\mathfrak{Q}$. Then $A = \mathfrak{Q} - \mathfrak{A}$ is complete. We may therefore regard $\mathfrak{A}$ as a border set of A; that is, a set of non-overlapping cubes $\{\mathfrak{q}_n\}$. We show now that $\mathrm{Cl} f \leq 1$ in any one of these cubes as $\mathfrak{q}$. To this end we show that the points $\mathfrak{B}_m$ of $\mathfrak{q}$ at which

$$a + \frac{1}{m} \leq f \leq b - \frac{1}{m}$$

form a complete set. For let $b_1, b_2 \cdots$ be points of $\mathfrak{B}_m$, which $\doteq \beta$. We wish to show that β lies in $\mathfrak{B}_m$. Suppose first that $b_s, b_{s+1} \cdots$ have all their coördinates except one, say x, the same as the coördinates of β. Since

$$a + \frac{1}{m} \leq f(b_{s+p}) \leq b - \frac{1}{m},$$

therefore

$$a + \frac{1}{m} < \lim_{p=\infty} f(b_{s+p}) \leq b - \frac{1}{m}.$$

As f is continuous in x_1, and as only the coördinate x_1 varies in b_{s+p}, we have

$$a + \frac{1}{m} \leq f(\beta) \leq b - \frac{1}{m}.$$

Hence β lies in $\mathfrak{B}_m$.

We suppose next that $b_s, b_{s+1} \cdots$ have all their coördinates the same as β except two, say x_1, x_2.

We may place each b_n at the center of an interval $\mathfrak{i}$ of length δ, parallel to the x_1 axis, such that

$$a + \frac{1}{m} - \epsilon \leq f(x) \leq b - \frac{1}{m} + \epsilon,$$

since f is uniformly continuous in x_1, by I, 352. These intervals cut an ordinate in the x_1, x_2 plane through β, in a set of points c_{s+p} which $\doteq \beta$. Then as before,

$$a + \frac{1}{m} - \epsilon \leq f(\beta) \leq b - \frac{1}{m} + \epsilon.$$

As ϵ is small at pleasure, β lies in $\mathfrak{B}_m$. In this way we may continue.

As $\mathrm{Cl} f \leq 1$ in each $\mathfrak{q}_n$, it is in $\mathfrak{A}$, by 481, 3.

484. (*Volterra.*) *Let $f_1, f_2 \cdots$ be at most pointwise discontinuous in the limited complete set $\mathfrak{A}$. Then there exists a point of $\mathfrak{A}$ at which all the f_n are continuous.*

For if $\mathfrak{A}$ contains an isolated point, the theorem is obviously true, since every function is continuous at an isolated point. Let us therefore suppose that $\mathfrak{A}$ is perfect.

Let $\epsilon_1 > \epsilon_2 > \cdots \doteq 0$. Let a_1 be a point of continuity of f_1. Then

$$\mathrm{Osc}\, f_1 < \epsilon \quad , \quad \text{in some } \mathfrak{A}_1 = V_{\delta_1}(a_1).$$

In $\mathfrak{A}_1$ there is a point b of continuity of f_1. Hence $\mathrm{Osc}\, f_1 < \epsilon_2$ in some $V_\eta(b)$, and we may take b so that $V_\eta(b) < \mathfrak{A}_1$. But in $V_\eta(b)$ there is a point a_2 at which f_2 is continuous. Hence

$$\mathrm{Osc}\, f_1 < \epsilon_2 \quad , \quad \mathrm{Osc}\, f_2 < \epsilon_1 \quad , \quad \text{in some } \mathfrak{A}_2 = V_{\delta_2}(a_2),$$

and we may take a_2 such that $\mathfrak{A}_2 < \mathfrak{A}_1$. Similarly there exists a point a_3 in $\mathfrak{A}_2$, such that

$$\mathrm{Osc}\, f_1 < \epsilon_3 \quad , \quad \mathrm{Osc}\, f_2 < \epsilon_2 \quad , \quad \mathrm{Osc}\, f_3 < \epsilon_1 \quad , \quad \text{in some } \mathfrak{A}_3 = V_{\delta_3}(a_3),$$

and we may take a_3 so that $\mathfrak{A}_3 < \mathfrak{A}_2$.

In this way we may continue. As the sets $\mathfrak{A}_n$ are obviously complete, $Dv\{\mathfrak{A}_n\}$ contains at least one point α of $\mathfrak{A}$. But at this point each f_m is continuous.

485. 1. *Let $\mathfrak{A} = \mathfrak{B} + \mathfrak{C}$ be complete, let $\mathfrak{B}$, $\mathfrak{C}$ be pantactic with reference to $\mathfrak{A}$. Then there exists no pair of functions f, g defined over $\mathfrak{A}$, such that if $\mathfrak{B}$ are the points of discontinuity of f in $\mathfrak{A}$, then $\mathfrak{B}$ shall be the points of continuity of g in $\mathfrak{A}$.*

This is a corollary of Volterra's theorem. For in any $V_\delta(a)$ of a point of $\mathfrak{A}$, there are points of $\mathfrak{B}$ and of $\mathfrak{C}$. Hence there are points of continuity of f and g. Hence f, g are at most pointwise discontinuous in $\mathfrak{A}$. Then by 484, there is a point in $\mathfrak{A}$ where f and g are both continuous, which contradicts the hypothesis.

2. *Let $\mathfrak{A} = \mathfrak{B} + \mathfrak{C}$ be complete, and let $\mathfrak{B}$, $\mathfrak{C}$ each be pantactic with reference to $\mathfrak{A}$. If $\mathfrak{B}$ is hypercomplete, $\mathfrak{C}$ is not.*

For if $\mathfrak{B}$, $\mathfrak{C}$ were the union of an enumerable set of complete sets, 473 shows that there exists a function f defined over $\mathfrak{A}$ which has $\mathfrak{B}$ as its points of discontinuity ; and also a function g

which has $\mathfrak{E}$ as its points of discontinuity. But no such pair of functions can exist by 1.

3. *The non-rational points $\mathfrak{J}$ in any cube $\mathfrak{Q}$ cannot be hypercomplete.*

For the rational points in $\mathfrak{Q}$ are hypercomplete.

4. As an application of 2 we can state:

The limited function $f(x_1 \cdots x_m)$ which is ≤ 0 at the irrational points of a cube $\mathfrak{Q}$, and > 0 at the other points $\mathfrak{J}$ of $\mathfrak{Q}$, cannot be of class 0 or 1 in $\mathfrak{Q}$.

For if $\text{Cl} f \leq 1$, the points of $\mathfrak{Q}$ where $f > 0$ must form a hypercomplete set, by 479, 4. But these are the points $\mathfrak{J}$.

486. 1. (*Baire.*) *If the class of $f(x_1 \cdots x_m)$ is 1 in the complete set $\mathfrak{A}$, it is at most pointwise discontinuous in any complete $\mathfrak{B} \leq \mathfrak{A}$.*

If $\text{Cl} f = 1$ in $\mathfrak{A}$, it is ≤ 1 in any complete $\mathfrak{B} < \mathfrak{A}$ by 481, 4.; we may therefore take $\mathfrak{B} = \mathfrak{A}$. Let a be any point of $\mathfrak{A}$. We shall show that in any $V = V_\delta(a)$ there is a point c of continuity of f. Let $\epsilon_1 > \epsilon_2 > \cdots = 0$. Using the notation of 479, 1, we saw that the sets $\mathfrak{A}_\iota = (m_\iota < f < m_{\iota+2})$ are hypercomplete. By 473, we can construct a function $\phi_\iota(x_1 \cdots x_m)$, defined over the m-way space $\mathfrak{R}_m$ which is discontinuous at the points $\mathfrak{A}_\iota$, and continuous elsewhere in $\mathfrak{R}_m$. These functions $\phi_1, \phi_2 \cdots$ are not all at most pointwise discontinuous in V. For then, by 484, there exists in V a point of continuity b, common to all the ϕ's. This point b must lie in some $\mathfrak{A}_\iota$, whose points are points of discontinuity of ϕ_ι.

Let us therefore suppose that ϕ_j is not at most pointwise discontinuous in V. Then there exists a point c_1 in V, and an η_1 such that $V_1 = V_{\eta_1}(c_1)$ contains no point of continuity of ϕ_j. Thus $V_1 \leq \mathfrak{A}_j$. But in $\mathfrak{A}_j$ and hence in V_1, $\text{Osc} f < \epsilon_1$. The same reasoning shows that in V_1 there exists a $V_2 = V_{\eta_2}(c_2)$, such that $\text{Osc} f < \epsilon_2$ in V_2. As $\mathfrak{A}$ is complete, $V_1 \geq V_2 \geq \cdots$ defines a point c in V at which f is continuous.

2. *If the class of $f(x_1 \cdots x_m)$ is 1 in the complete set $\mathfrak{A}$, its points of discontinuity $\mathfrak{D}$ form a set of the first category.*

For by 462, 3, the points $\mathfrak{D}_n$ of $\mathfrak{D}$ at which $\mathrm{Osc} f \geq \frac{1}{n}$ form a
mplete set. Each $\mathfrak{D}_n$ is apantactic, since f is at most pointwise
scontinuous, and $\mathfrak{D}_n$ is complete. Hence $\mathfrak{D} = \{\mathfrak{D}_n\}$ is the union
an enumerable set of apantactic sets, and is therefore of the 1°
tegory.

487. 1. *Let f be defined over the limited complete set $\mathfrak{A}$. If
ass f is not ≤ 1, there exists a perfect set $\mathfrak{D}$ in $\mathfrak{A}$, such that f is
tally discontinuous in $\mathfrak{D}$.*

For if $\mathrm{Cl} f$ is not ≤ 1 there exists, by 480, an ϵ such that for
is ϵ, $\mathfrak{A}$ is not an $\mathfrak{E}_\epsilon$ set. Let now e be a point of $\mathfrak{A}$ such that
e points $\mathfrak{a}$ of $\mathfrak{A}$ which lie within some cube $\mathfrak{q}$, whose center is e,
rm an $\mathfrak{E}_\epsilon$ set. Let $\mathfrak{B} = \{\mathfrak{a}\}$, $\mathfrak{E} = \{e\}$.

Then $\mathfrak{B} = \mathfrak{E}$. For obviously $\mathfrak{E} < \mathfrak{B}$, since each e is in some
On the other hand, $\mathfrak{B} \leq \mathfrak{E}$. For any point b of $\mathfrak{B}$ lies within
me $\mathfrak{q}$. Thus b is the center of a cube $\mathfrak{q}'$ within $\mathfrak{q}$. Obviously
e points of $\mathfrak{A}$ within $\mathfrak{q}'$ form an $\mathfrak{E}_\epsilon$ set.

By Borel's theorem, each point e lies within an enumerable set
cubes $\{\mathfrak{c}_n\}$, such that each $\mathfrak{c}$ lies within some $\mathfrak{q}$. Thus the
ints $\mathfrak{a}_n$ of $\mathfrak{A}$ in $\mathfrak{c}_n$, form an $\mathfrak{E}_\epsilon$ set. As $\mathfrak{E} = \{\mathfrak{a}_n\}$, $\mathfrak{E}$ is an $\mathfrak{E}_\epsilon$ set.

Let $\mathfrak{D} = \mathfrak{A} - \mathfrak{E}$. If $\mathfrak{D}$ were 0, $\mathfrak{A} = \mathfrak{E}$ and $\mathfrak{A}$ would be an $\mathfrak{E}_\epsilon$ set
ntrary to hypothesis. Thus $\mathfrak{D} > 0$.

$\mathfrak{D}$ *is complete.* For if l were a limiting point of $\mathfrak{D}$ in $\mathfrak{E}$, l must
in some $\mathfrak{c}$. But every point of $\mathfrak{A}$ in $\mathfrak{c}$ is a point of $\mathfrak{E}$ as we saw.
nus l cannot lie in $\mathfrak{E}$.

We show finally that at any point d of $\mathfrak{D}$,

$$\mathrm{Osc} f \geq \epsilon, \text{ with respect to } \mathfrak{D}.$$

If not, $\mathrm{Osc} f < \epsilon$ with respect to the points $\mathfrak{d}$ of $\mathfrak{D}$ within
me cube $\mathfrak{q}$ whose center is d. Then $\mathfrak{d}$ is an $\mathfrak{E}_\epsilon$ set. Also the
ints $\mathfrak{e}$ of $\mathfrak{E}$ in $\mathfrak{q}$ form an $\mathfrak{E}_\epsilon$ set. Thus the points $\mathfrak{d} + \mathfrak{e}$, that is,
e points of $\mathfrak{A}$ in $\mathfrak{q}$ form an $\mathfrak{E}_\epsilon$ set. Hence d belongs to $\mathfrak{E}$, and
t to $\mathfrak{D}$. As $\mathrm{Osc} f \geq \epsilon$ at each point of $\mathfrak{D}$, each point of $\mathfrak{D}$ is a
int of discontinuity with respect to $\mathfrak{D}$. Thus f is totally discon-
nuous in $\mathfrak{D}$.

This shows that $\mathfrak{D}$ can contain no isolated points. Hence $\mathfrak{D}$ is
rfect.

2. *Let f be defined over the limited complete set $\mathfrak{A}$. If f is at most pointwise discontinuous in any perfect $\mathfrak{B} \leq \mathfrak{A}$, its class is ≤ 1 in $\mathfrak{A}$.*

This is a corollary of 1. For if Class f were not 0, or 1, there exists a perfect set $\mathfrak{D}$ such that f is totally discontinuous in $\mathfrak{D}$.

488. *If the class of $f, g \leq 1$ in the limited complete set $\mathfrak{A}$, the class of their sum, difference, or product is ≤ 1. If $f > 0$ in $\mathfrak{A}$, the class of $\phi = 1/f$ is ≤ 1.*

For example, let us consider the product $h = fg$. If Cl h is not ≤ 1, there exists a perfect set $\mathfrak{D}$ in $\mathfrak{A}$, as we saw in 487, 1, such that h is totally discontinuous in $\mathfrak{D}$. But f, g being of class ≤ 1, are at most pointwise discontinuous in $\mathfrak{D}$ by 486. Then by 484, there exists a point of $\mathfrak{D}$ at which f, g are both continuous. Then h is continuous at this point, and is therefore not totally discontinuous in $\mathfrak{D}$.

Let us consider now the quotient ϕ. If Cl ϕ is not ≤ 1, ϕ is totally discontinuous in some perfect set $\mathfrak{D}$ in $\mathfrak{A}$. But since $f > 0$ in $\mathfrak{D}$, f must also be totally discontinuous in $\mathfrak{D}$. This contradicts 486.

489. 1. *Let $F = \Sigma f_{\iota_1 \cdots \iota_s}(x_1 \cdots x_m)$ converge uniformly in the complete set $\mathfrak{A}$. Let the class of each term f_ι be ≤ 1, then* Class $F \leq 1$ *in $\mathfrak{A}$.*

For setting as usual [117],

$$F = F_\lambda + \overline{F}_\lambda \tag{1}$$

there exists for each $\epsilon > 0$, a fixed rectangular cell R_λ, such that

$$|\overline{F}_\lambda| < \epsilon, \qquad \text{as } x \text{ ranges over } \mathfrak{A}. \tag{2}$$

As the class of each term in F_λ is ≤ 1, Cl $F_\lambda \leq 1$ in $\mathfrak{A}$. Hence $\mathfrak{A}$ is an $\mathfrak{E}_\sigma$ set with respect to F_λ.

From 1), 2) it follows that $\mathfrak{A}$ is an $\mathfrak{E}_\sigma$ set with respect to F.

2. *Let $F = \Pi f_{\iota_1 \cdots \iota_s}(x_1 \cdots x_m)$ converge uniformly in the complete set $\mathfrak{A}$. If the class of each f_ι is ≤ 1, then* Cl $F \leq 1$ *in $\mathfrak{A}$.*

Semicontinuous Functions

490. Let $f(x_1 \cdots x_m)$ be defined over $\mathfrak{A}$. If a is a point of $\mathfrak{A}$, ax f in $V_\delta(a)$ exists, finite or infinite, and may be regarded as a nction of δ. When finite, it is a monotone decreasing function δ. Thus its limit as $\delta \doteq 0$ exists, finite or infinite. We call s limit the *maximum of f at $x = a$*, and we denote it by

$$\operatorname*{Max}_{x=a} f.$$

Similar remarks apply to the minimum of f in $V_\delta(a)$. Its limit, ite or infinite, as $\delta \doteq 0$, we call the *minimum of f at $x = a$*, and denote it by

$$\operatorname*{Min}_{x=a} f.$$

The maximum and minimum of f in $V_\delta(a)$ may be denoted by

$$\operatorname*{Max}_{a,\,\delta} f \quad , \quad \operatorname*{Min}_{a,\,\delta} f.$$

Obviously,
$$\operatorname*{Max}_{x=a} (-f) = -\operatorname*{Min}_{x=a} f,$$

$$\operatorname*{Min}_{x=a} (-f) = -\operatorname*{Max}_{x=a} f.$$

91. *Example 1.*
$$f(x) = \frac{1}{x} \text{ in } (-1, 1) \quad , \quad \text{for } x \neq 0$$
$$= 0 \quad , \quad \text{for } x = 0.$$

Then
$$\operatorname*{Max}_{x=0} f = +\infty \quad , \quad \operatorname*{Min}_{x=0} f = -\infty.$$

Example 2.
$$f(x) = \sin\frac{1}{x} \text{ in } (-1, 1) \quad , \quad \text{for } x \neq 0$$
$$= 0 \quad , \quad \text{for } x = 0.$$

Then
$$\operatorname*{Max}_{x=0} f = 1 \quad , \quad \operatorname*{Min}_{x=0} f = -1.$$

Example 3.
$$f(x) = 1 \text{ in } (-1, 1) \quad , \quad \text{for } x \neq 0$$
$$= 2 \quad , \quad \text{for } x = 0.$$

Then
$$\operatorname*{Max}_{x=0} f = 2 \quad , \quad \operatorname*{Min}_{x=0} f = 1.$$

We observe that in Exs. 1 and 2,

$$\overline{\lim_{x=0}} f = \operatorname*{Max}_{x=0} f \quad , \quad \underline{\lim_{x=0}} f = \operatorname*{Min}_{x=0} f ;$$

while in Ex. 3,

$$\overline{\lim_{x=0}} f = 1 \quad , \quad \text{and hence } \operatorname*{Max}_{x=0} f > \overline{\lim_{x=0}} f.$$

Also

$$\underline{\lim_{x=0}} f = \operatorname*{Min}_{x=0} f.$$

Example 4.

$$f(x) = (x^2+1)\sin\frac{1}{x} \text{ in } (-1, 1) \quad , \quad \text{for } x \neq 0$$

$$= -2 \quad , \quad \text{for } x = 0.$$

Here

$$\operatorname*{Max}_{x=0} f = 1 \quad , \quad \operatorname*{Min}_{x=0} f = -2,$$

$$\overline{\lim_{x=0}} f = 1 \quad , \quad \underline{\lim_{x=0}} f = -1.$$

Example 5. Let

$$f(x) = x \quad , \quad \text{for rational } x \text{ in } (0, 1)$$

$$= 1 \quad , \quad \text{for irrational } x.$$

Here

$$\operatorname*{Max}_{x=0} f = 1 \quad , \quad \operatorname*{Min}_{x=0} f = 0,$$

$$\overline{\lim_{x=0}} f = 1.$$

492. 1. *For M to be the maximum of f at $x = a$, it is necessary and sufficient that*

1° $\epsilon > 0, \quad \delta > 0, \quad f(x) < M + \epsilon,$ *for any x in $V_\delta(a)$;*

2° *there exists for each $\epsilon > 0$, and in any $V_\delta(a)$, a point α such that*

$$M - \epsilon < f(\alpha).$$

These conditions are *necessary*. For M is the limit of $\operatorname{Max} f$ in $V_\delta(a)$, as $\delta \doteq 0$. Hence

$$\epsilon > 0, \qquad \delta > 0, \qquad \operatorname*{Max}_{a, \delta} f < M + \epsilon.$$

But for any x in $V_\delta(a)$,

$$f(x) \leq \operatorname*{Max}_{a, \delta} f.$$

Hence $$f(x) < M + \epsilon \quad , \quad x \text{ in } V_\delta(a),$$
hich is condition 1°.

As to 2°, we remark that for each $\epsilon > 0$, and in any $V_\delta(a)$, ere is a point α, such that

$$-\epsilon + \operatorname*{Max}_{a, \delta} f < f(\alpha).$$

But
$$M \leq \operatorname*{Max}_{a, \delta} f.$$

Hence
$$-\epsilon + M < f(\alpha),$$
hich is 2°.

hese conditions are *sufficient*. For from 1° we have

$$\operatorname*{Max}_{a, \delta} f \leq M + \epsilon,$$

d hence letting $\delta \doteq 0$,
$$\operatorname*{Max}_{x=a} f \leq M, \tag{1}$$

nce $\epsilon > 0$ is small at pleasure.

From 2° we have
$$\operatorname*{Max}_{a, \delta} f \geq M - \epsilon,$$

d hence letting $\delta \doteq 0$,
$$\operatorname*{Max}_{x=a} f \geq M. \tag{2}$$

From 1), 2) we have $M = \operatorname*{Max}_{x=a} f$.

2. *For m to be the minimum of f at $x = a$, it is necessary and fficient that*

1° $\epsilon > 0, \qquad \delta > 0, \qquad m - \epsilon < f(x),$ *for any x in $V_\delta(a)$;*

2° *that there exists for each $\epsilon > 0$, and in any $V_\delta(a)$, a point α ch that*

$$f(\alpha) < m + \epsilon.$$

493. When $\operatorname*{Max}_{x=a} f = f(a)$, we say f is *supracontinuous* at $x = a$. hen $\operatorname*{Min}_{x=a} f = f(a)$, we say f is *infracontinuous* at a. When f is pra (infra) continuous at each point of $\mathfrak{A}$, we say f is supra nfra) continuous in $\mathfrak{A}$. When f is either supra or infracontinu- s at a and we do not care to specify which, we say it is *semi- ntinuous* at a.

The function which is equal to Max f at each point x of $\mathfrak{A}$ we call the *maximal* function of f, and denote it by a dash above, *viz.* $\overline{f}(x)$. Similarly the *minimal* function $\underline{f}(x)$ is defined as the value of Min f at each point of $\mathfrak{A}$.

Obviously
$$\operatorname*{Osc}_{x=a} f = \operatorname*{Max}_{x=a} f - \operatorname*{Min}_{x=a} f = \operatorname*{Disc}_{x=a} f.$$

We call
$$\omega(x) = \overline{f}(x) - \underline{f}(x)$$
the *oscillatory function.*

We have at once the theorem:

For f to be continuous at $x = a$, it is necessary and sufficient that
$$f(a) = \overline{f}(a) = \underline{f}(a).$$

For
$$\operatorname*{Min}_{a,\,\delta} f \leq f(a) \leq \operatorname*{Max}_{a,\,\delta} f.$$

Passing to the limit $x = a$, we have
$$\operatorname*{Min}_{x=a} f \leq f(a) \leq \operatorname*{Max}_{x=a} f,$$
or
$$\underline{f}(a) \leq f(a) \leq \overline{f}(a).$$

But for f to be continuous at $x = a$, it is necessary and sufficient that
$$\omega(a) = \operatorname*{Osc}_{x=a} f = 0.$$

494. 1. *For f to be supracontinuous at $x = a$, it is necessary and sufficient that for each $\epsilon > 0$, there exists a $\delta > 0$, such that*
$$f(x) < f(a) + \epsilon \quad , \quad \textit{for any } x \textit{ in } V_\delta(a). \tag{1}$$

Similarly the condition for infracontinuity is
$$f(a) - \epsilon < f(x) \quad , \quad \textit{for any } x \textit{ in } V_\delta(a). \tag{2}$$

Let us prove 1). It is *necessary*. For when f is supracontinuous at a,
$$f(a) = \operatorname*{Max}_{x=a} f(x).$$

Then by 492, 1,
$$\epsilon > 0 \quad , \quad \delta > 0 \quad , \quad f(x) < f(a) + \epsilon \quad , \quad \text{for any } x \text{ in } V_\delta(a),$$
which is 1).

It is *sufficient*. For 1) is condition 1° of 492, 1. The condition 2° is satisfied, since for α we may take the point a.

2. *The maximal function $\overline{f}(x)$ is supracontinuous; the minimal function $\underline{f}(x)$ is infracontinuous, in $\mathfrak{A}$.*

To prove that $\overline{f}$ is supracontinuous we use 1, showing that

$$\overline{f}(x) < \overline{f}(a) + \epsilon \quad , \quad \text{for any } x \text{ in some } V_\delta(a).$$

Now by 492, 1,

$$\epsilon' > 0,\ \delta > 0 \quad , \quad f(x) < \overline{f}(a) + \epsilon' \quad , \quad \text{for any } x \text{ in } V_\delta(a).$$

Thus if $\epsilon' < \epsilon$

$$\overline{f}(x) < \overline{f}(a) + \epsilon \quad , \quad \text{for any } x \text{ in } V_\eta(a) \quad , \quad \eta = \frac{\delta}{2}.$$

3. *The sum of two supra (infra) continuous functions in $\mathfrak{A}$ is a supra (infra) continuous function in $\mathfrak{A}$.*

For let f, g be supracontinuous in $\mathfrak{A}$; let $f + g = h$. Then by 1,

$$f(x) < f(a) + \frac{\epsilon}{2},$$

$$g(x) < g(a) + \frac{\epsilon}{2},$$

for any x in some $V_\delta(a)$; hence

$$h(x) < h(a) + \epsilon.$$

This, by 1, shows that h is supracontinuous at $x = a$.

4. *If $f(x)$ is supra (infra) continuous at $x = a$, $g(x) = -f(x)$ is infra (supra) continuous.*

Let us suppose that f is supracontinuous. Then by 1,

$$f(x) < f(a) + \epsilon \quad , \quad \text{for any } x \text{ in some } V_\delta(a).$$

Hence

$$-f(a) - \epsilon < -f(x),$$

or

$$g(a) - \epsilon < g(x) \quad , \quad \text{for any } x \text{ in } V_\delta(a).$$

Thus by 1, g is infracontinuous at a.

495. *If $f(x_1 \cdots x_m)$ is supracontinuous in the limited comple set $\mathfrak{A}$, the points $\mathfrak{B}$ of $\mathfrak{A}$ at which $f \geq c$ an arbitrary constant form complete set.*

For let $f \geq c$ at $b_1, b_2 \cdots$ which $\doteq b$; we wish to show that b li in $\mathfrak{B}$.

Since f is supracontinuous, by 494, 1,

$$f(x) < f(b) + \epsilon \quad , \quad \text{for any } x \text{ in some } V_\delta(b) = V.$$

But $c \leq f(b_n)$, by hypothesis; and b_n lies in V, for $n >$ some r Hence

$$c \leq f(b_n) < f(b) + \epsilon,$$

or

$$c - \epsilon < f(b).$$

As $\epsilon > 0$ is small at pleasure,

$$f(b) \geq c,$$

and b lies in $\mathfrak{B}$.

496. 1. *The oscillatory function $\omega(x)$ is supracontinuous.*

For by 493,
$$\omega(x) = \mathrm{Max}\, f - \mathrm{Min}\, f$$
$$= \mathrm{Max}\, f + \mathrm{Max}\, (-f).$$

But these two maximal functions are supracontinuous by 494, Hence by 494, 3, their sum ω is supracontinuous.

2. The oscillatory function ω is not necessarily infracon tinuous, as is shown by the following

Example. $f = 1$ in $(-1, 1)$, except for $x = 0$, where $f =$ 2 Then $\omega(x) = 0$, except at $x = 0$, where $\omega = 1$. Thus

$$\underset{x=0}{\mathrm{Min}}\, \omega(x) = 0 \quad , \quad \text{while } \omega(0) = 1.$$

Hence $\omega(x)$ is not infracontinuous at $x = 0$.

3. *Let $\omega(x)$ be the oscillatory function of $f(x_1 \cdots x_m)$ in $\mathfrak{A}$. Fo f to be at most pointwise discontinuous in $\mathfrak{A}$, it is necessary tha $\mathrm{Min}\, \omega = 0$ at each point of $\mathfrak{A}$. If $\mathfrak{A}$ is complete, this condition sufficient.*

It is *necessary*. For let a be a point of $\mathfrak{A}$. As f is at m
pointwise discontinuous, there exists a point of continuity in a
$V_\delta(a)$. Hence $\mathrm{Min}\, \omega(x) = 0$, in $V_\delta(a)$. Hence $\underset{x=a}{\mathrm{Min}}\, \omega(x) = 0.$

It is *sufficient*. For let $\epsilon_1 > \epsilon_2 > \cdots \doteq 0$. Since $\underset{x=a}{\mathrm{Min}}\, \omega(x) =$
there exists in any $V_\delta(a)$ a point α_1 such that $\omega(\alpha_1) < \frac{1}{2}$
Hence $\omega(x) < \epsilon_1$ in some $V_{\delta_1}(\alpha_1) < V_\delta$. In V_{δ_1} there exists a po
α_2 such that $\omega(x) < \epsilon_2$ in some $V_{\delta_2}(\alpha) < V_{\delta_1}$, etc. Since $\mathfrak{A}$ is co
plete and since we may let $\delta_n \doteq 0$,

$$V_{\delta_1} > V_{\delta_2} > \cdots \doteq \text{a point } \alpha \text{ of } \mathfrak{A},$$

at which f is obviously continuous. Thus in each $V_\delta(a)$ is a pc
of continuity of f. Hence f is at most pointwise discontinuous

497. 1. *At each point x of $\mathfrak{A}$,*

$$\phi = \mathrm{Min}\, \{\overline{f}(x) - f(x)\}, \text{ and } \psi = \mathrm{Min}\, \{f(x) - \underline{f}(x)\}$$

are both $= 0$.

Let us show that $\phi = 0$ at an arbitrary point a of $\mathfrak{A}$. By 4
2, $\overline{f}(x)$ is supracontinuous; hence by 494, 1,

$$\overline{f}(x) < \overline{f}(a) + \epsilon \quad , \quad \text{for any } x \text{ in some } V_\delta(a) = V.$$

Also there exists a point α in V such that

$$-\epsilon + \overline{f}(a) < f(\alpha).$$

Also by definition

$$f(\alpha) \leq \overline{f}(\alpha).$$

If in 1) we replace x by α we get

$$\overline{f}(\alpha) < \overline{f}(a) + \epsilon.$$

From 2), 3), 4) we have

$$-\epsilon + \overline{f}(a) < f(\alpha) \leq \overline{f}(\alpha) < \overline{f}(a) + \epsilon,$$

or

$$0 \leq \overline{f}(\alpha) - f(\alpha) < 2\,\epsilon.$$

As $\epsilon > 0$ is small at pleasure, this gives

$$\phi(a) = 0.$$

2. *If f is semicontinuous in the complete set $\mathfrak{A}$, it is at most pointwise discontinuous in $\mathfrak{A}$.*

For

$$\begin{aligned}\omega(x) &= \overline{f}(x) - \underline{f}(x)\\ &= [\overline{f}(x) - f(x)] + [f(x) - \underline{f}(x)] \quad (1\\ &= \phi(x) + \psi(x).\end{aligned}$$

To fix the ideas let f be supracontinuous. Then $\phi = 0$ in $\mathfrak{A}$. Hence 1) gives

$$\text{Min}\ \omega(x) = \text{Min}\ \psi(x) = 0, \qquad \text{by 1.}$$

Thus by 496, 3, f is at most pointwise discontinuous in $\mathfrak{A}$.

CHAPTER XV

DERIVATES, EXTREMES, VARIATION

Derivates

498. Suppose we have given a one-valued continuous function $f(x)$ spread over an interval $\mathfrak{A} = (a < b)$. We can state various properties which it enjoys. For example, it is limited, it takes on its extreme values, it is integrable. On the other hand, we do not know 1° how it oscillates in $\mathfrak{A}$, or 2° if it has a differential coefficient at each point of $\mathfrak{A}$. In this chapter we wish to study the behavior of continuous functions with reference to these last two properties. In Chapters VIII and XI of volume I this subject was touched upon; we wish here to develop it farther.

499. In I, 363, 364, we have defined the terms difference quotient, differential coefficient, derivative, right- and left-hand differential coefficients and derivatives, unilateral differential coefficients and derivatives. The corresponding symbols are

$$\frac{\Delta f}{\Delta x} \quad , \quad f'(a) \quad , \quad f'(x) \quad , \quad Rf'(a) \quad ,$$

$$Lf'(a) \quad , \quad Rf'(x) \quad , \quad Lf'(x).$$

The unilateral differential coefficient and derivative may be denoted by

$$Uf'(a) \quad , \quad Uf'(x). \tag{1}$$

When

$$\lim_{h=0} \frac{\Delta f}{\Delta x}$$

does not exist, finite or infinite, we may introduce its upper and lower limits. Thus

$$\overline{f}'(a) = \overline{\lim_{h=0}} \frac{\Delta f}{\Delta x} \quad , \quad \underline{f}'(a) = \underline{\lim_{h=0}} \frac{\Delta f}{\Delta x} \tag{2}$$

always exist, finite or infinite. We call them the upper and lower differential coefficients at the point $x = a$. The aggregate of values

that 2) take on define the upper and lower derivatives of $f(x)$, as in I, 363.

In a similar manner we introduce the upper and lower right- and left-hand differential coefficients and derivatives,

$$\overline{R}f' \quad , \quad \underline{R}f' \quad , \quad \overline{L}f' \quad , \quad \underline{L}f'. \tag{3}$$

Thus, for example,

$$\overline{R}f'(a) = R\overline{\lim_{h=0}}\frac{f(a+h)-f(a)}{h},$$

finite or infinite. Cf. I, 336 *seq.*

If $f(x)$ is defined *only* in $\mathfrak{A} = (\alpha < \beta)$, the points a, $a+h$ must lie in $\mathfrak{A}$. Thus there is no upper or lower right-hand differential coefficient at $x = \beta$; also no upper or lower left-hand differential coefficient at $x = \alpha$. This fact must be borne in mind. We call the functions 3) *derivates* to distinguish them from the derivatives Rf', Lf'. When $\overline{R}f'(a) = \underline{R}f'(a)$, finite or infinite, $Rf'(a)$ exists also finite or infinite, and has the same value. A similar remark applies to the left-hand differential coefficient.

To avoid such repetition as just made, it is convenient to introduce the terms upper and lower unilateral differential coefficients and derivatives, which may be denoted by

$$\overline{U}f' \quad , \quad \underline{U}f'. \tag{4}$$

The symbol U should of course refer to the same side, if it is used more than once in an investigation.

When no ambiguity can arise, we may abbreviate the symbols 3), 4) thus:

$$\overline{R} \quad , \quad \underline{R} \quad , \quad \overline{L} \quad , \quad \underline{L} \quad , \quad \overline{U} \quad , \quad \underline{U}.$$

The value of one of these derivates as $\overline{R}$ at a point $x = \alpha$ may similarly be denoted by

$$\overline{R}(\alpha).$$

The difference quotient

$$\frac{f(a)-f(b)}{a-b}$$

may be denoted by

$$\Delta(a, b).$$

Example 1.
$$f(x) = x \sin \frac{1}{x} \quad , \quad x \neq 0 \text{ in } (-1, 1)$$
$$= 0 \quad , \quad x = 0.$$

Here for $x = 0$, $\displaystyle \frac{\Delta f}{\Delta x} = \frac{h \sin \frac{1}{h}}{h} = \sin \frac{1}{h}.$

Hence
$$\overline{R}f'(0) = +1 \quad , \quad \underline{R}f'(0) = -1,$$
$$\overline{L}f'(0) = +1 \quad , \quad \underline{L}f'(0) = -1,$$
$$\overline{f}'(0) = +1 \quad , \quad \underline{f}'(0) = -1.$$

Example 2.
$$f(x) = x^{\frac{1}{3}} \sin \frac{1}{x} \quad , \quad x \neq 0 \text{ in } (-1, 1)$$
$$= 0 \quad , \quad x = 0.$$

Here for $x = 0$, $\displaystyle \frac{\Delta f}{\Delta x} = \frac{\sin \frac{1}{h}}{h^{\frac{2}{3}}}.$

Hence
$$\overline{R}f'(0) = +\infty \quad , \quad \underline{R}f'(0) = -\infty,$$
$$\overline{L}f'(0) = +\infty \quad , \quad \underline{L}f'(0) = -\infty,$$
$$\overline{f}'(0) = +\infty \quad , \quad \underline{f}'(0) = -\infty.$$

Example 3.
$$f(x) = x \sin \frac{1}{x} \quad , \quad \text{for } 0 < x \leq 1$$
$$= x^{\frac{1}{3}} \sin \frac{1}{x} \quad , \quad \text{for } -1 \leq x < 0$$
$$= 0 \quad , \quad \text{for } x = 0.$$

Here
$$\overline{R}f'(0) = +1 \quad , \quad \underline{R}f'(0) = -1,$$
$$\overline{L}f'(0) = +\infty \quad , \quad \underline{L}f'(0) = -\infty,$$
$$\overline{f}'(0) = +\infty \quad , \quad \underline{f}'(0) = -\infty.$$

500. 1. Before taking up the general theory it will be well for the reader to have a few examples in mind to show him how complicated matters may get. In I, 367 *seq.*, we have exhibited functions which oscillate infinitely often about the points of a set

of the 1° species, and which may or may not have differential coefficients at these points.

The following theorem enables us to construct functions which do not possess a differential coefficient at the points of an enumerable set.

2. *Let $\mathfrak{E} = \{e_n\}$ be an enumerable set lying in the interval $\mathfrak{A}$. For each x in $\mathfrak{A}$, and e_n in $\mathfrak{E}$, let $x - e_n$ lie in an interval $\mathfrak{B}$ containing the origin. Let $g(x)$ be continuous in $\mathfrak{B}$. Let $g'(x)$ exist and be numerically $\leq M$ in $\mathfrak{B}$, except at $x = 0$, where the difference quotients are numerically $\leq M$. Let $A = \Sigma a_n$ converge absolutely. Then*

$$F(x) = \Sigma a_n g(x - e_n)$$

is a continuous function in $\mathfrak{A}$, having a derivative in $\mathfrak{C} = \mathfrak{A} - \mathfrak{E}$. At the points of $\mathfrak{E}$, the difference quotient of F behaves essentially as that of g at the origin.

For $g(x)$ being continuous in $\mathfrak{B}$, it is numerically $<$ some constant in $\mathfrak{A}$. Thus F converges uniformly in $\mathfrak{A}$. As each term $g(x - e_n)$ is continuous in $\mathfrak{A}$, F is continuous in $\mathfrak{A}$.

Let us consider its differential coefficient at a point x of $\mathfrak{C}$. Since $g'(x - e_n)$ exists and is numerically $\leq M$,

$$F'(x) = \Sigma a_n g'(x - e_n) \quad , \quad \text{by 156, 2.}$$

Let now $x = e_m$, a point of $\mathfrak{E}$,

$$F(x) = a_m g(x - e_m) + \Sigma^* a_n g(x - e_n)$$

$$= a_m g(x - e_m) + G(x).$$

The summation in Σ^* extends over all $n \neq m$. Hence by what has just been shown, G has a differential coefficient at $x = e_m$. Thus $\dfrac{\Delta F}{\Delta x}$ behaves at $x = e_m$, essentially as $\dfrac{\Delta g}{\Delta x}$ at $x = 0$. Hence

$$\overline{\underline{U}}F'(e_m) = a_m \overline{\underline{U}}g'(0) + G'(e_m). \tag{1}$$

501. *Example 1.* Let

$$\begin{aligned} g(x) &= ax \quad , \quad x \geq 0 \\ &= bx \quad , \quad x < 0, \end{aligned} \qquad b < 0 < a.$$

Then

$$F(x) = \sum \frac{1}{n^2} g(x - e_n)$$

continuous in any interval $\mathfrak{A}$, and has a derivative

$$F'(x) = \sum \frac{1}{n^2} g'(x - e_n)$$

the points of $\mathfrak{A}$ not in $\mathfrak{E}$. At the point e_m,

$$RF'(x) = a_m a + \sum{}^* \frac{1}{m^2} g'(e_m - e_n),$$

$$LF'(x) = a_m b + \sum{}^* \frac{1}{m^2} g'(e_m - e_n).$$

Let $\mathfrak{E}$ denote the rational points in $\mathfrak{A}$. The graph of $F(x)$ is a ntinuous curve having tangents at a pantactic set of points; d at another pantactic set, viz. the set $\mathfrak{E}$, angular points (I, 366).

A simple example of a g function is

$$g(x) = |x| = +\sqrt{x^2}.$$

Example 2. Let $g(x) = x^2 \sin \frac{\pi}{x} \quad , \quad x \neq 0$

$$= 0 \quad , \quad x = 0.$$

This function has a derivative

$$g'(x) = 2x \sin \frac{\pi}{x} - \pi \cos \frac{\pi}{x} \quad , \quad x \neq 0$$

$$= 0 \quad , \quad x = 0.$$

Thus if Σc_n is an absolutely convergent series, and $\mathfrak{E} = \{e_n\}$ an umerable set in the interval $\mathfrak{A} = (0, 1)$,

$$F(x) = \Sigma c_n g(x - e_n)$$

a continuous function whose derivative in $\mathfrak{A}$ is

$$F'(x) = \Sigma c_n g'(x - e_n).$$

Thus F has a derivative which is continuous in $\mathfrak{A} - \mathfrak{E}$, and at e point $x = e_m$

$$\text{Disc } F' = 2\, c_m \pi,$$

ce

$$\underset{x=0}{\text{Disc}}\ g'(x) = 2\,\pi.$$

If $\mathfrak{E}$ is the set of rational points in $\mathfrak{A}$, the graph of $F(x)$ is a continuous curve having at each point of $\mathfrak{A}$ a tangent which does not turn continuously as the point of contact ranges over the curve; indeed the points of abrupt change in the direction of the tangent are pantactic in $\mathfrak{A}$.

Example 3. Let
$$g(x) = x \sin \log x^2 \quad , \quad x \neq 0$$
$$= 0 \quad , \quad x = 0.$$

Then
$$g'(x) = \sin \log x^2 + 2 \cos \log x^2 \quad , \quad x \neq 0.$$

At $x = 0$,
$$\frac{\Delta g}{\Delta x} = \sin \log h^2$$

which oscillates infinitely often between ± 1, as $h = \Delta x \doteq 0$. Let $\mathfrak{E} = \{e_n\}$ denote the rational points in an interval $\mathfrak{A}$. The series

$$F = \sum \frac{1}{n^2}(x - e_n) \sin \log (x - e_n)^2$$

satisfies the condition of our theorem. Hence $F(x)$ is a continuous function in $\mathfrak{A}$ which has a derivative in $\mathfrak{A} - \mathfrak{E}$. At $x = e_m$,

$$\overline{U}F'(x) = \frac{1}{m^2} + G'(e_m) \quad , \quad \underline{U}F'(x) = -\frac{1}{m^2} + G'(e_m).$$

Thus the graph of F is a continuous curve which has tangents at a pantactic set of points in $\mathfrak{A}$, and at another pantactic set it has neither right- nor left-hand tangents.

502. *Weierstrass' Function.* For a long time mathematicians thought that a continuous function of x must have a derivative, at least after removing certain points. The examples just given show that these exceptional points may be pantactic. Weierstrass called attention to a continuous function which has at no point a differential coefficient. This celebrated function is defined by the series

$$F(x) = \sum_0 a^n \cos b^n \pi x = \cos \pi x + a \cos b\pi x + a^2 \cos b^2 \pi x + \cdots \quad (1$$

where $0 < a < 1$; b is an odd integer so chosen that

$$ab > 1 + \tfrac{3}{2}\pi. \quad (2$$

The series F converges absolutely and uniformly in any interval $\mathfrak{A}$, since

$$|a^n \cos b^n \pi x| \leq a^n.$$

Hence F is a continuous function in $\mathfrak{A}$. Let us now consider the series obtained by differentiating 1) termwise,

$$G(x) = -\pi\Sigma(ab)^n \sin b^n \pi x.$$

If $ab < 1$, this series also converges absolutely and uniformly, and

$$F'(x) = G(x),$$

by 155, 1. In this case the function has a finite derivative in $\mathfrak{A}$. Let us suppose, however, that the condition 2) holds. We have

$$\frac{\Delta F}{\Delta x} = Q = \sum_0^\infty \frac{a^n}{h}\{\cos b^n \pi(x+h) - \cos b^n \pi x\} = Q_m + \overline{Q}_m. \qquad (3$$

Now

$$Q_m = \sum_0^{m-1} \frac{a^n}{h}\{\cos b^n \pi(x+h) - \cos b^n \pi x\}$$

$$= -\pi \sum_0^{m-1} \frac{(ab)^n}{h} \int_x^{x+h} \sin b^n \pi u du.$$

Since

$$\left|\int_x^{x+h} \sin b^n \pi u du\right| < \left|\int_x^{x+h} du\right| = |h|,$$

$$|Q_m| < \pi \sum_0^{m-1} (ab)^n = \pi \frac{1-(ab)^m}{1-ab} < \pi \frac{(ab)^m}{ab-1}, \text{ if } ab > 1.$$

Consider now

$$\overline{Q}_m = \sum_m^\infty \frac{a^n}{h}\{\cos b^n \pi(x+h) - \cos b^n \pi x\}.$$

Up to the present we have taken h arbitrary. Let us now take it as follows; the reason for this choice will be evident in a moment.

Let

$$b^m x = \iota_m + \xi_m,$$

where ι_m is the nearest integer to $b^m x$. Thus

$$-\tfrac{1}{2} \leq \xi_m \leq \tfrac{1}{2}.$$

Then

$$b^m(x+h) = \iota_m + \xi_m + hb^m = \iota_m + \eta_m.$$

We choose h so that

$$\eta_m = \xi_m + hb^m \text{ is } \pm 1, \text{ at pleasure.}$$

Then

$$h = \frac{\eta_m - \xi_m}{b^m} \doteq 0, \text{ as } m \doteq \infty;$$

moreover

$$\operatorname{sgn} h = \operatorname{sgn} \eta_m \quad , \quad \text{and } |\eta_m - \xi_m| \leq \tfrac{3}{2}.$$

This established, we note that

$$\cos b^n\pi(x+h) = \cos b^{n-m}\pi \cdot b^m(x+h) = \cos b^{n-m}(\iota_m + \eta_m)\pi$$

$$= \cos(\iota_m + \eta_m)\pi \quad , \quad \text{since } b \text{ is odd}$$

$$= (-1)^{\iota_m+1} \quad , \quad \text{since } \eta_m \text{ is odd.}$$

Also

$$\cos b^n\pi x = \cos b^{n-m}(\iota_m + \xi_m)\pi$$

$$= (-1)^{\iota_m} \cos b^{n-m}\xi_m\pi.$$

Thus

$$\overline{Q}_m = e_m \sum_m^\infty \frac{a^n}{h}\{1 + \cos b^{n-m}\xi_m\pi\},$$

where

$$e_m = (-1)^{\iota_m+1}.$$

Now each $\{\ \} \geq 0$ and in particular the first is > 0. Thus

$$\operatorname{sgn} \overline{Q}_m = \operatorname{sgn} \frac{e_m}{h} = \operatorname{sgn} e_m\eta_m,$$

and

$$|\overline{Q}_m| > \frac{a^m}{h} = \frac{(ab)^m}{\eta_m - \xi_m} \geq \tfrac{2}{3}(ab)^m.$$

Thus if 2) holds, $|\overline{Q}_m| > |Q_m|$. Hence from 3),

$$\operatorname{sgn} Q = \operatorname{sgn} \overline{Q}_m = \operatorname{sgn} e_m\eta_m,$$

and

$$|Q| > (ab)^m\left(\frac{2}{3} - \frac{\pi}{ab-1}\right).$$

Let now $m \doteq \infty$. Since $\eta_m = \pm 1$ at pleasure, we can make $Q \doteq +\infty$, or to $-\infty$, or oscillate between $\pm\infty$, without becoming definitely infinite. *Thus $F(x)$ has at no point a finite or infinite differential coefficient. This does not say that the graph of F does not have tangents; but when they exist, they must be cuspidal tangents.*

503. 1. *Volterra's Function.*

In the interval $\mathfrak{A} = (0, 1)$, let $\mathfrak{H} = \{\eta\}$ be a Harnack set of measure $0 < h < 1$. Let $\Delta = \{\delta_n\}$ be the associate set of black intervals. In each of the intervals $\delta_n = (\alpha < \beta)$, we define an auxiliary function f_n as follows:

$$f_n(x) = (x - \alpha)^2 \sin \frac{1}{x - \alpha} \quad , \quad \text{in } (\alpha^*, \gamma), \tag{1}$$

where γ is the largest value of x corresponding to a maximum of the function on the right of 1), such that γ lies to the left of the middle point μ of δ_n. If the value of $f_n(x)$ at γ is g, we now make

$$f_n(x) = g \quad , \quad \text{in } (\gamma, \mu).$$

Finally $f_n(\alpha) = 0$. This defines $f_n(x)$ for one half of the interval δ_n. We define $f_n(x)$ for the other half of δ_n by saying that if $x < x'$ are two points of δ_n at equal distances from the middle point μ, then

$$f_n(x) = f_n(x').$$

With Volterra we now define a function $f(x)$ in $\mathfrak{A}$ as follows:

$$\begin{aligned} f(x) &= f_n(x) \quad , \quad \text{in } \delta_n \quad , \quad n = 1, 2, \cdots \\ &= 0 \quad , \quad \text{in } \mathfrak{H}. \end{aligned}$$

Obviously $f(x)$ is continuous in $\mathfrak{A}$.

At a point x of $\mathfrak{A}$ not in $\mathfrak{H}$, $f'(x)$ behaves as

$$2\, x \sin \frac{1}{x} - \cos \frac{1}{x},$$

as is seen from 1). Thus as x converges in δ_n toward one of its end points α, β, we see that $f'(x)$ oscillates infinitely often between limits which $\doteq \pm 1$. Thus

$$R \varlimsup_{x=\alpha} f'(x) = +1 \quad , \quad R \varliminf_{x=\alpha} f'(x) = -1;$$

similar limits exist for the points β.

Let us now consider the differential coefficient at a point η of $\mathfrak{H}$. We have

$$\frac{\Delta f}{\Delta x} = \frac{f(\eta + k) - f(\eta)}{k} = \frac{f(\eta + k)}{k} \quad , \quad \text{since } f(\eta) = 0.$$

If $\eta + k$ is a point of $\mathfrak{H}$, $f(\eta + k) = 0$. If not, $\eta + k$ lies in some interval δ_m. Let $x = e$ be the end point of δ_m nearest $\eta + k$. Then

$$\left|\frac{\Delta f}{\Delta x}\right| \leq \frac{|\eta + k - e|^2}{|k|} \leq |k| \doteq 0 \quad , \quad \text{as } k \doteq 0.$$

Thus $f'(\eta) = 0$. Hence Volterra's function $f(x)$ has a differential coefficient at each point of $\mathfrak{A}$; moreover $f'(x)$ is limited in $\mathfrak{A}$. Each point η of $\mathfrak{H}$ is a point of discontinuity of $f'(x)$, and

$$\underset{x=\eta}{\mathrm{Disc}} f'(x) \geq 2.$$

Hence $f'(x)$ is not R-integrable, as $\overline{\overline{\mathfrak{H}}} = h > 0$.

We have seen, in I, 549, that not every limited R-integrable function has a primitive. Volterra's function illustrates conversely the remarkable fact that *Not every limited derivative is R-integrable.*

2. It is easy to show, however, that *The derivative of Volterra's function is L-integrable.*

For let $\mathfrak{A}_\lambda$ denote the points of $\mathfrak{A}$ at which $f'(x) \geq \lambda$. Then when $\lambda > 1/m$, $m = 1, 2, \cdots$ $\mathfrak{A}_\lambda$ consists of an enumerable set of intervals. Hence in this case $\mathfrak{A}_\lambda$ is measurable. Hence $\mathfrak{A}_\lambda$, $\lambda > 0$, is measurable. Now $\mathfrak{A}$, $\lambda \geq 0$, differs from the foregoing by adding the points $\mathfrak{Z}_n$ in each δ_n at which $f'(x) = 0$, and the points $\mathfrak{H}$. But each $\mathfrak{Z}_n$ is enumerable, and hence a null set, and $\mathfrak{H}$ is measurable, as it is perfect. Thus $\mathfrak{A}_\lambda$, $\lambda \geq 0$, is measurable. In the same way we see $\mathfrak{A}_\lambda$ is measurable when λ is negative. Thus $\mathfrak{A}_\lambda$ is measurable for any λ, and hence L-integrable.

504. 1. We turn now to general considerations and begin by considering the upper and lower limits of the sum, difference, product, and quotient of two functions at a point $x = a$.

Let us note first the following theorem :

Let $f(x_1 \cdots x_m)$ be limited or not in $\mathfrak{A}$ which has $x = a$ as a limiting point. Let $\Phi_\delta = \mathrm{Max}\, f$, $\phi_\delta = \mathrm{Min}\, f$ in $V_\delta^*(a)$. Then

$$\underline{\lim_{x=a}} f = \lim_{\delta=0} \phi_\delta \quad , \quad \overline{\lim_{x=a}} f = \lim_{\delta=0} \Phi_\delta.$$

This follows at once from I, 338.

2. *Let $f(x_1 \cdots x_m)$, $g(x_1 \cdots x_m)$ be limited or not in $\mathfrak{A}$ which has $x = a$ as limiting point.*

Let

$$\underline{\lim}\, f = \alpha \quad , \quad \underline{\lim}\, g = \beta$$

$$\overline{\lim}\, f = A \quad , \quad \overline{\lim}\, g = B$$

as $x \doteq a$. Then, these limits being finite,

$$\alpha + \beta \leq \overline{\underline{\lim}}\, (f + g) \leq A + B, \tag{1}$$

$$\alpha - B \leq \overline{\underline{\lim}}\, (f - g) \leq A - \beta. \tag{2}$$

For in any $V_\delta^*(a)$,

$$\mathrm{Min}\, f + \mathrm{Min}\, g \leq \mathrm{Min}\,(f + g) \leq \mathrm{Max}\,(f + g) \leq \mathrm{Max}\, f + \mathrm{Max}\, g.$$

Letting $\delta \doteq 0$, we get 1).

Also in $V_\delta^*(a)$,

$$\mathrm{Min}\, f - \mathrm{Max}\, g \leq \mathrm{Min}\,(f - g) \leq \mathrm{Max}\,(f - g) \leq \mathrm{Max}\, f - \mathrm{Min}\, g.$$

Letting $\delta \doteq 0$, we get 2).

3. *If*

$$f(x) \geq 0 \quad , \quad g(x) \geq 0,$$

$$\alpha\beta \leq \overline{\underline{\lim}}\, fg \leq AB. \tag{3}$$

If

$$f(x) \geq 0 \quad , \quad \beta \leq 0 \leq B,$$

$$A\beta \leq \overline{\underline{\lim}}\, fg \leq AB. \tag{4}$$

4. *If*

$$f(x) \geq 0 \quad , \quad g(x) \geq k > 0,$$

$$\frac{\alpha}{B} \leq \overline{\underline{\lim}} \frac{f}{g} \leq \frac{A}{\beta}. \tag{5}$$

If

$$\alpha < 0 < A \quad , \quad g(x) \geq k > 0,$$

$$\frac{\alpha}{\beta} \leq \overline{\underline{\lim}} \frac{f}{g} \leq \frac{A}{\beta}. \tag{6}$$

The relations 3), 4), 5), 6) may be proved as in 2. For example, to prove 5), we observe that in $V_\delta^*(a)$,

$$\frac{\mathrm{Min}\, f}{\mathrm{Max}\, g} \leq \mathrm{Min}\, \frac{f}{g} \leq \mathrm{Max}\, \frac{f}{g} \leq \frac{\mathrm{Max}\, f}{\mathrm{Min}\, g}.$$

2. *Let $f(x)$ be continuous in the interval $\mathfrak{A}=(a<b)$. Then $U f'(x)$ cannot be constantly $+\infty$, or constantly $-\infty$ in $\mathfrak{A}$.*

For consider

$$\phi(x)=f(x)-f(a)-\frac{x-a}{b-a}\{f(b)-f(a)\},$$

which is continuous, and vanishes for $x=a$, $x=b$. We observe that $\phi(x)$ differs from $f(x)$ only by a linear function. If now $Uf'(x)=+\infty$ constantly, obviously $U\phi'(x)=+\infty$ also. Thus ϕ is a univariant function in $\mathfrak{A}$. This is not possible, since ϕ has the same value at a and b.

3. *Let $f(x)$ be continuous in $\mathfrak{A}=(a<b)$, and have a derivative, finite or infinite, in $\mathfrak{A}=(a^*, b)$. Then*

$$\operatorname{Min} f'(x) \leq \bar{R}f'(a) \leq \operatorname{Max} f'(x) \quad , \quad in\ \mathfrak{A}.$$

For the Law of the Mean holds, hence

$$\frac{f(a+h)-f(a)}{h}=f'(\alpha) \quad , \quad a<\alpha<a+h.$$

Letting now $h \doteq 0$, we get the theorem.

Remark. This theorem answers the question: Can a continuous curve have a vertical tangent at a point $x=a$, if the derivatives remain $<M$ in $V^*(a)$? The answer is, No.

4. *Let $f(x)$ be continuous in $\mathfrak{A}=(a<b)$, and have a derivative, finite or infinite, in $\mathfrak{A}^*=(a^*, b)$. If $f'(a)$ exists, finite or infinite, there exists a sequence $\alpha_1>\alpha_2>\cdots \doteq a$ in $\mathfrak{A}$, such that*

$$f'(a)=\lim_{n=\infty} f'(\alpha_n). \tag{1}$$

For

$$\frac{f(a+h)-f(a)}{h}=f'(\alpha_h) \quad , \quad a<\alpha_h<a+h. \tag{2}$$

Let now h range over $h_1>h_2>\cdots \doteq 0$. If we set $\alpha_n=\alpha_{h_n}$, the relation 1) follows at once from 2), since $f'(a)$ exists by hypothesis.

510. 1. *A right-hand derivate of a continuous function $f(x)$ cannot have a discontinuity of the $1°$ kind on the right. A similar statement holds for the other derivates.*

For let $R(x)$ be one of the right-hand derivates. It it has a discontinuity of the 1° kind on the right at $x=a$, there exists a number l such that

$$l-\epsilon \leq R(x) \leq l+\epsilon \quad , \quad \text{in some } (a < a+\delta).$$

Then by 506, 1,

$$l-\epsilon \leq \frac{f(a+h)-f(a)}{h} \leq l+\epsilon \quad , \quad 0 < h < \delta.$$

Hence
$$R(a)=l,$$

and $R(x)$ is continuous on the right at $x=a$, which is contrary to hypothesis.

2. It can, however, have a discontinuity of the 1° kind on the left, as is shown by the following

Example. Let $f(x)=|x|=+\sqrt{x^2}$, in $\mathfrak{A}=(-1, 1)$.

Here
$$R(x)=+1 \quad , \quad \text{for } x \geq 0 \text{ in } \mathfrak{A}$$
$$=-1 \quad , \quad \text{for } x < 0.$$

Thus at $x=0$, R is continuous on the right, but has a discontinuity of the 1° kind on the left.

3. *Let $f(x)$ be continuous in $\mathfrak{A}=(a, b)$, and have a derivative, finite or infinite, in $\mathfrak{A}^*=(a^*, b^*)$. Then the discontinuities of $f'(x)$ in $\mathfrak{A}$, if any exist, must be of the second kind.*

This follows from 1.

Example.
$$f(x)=x^2 \sin\frac{1}{x} \quad , \quad \text{for } x \neq 0 \text{ in } \mathfrak{A}=(0, 1)$$
$$=0 \quad , \quad \text{for } x=0.$$

Then
$$f'(x)=2x\sin\frac{1}{x}-\cos\frac{1}{x} \quad , \quad x \neq 0$$
$$=0 \quad , \quad x=0.$$

The discontinuity of $f'(x)$ at $x=0$, is in fact of the 2° kind.

4. *Let $f(x)$ be continuous in $\mathfrak{A}=(a<b)$, except at $x=a$, which is a point of discontinuity of the 2° kind. Let $f'(x)$ exist, finite or infinite, in (a^*, b). Then $x=a$ is a point of infinite discontinuity of $f'(x)$.*

We introduce the auxiliary function

$$\phi(x) = f(x) - (M + c)x, \tag{2}$$

where

$$0 < c < e = c + \delta.$$

Then
$$\frac{\phi(\beta) - \phi(\alpha)}{\beta - \alpha} = \frac{f(\beta) - f(\alpha)}{\beta - \alpha} - (M + c) = \delta.$$

Hence
$$\phi(\beta) - \phi(\alpha) = \delta(\beta - \alpha) = \eta.$$

Consider now the equation

$$\phi(\beta) - \phi(x) = \eta.$$

It is satisfied for $x = \alpha$. If it is satisfied for any other x in the interval $(\alpha\beta)$, there is a last point, say $x = \gamma$, where it is satisfied, by 458, 3.

Thus for $x > \gamma$,
$$\phi(x) \text{ is } > \phi(\alpha).$$

Hence
$$\overline{R}\phi'(\gamma) \geq 0. \tag{3}$$

Now from 2) we have

$$\overline{R}f'(\gamma) = \overline{R}\phi'(\gamma) + M + c$$
$$> M.$$

Hence M is not the maximum of $\overline{R}f'(x)$ in $\mathfrak{A}$. Similarly the other half of 1) is established. The case that m or M is infinite is obviously true.

2. *Let $f(x)$ be defined over $\mathfrak{A} = (a < b)$. Let $a_1 < a_2 < \cdots < a_n$ lie in $\mathfrak{A}$. Let m and M denote the minimum and maximum of the difference quotients*

$$\Delta(a_1, a_2) \quad , \quad \Delta(a_2, a_3) \quad , \quad \cdots \Delta(a_{n-1}, a_n).$$

Then
$$m \leq \Delta(a_1, a_n) \leq M. \tag{1}$$

For let us first take three points $\alpha < \beta < \gamma$ in $\mathfrak{A}$. We have identically

$$\Delta(\alpha, \gamma) = \frac{\alpha - \beta}{\alpha - \gamma} \cdot \Delta(\alpha, \beta) + \frac{\beta - \gamma}{\alpha - \gamma} \cdot \Delta(\beta, \gamma).$$

Now the coefficients of Δ on the right lie between 0 and 1. Hence 1) is true in this case. The general case is now obvious.

507. 1. *Let $f(x)$ be continuous in $\mathfrak{A}=(a<b)$. The four derivates of f have the same extremes in $\mathfrak{A}$.*

To fix the ideas let

$$\text{Min } \underline{L}=m \quad , \quad \text{Min } \overline{R}=\mu, \quad \text{in } \mathfrak{A}.$$

We wish to show that $m=\mu$. To this end *we first show that*

$$\mu \leq m. \tag{1}$$

For there exists an α in $\mathfrak{A}$, such that

$$\underline{L}(\alpha)<m+\epsilon.$$

There exists therefore a $\beta<\alpha$ in $\mathfrak{A}$, such that

$$q=\frac{f(\alpha)-f(\beta)}{\alpha-\beta}<m+\epsilon', \qquad 0<\epsilon'<\epsilon.$$

Now by 506, 1,

$$\mu=\text{Min } \overline{R}\leq q.$$

Hence $$\mu \leq m,$$
as $\epsilon>0$ is small at pleasure.

We show now that

$$m\leq\mu. \tag{2}$$

For there exists an α in $\mathfrak{A}$, such that

$$\overline{R}(\alpha)<\mu+\epsilon.$$

There exists therefore a $\beta>\alpha$ in $\mathfrak{A}$, such that

$$q=\frac{f(\alpha)-f(\beta)}{\alpha-\beta}<\mu+\epsilon', \qquad 0<\epsilon>'\epsilon.$$

Thus by 506, 1,

$$m=\text{Min } \underline{L}\leq q.$$

Hence as before $m\leq\mu$. From 1), 2) we have $m=\mu$.

2. In 499, we emphasized the fact that the left-hand derivates are not defined at the left-hand end point of an interval, and the right-hand derivates at the right-hand end point of an interval for which we are considering the values of a function. The following example shows that our theorems may be at fault if this fact is overlooked.

Example. Let $f(x) = |x|$.

If we restrict x to lie in $\mathfrak{A} = (0, 1)$, the four derivates $= 1$ when they are defined. Thus the theorem 1 holds in this case. If, however, we regarded the left-hand derivates as defined at $x = 0$, and to have the value

$$Lf'(0) = -1,$$

as they would have if we considered values of f to the left of $\mathfrak{A}$, the theorem 1 would no longer be true.

For then $$\text{Min } \overline{\underline{L}} = -1 \quad , \quad \text{Min } \overline{\underline{R}} = +1,$$

and the four derivates do not have the same minimum in $\mathfrak{A}$.

3. *Let $f(x)$ be continuous about the point $x = c$. If one of its four derivates is continuous at $x = c$, all the derivates defined at this point are continuous, and all are equal.*

For their extremes in any $V_\delta(c)$ are the same. If now $\overline{R}$ is continuous at $x = c$,

$$\overline{R}(c) - \epsilon < \overline{R}(x) < \overline{R}(c) + \epsilon,$$

for any x in some $V_\delta(c)$.

4. *Let $f(x)$ be continuous about the point $x = c$. If one of its four derivates is continuous at $x = c$, the derivative exists at this point.*

This follows at once from 3.

Remark. We must guard against supposing that the derivative is continuous at $x = c$, or even exists in the vicinity of this point.

Example. Let $F(x)$ be as in 501, Ex. 1. Let

$$\mathfrak{A} = (0, 1) \text{ and } \mathfrak{E} = \left\{\frac{1}{n}\right\}.$$

Let $$H(x) = x^2 F(x).$$

Then $$RH'(x) = 2\,xF(x) + x^2 RF'(x),$$

$$LH'(x) = 2\,xF(x) + x^2 LF'(x).$$

Obviously both RH' and LH' are continuous at $x = 0$ and $H'(0) = 0$. But H' does not exist at the points of $\mathfrak{E}$, and hence

does not exist in any vicinity $(0, \delta)$ of the origin, however small $\delta > 0$ is taken.

5. *If one of the derivates of the continuous function $f(x)$ is continuous in an interval $\mathfrak{A}$, the derivative $f'(x)$ exists, and is continuous in $\mathfrak{A}$.*

This follows from 3.

6. *If one of the four derivates of the continuous function $f(x)$* is $= 0$ *in an interval $\mathfrak{A}$, $f(x) =$* const *in $\mathfrak{A}$.*

This follows from 3.

508. 1. *If one of the derivates of the continuous function $f(x)$ is* ≥ 0 in $\mathfrak{A} = (a < b)$, *$f(x)$ is monotone increasing in $\mathfrak{A}$.*

For then $m = \text{Min}\, \bar{R}f' \geq 0$, in $(a < x)$. Thus by 506, 1,

$$f(x) - f(a) \geq 0.$$

2. *If one of the derivates of the continuous function $f(x)$ is ≥ 0 in $\mathfrak{A}$, $f(x)$ is monotone decreasing.*

3. *If one of the derivates of the continuous function $f(x)$ is ≥ 0 in $\mathfrak{A}$, without being constantly 0 in any little interval of $\mathfrak{A}$, $f(x)$ is an increasing function in $\mathfrak{A}$. Similarly f is a decreasing function in $\mathfrak{A}$, if one of the derivates is ≤ 0, without being constantly 0 in any little interval of $\mathfrak{A}$.*

The proof is analogous to I, 403.

509. 1. *Let $f(x)$ be continuous in the interval $\mathfrak{A}$, and have a derivative, finite or infinite, within $\mathfrak{A}$. Then the points where the derivative is finite form a pantactic set in $\mathfrak{A}$.*

For let $\alpha < \beta$ be two points of $\mathfrak{A}$. Then by the Law of the Mean,

$$f'(\gamma) = \frac{f(\beta) - f(\alpha)}{\beta - \alpha}, \quad \alpha < \gamma < \beta.$$

As the right side has a definite value, the left side must have. Thus in any interval (α, β) in $\mathfrak{A}$, there is a point γ where the differential coefficient is finite.

For if

$$p = R\overline{\lim_{x=a}} f(x) \quad , \quad q = R\underline{\lim_{x=a}} f(x),$$

there exists a sequence of points $\alpha_1 > \alpha_2 > \cdots \doteq a$, such that $f(\alpha_n) \doteq p$; and another sequence $\beta_1 > \beta_2 > \cdots \doteq \alpha$, such that $f(\beta_n) \doteq q$. We may suppose

$$\alpha_n > \beta_n \quad , \quad \text{or } \alpha_n < \beta_n \quad , \quad n = 1, 2, \cdots$$

Then the Law of the Mean gives

$$Q_n = \frac{f(\alpha_n) - f(\beta_n)}{\alpha_n - \beta_n} = f'(\gamma_n),$$

where γ_n lies between α_n, β_n. Now the numerator $\doteq p - q$, while the denominator $\doteq 0$. Hence $Q_n \doteq +\infty$, or $-\infty$, as we choose.

5. *Let $f(x)$ have a finite unilateral differential coefficient U at each point of the interval $\mathfrak{A}$. Then U is at most pointwise discontinuous in $\mathfrak{A}$.*

For by 474, 3, U is a function of class 1. Hence, by 486, 1, it is at most pointwise discontinuous in $\mathfrak{A}$.

511. *Let $f(x)$ be continuous in the interval $(a < b)$. Let $R(x)$ denote one of the right-hand derivates of $f(x)$. If R is not continuous on the right at a, then*

$$l \leq R(a) \leq m, \tag{1}$$

where

$$l = R\underline{\lim}\, R(x) \quad , \quad m = R\overline{\lim}\, R(x) \quad , \quad x \doteq a.$$

To fix the ideas let R be the upper right-hand derivate. Let us suppose that $\alpha = \overline{R}f'(a)$ were $> m$. Let us choose η, and c such that

$$m + \eta < c < \alpha. \tag{2}$$

We introduce the auxiliary function

$$\phi(x) = cx - f(x).$$

Then

$$\overline{R}\phi'(x) = c - \underline{R}f'(x) \quad , \quad \underline{R}\phi'(x) = c - \overline{R}f'(x). \tag{3}$$

Now if $\delta > 0$ is sufficiently small,

$$\overline{R}f'(x) < m + \eta \quad , \quad \text{for any } x \text{ in } \mathfrak{A}^* = (a^*, a + \delta).$$

Thus 2), 3), show that

$$\underline{R}\phi'(x) \geq \sigma \quad , \quad \sigma > 0.$$

Hence $\phi(x)$ is an increasing function in $\mathfrak{A}^*$. But, on the other hand,

$$\overline{R}f'(a) = \underline{R}f'(a),$$

since $\alpha > m$. Hence

$$\overline{R}\phi'(a) = c - \overline{R}f'(a) = c - \alpha < 0.$$

Hence ϕ is a decreasing function at $x = a$. This is impossible since ϕ is continuous at a. Thus $\alpha \leq m$.

Similarly we may show that $l \leq \alpha$.

512. 1. *Let $f(x)$ be continuous in $\mathfrak{A} = (a < b)$, and have a derivative, finite or infinite. If $\alpha = f'(a)$, $\beta = f'(b)$, then $f'(x)$ takes on all values between α, β, as x ranges over $\mathfrak{A}$.*

For let $\alpha < \gamma < \beta$, and let

$$Q(x, h) = \frac{f(x+h) - f(x)}{h} \quad , \quad h > 0.$$

We can take h so small that

$$Q(a, h) < \gamma \quad , \quad \text{and } Q(b, -h) > \gamma.$$

Now

$$Q(b, -h) = Q(b-h, h).$$

Hence

$$Q(b-h, h) > \gamma.$$

If now we fix h, $Q(x, h)$ is a continuous function of x. As Q is $< \gamma$, for $x = a$, and $> \gamma$, for $x = b - h$, it takes on the value γ for some x, say for $x = \xi$, between a, $b - h$. Thus

$$Q(\xi, h) = \gamma.$$

But by the Law of the Mean,

$$Q(\xi, h) = f'(\eta),$$

where

$$a < \xi < \eta < \xi + h < b.$$

Thus $f'(x) = \gamma$, at $x = \eta$ in $\mathfrak{A}$.

2. *Let $f(x)$ be continuous in the interval $\mathfrak{A}$, and admit a derivative, finite or infinite. If $f'(x) = 0$ in $\mathfrak{A}$, except possibly at an enumerable set $\mathfrak{E}$, then $f' = 0$ also in $\mathfrak{E}$.*

For if $f'(\alpha) = 0$, and $f'(\beta) = b \neq 0$, then $f'(x)$ ranges over all values in $(0, b)$, as x passes from α to β. But this set of values has the cardinal number $\mathfrak{c}$. Hence there is a set of values in (α, β) whose cardinal number is $\mathfrak{c}$, where $f'(x) \neq 0$. This is contrary to the hypothesis.

3. *Let $f(x)$, $g(x)$ be continuous and have derivatives, finite or infinite, in the interval $\mathfrak{A}$. If in $\mathfrak{A}$ there is an α for which*

$$f'(\alpha) > g'(\alpha),$$

and a β for which

$$f'(\beta) < g'(\beta),$$

then there is a γ for which

$$f'(\gamma) = g'(\gamma),$$

provided

$$\delta(x) = f(x) - g(x)$$

has a derivative, finite or infinite.

For by hypothesis

$$\delta'(\alpha) > 0 \quad , \quad \delta'(\beta) < 0.$$

Hence by 1 there is a point where $\delta' = 0$.

513. 1. *If one of the four derivates of the continuous function $f(x)$ is limited in the interval $\mathfrak{A}$, all four are, and they have the same upper and lower R-integrals.*

The first part of the theorem is obvious from 507, 1. Let us effect a division of $\mathfrak{A}$ of norm d. Then

$$\overline{\int_{\mathfrak{A}}} \overline{R} = \lim_{d=0} \Sigma M_\iota d_\iota \quad , \quad M_\iota = \operatorname{Max} \overline{R}_\iota \text{ in } d_\iota .$$

But the maximum of the three other derivates in d_ι is also M_ι by 507, 1. Hence the last part of the theorem.

2. *Let $f(x)$ be continuous and have a limited unilateral derivate as $\overline{R}$ in $\mathfrak{A} = (a < b)$. Then*

$$\underline{\int_a^b} \overline{R} dx \leq f(b) - f(a) \leq \overline{\int_a^b} \overline{R} dx. \tag{1}$$

For let $a < a_1 < a_2 < \cdots < b$ determine a division of $\mathfrak{A}$, of norm d.

hen by 506, 1,

$$\text{Min } \overline{R} \leq \frac{f(a_{m+1}) - f(a_m)}{a_{m+1} - a_m} < \text{Max } \overline{R},$$

the interval $(a_m, a_{m+1}) = d_m$.

Hence

$$\Sigma d_m \text{ Min } \overline{R} \leq f(b) - f(a) \leq \Sigma d_m \text{ Max } \overline{R}.$$

Letting $d \doteq 0$, we get 1).

3. *If $f(x)$ is continuous, and $\overline{\underline{U}}f'$ is limited and R-integrable in* $= (a < b)$, *then*

$$\int_a^b \overline{\underline{U}}f' = f(b) - f(a).$$

514. 1. *Let $f(x)$ be limited in $\mathfrak{A} = (a < b)$, and*

$$F(x) = \overline{\underline{\int_a^x}} f dx \quad , \quad a \leq x \leq b.$$

Then $$U \underline{\lim}_{x=u} f \leq \overline{\underline{U}}F'(u) \leq U \overline{\lim}_{x=u} f, \tag{1}$$

r any u within $\mathfrak{A}$.

To fix the ideas let us take a right-hand derivate at $x = u$. Then

$$h \text{ Min} f \leq \overline{\underline{\int_u^{u+h}}} f dx \leq h \text{ Max} f \quad , \quad \text{in } (u^*, u+h), \ h > 0.$$

Thus $$\text{Min} f \leq \frac{\Delta F}{\Delta x} \leq \text{Max} f.$$

Letting $h \doteq 0$, we get

$$R \underline{\lim}_{x=u} f \leq \overline{\underline{R}}F'(u) \leq R \overline{\lim}_{x=u} f,$$

hich is 1) for this case.

2. *Let $f(x)$ be limited in the interval $\mathfrak{A} = (a < b)$. If $f(x+0)$ ists,*

$$R \text{ derivative} \int_a^x f dx = f(x+0);$$

d if $f(x-0)$ exists, $\quad a < x < b$

$$L \text{ derivative} \int_a^x f dx = f(x-0).$$

3. *Let $f(x)$ be limited and R-integrable in $\mathfrak{A} = (a < b)$. The points where*

$$F(x) = \int_a^x f dx \quad , \quad a \leq x \leq b$$

does not have a differential coefficient in $\mathfrak{A}$ form a null set.

For

$$F'(x) = f(x) \qquad \text{by I, 537, 1,}$$

when f is continuous at x. But by 462, 6, the points where f is not continuous form a null set.

515. In I, 400, we proved the theorem:

Let $f(x)$ be continuous in $\mathfrak{A} = (a < b)$, and let its derivative $= 0$ within $\mathfrak{A}$. Then f is a constant in $\mathfrak{A}$. This theorem we have extended in 507, 6, to a derivate of $f(x)$. It can be extended still farther as follows:

1. *(L. Scheefer). If $f(x)$ is continuous in $\mathfrak{A} = (a < b)$, and if one of its derivates $= 0$ in $\mathfrak{A}$ except possibly at the points of an enumerable set $\mathfrak{E}$, then $f =$ constant in $\mathfrak{A}$.*

If f is a constant, the theorem is of course true. We show that the contrary case leads to an absurdity, by showing that Card $\mathfrak{E}$ would $= \mathfrak{c}$, the cardinal number of an interval.

For if f is not a constant, there is a point c in $\mathfrak{A}$ where $p = f(c) - f(a)$ is $\neq 0$. To fix the ideas let $p > 0$; also let us suppose the given derivate is $\overline{R} = \overline{R}f'(x)$.

Let

$$g(x, t) = f(x) - f(a) - t(x - a) \quad , \quad t > 0.$$

Obviously $|g|$ is the distance f is above or below the secant line,

$$y = t(x - a) + f(a).$$

Thus in particular for any t,

$$g(a, t) = 0 \quad , \quad g(c, t) = p - t(c - a).$$

Let $q > 0$ be an arbitrary but fixed number $< p$. Then

$$g(c, t) - q = p - q - t(c - a)$$

$$= (p - q)\left\{1 - t\frac{c - a}{p - q}\right\} > 0,$$

if $t < T$, where

$$T = \frac{p - q}{c - a}.$$

Hence

$$g(c, t) > q$$

any t in the interval $\mathfrak{T} = (\tau, T)$, $0 < \tau < T$. We note that

$$\text{Card } \mathfrak{T} = \mathfrak{c}.$$

Since for any t in $\mathfrak{T}$, $g(a, t) = 0$, and $g(c, t) > q$, let $x = e_t$ be maximum of the points $< c$ where $g(x, t) = q$. Then $e < c$, d for any h such that $e + h$ lies in (e, c),

$$0 < \frac{g(e+h) - g(e)}{h} = \frac{f(e+h) - f(e)}{h} - t.$$

Hence

$$\overline{R}f'(e) > 0.$$

Thus for any t in $\mathfrak{T}$, e_t lies in $\mathfrak{E}$. As t ranges over $\mathfrak{T}$, let e_t ge over $\mathfrak{E}_1 \leq \mathfrak{E}$. To each point e of $\mathfrak{E}_1$ corresponds but one nt t of $\mathfrak{T}$. For

$$0 = g(e, t) - g(e, t') = (t - t')(e - a).$$

Hence

$$t = t' \quad , \quad \text{as } e > a.$$

Thus

$$\text{Card } \mathfrak{T} = \text{Card } \mathfrak{E}_1 \leq \text{Card } \mathfrak{E},$$

ich is absurd.

. *Let $f(x)$ be continuous in $\mathfrak{A} = (a < b)$. Let $\mathfrak{E}$ denote the nts of $\mathfrak{A}$ where one of the derivates has one sign. If $\mathfrak{E}$ exists,*

Card $\mathfrak{E} = \mathfrak{c}$, the cardinal number of the continuum.

The proof is entirely similar to that in 1. For let c be a point $\mathfrak{E}$. Then there exists a $d > c$ such that

$$f(d) - f(c) = p > 0.$$

We now introduce the function

$$g(x, t) = f(x) - f(c) - t(x - c) \quad , \quad t > 0,$$

reason on this as we did on the corresponding g in 1, using the interval (c, d) instead of (a, b). We get

$$\text{Card } \mathfrak{E}_1 = \text{Card } \mathfrak{T} = \mathfrak{c}.$$

. *Let $f(x)$, $g(x)$ be continuous in the interval $\mathfrak{A}$. Let a pair of responding derivates as $\overline{R}f'$, $\overline{R}g'$ be finite and equal, except posy at an enumerable set $\mathfrak{E}$. Then $f = g + C$, in $\mathfrak{A}$, where C is a tant.*

For let
$$\phi = f - g \quad , \quad \psi = g - f.$$
Then in
$$A = \mathfrak{A} - \mathfrak{E},$$
$$\overline{R}\phi' \geq \overline{R}f' - \overline{R}g' = 0 \quad , \quad \overline{R}\psi' \geq 0.$$

But if $\overline{R}\phi' < 0$ at one point in $\mathfrak{A}$, it is < 0 at a set of points $\mathfrak{B}$ whose cardinal number is $\mathfrak{c}$. But $\mathfrak{B}$ lies in $\mathfrak{E}$. Hence $\overline{R}\phi$ is never < 0, in $\mathfrak{A}$. The same holds for ψ. Hence, by 508, ϕ and ψ are both monotone increasing. This is impossible unless $\phi =$ a constant.

516. The preceding theorem states that the continuous function $f(x)$ in the interval $\mathfrak{A}$ is known in $\mathfrak{A}$, aside from a constant, when $f'(x)$ is finite and known in $\mathfrak{A}$, aside from an enumerable set.

Thus $f(x)$ is known in $\mathfrak{A}$ when f' is finite and known at each *irrational* point of $\mathfrak{A}$.

This is not the case when f' is finite and known at each *rational* point only in $\mathfrak{A}$.

For the rational points in $\mathfrak{A}$ being enumerable, let them be
$$r_1, \; r_2, \; r_3 \cdots \tag{1}$$
Let
$$l = l_1 + l_2 + l_3 + \cdots$$
be a positive term series whose sum l is $< \widehat{\mathfrak{A}}$. Let us place r_1 within an interval δ_1 of length $\leq l_1$. Let r_{ι_2} be the first number in 1) not in δ_1. Let us place it within a non-overlapping interval δ_2 of length $\leq l_2$, etc.

We now define a function $f(x)$ in $\mathfrak{A}$ such that the value of f at any x is the length of all the intervals and part of an interval lying to the left of x. Obviously $f(x)$ is a continuous function of x in $\mathfrak{A}$. At each rational point $f'(x) = 1$. But $f(x)$ is not determined aside from a constant. For $\Sigma\delta_n \leq l$. Therefore when l is small enough we may vary the position and lengths of the δ-intervals, so that the resulting f's do not differ from each other only by a constant.

517. 1. *Let $f(x)$ be continuous in $\mathfrak{A} = (a < b)$ and have a finite derivate, say $\overline{R}f'$, at each point of $\mathfrak{A}$. Let $\mathfrak{E}$ denote the points of $\mathfrak{A}$ where $\overline{R}$ has one sign, say > 0. If $\mathfrak{E}$ exists, it cannot be a null set.*

For let c be a point of $\mathfrak{C}$, then there exists a point $d > c$ such
at

$$f(d) - f(c) = p > 0. \quad (1$$

Let $\mathfrak{C}_n$ denote the points of $\mathfrak{C}$ where

$$n - 1 \leq \overline{R}f' < n. \quad (2$$

Then $\mathfrak{C} = \mathfrak{C}_1 + \mathfrak{C}_2 + \cdots$ Let $0 < q < p$. We take the positive
nstants $q_1, q_2 \cdots$ such that

$$q_1 + 2q_2 + 3q_3 + \cdots \leq q.$$

If now $\mathfrak{C}$ is a null set, each $\mathfrak{C}_m$ is also. Hence the points of $\mathfrak{C}_m$
n be inclosed *within* a set of intervals δ_{mn} such that $\sum_n \delta_{mn} < q_m$.
t now $q_m(x)$ be the sum of the intervals and parts of intervals
$_n$, $n = 1, 2 \cdots$ which lie in the interval $(a \leq x)$. Let

$$Q(x) = \sum_m m q_m(x).$$

Obviously $Q(x)$ is a monotone increasing function, and

$$0 \leq Q(x) \leq q. \quad (3$$

Consider now

$$P(x) = f(x) - f(a) - Q(x).$$

We have at a point of $\mathfrak{A} - \mathfrak{C}$,

$$\frac{\Delta P}{\Delta x} = \frac{\Delta f}{\Delta x} - \frac{\Delta Q}{\Delta x} \leq \frac{\Delta f}{\Delta x} \quad , \quad \Delta x > 0.$$

Hence at such a point

$$\overline{R}P' \leq \overline{R}f' \leq 0.$$

But at a point x of $\mathfrak{C}$, $\overline{R}P' < 0$ also. For x must lie in some
, and hence within some δ_{mn}. Thus $q_m(x)$ increases by at least
when x is increased to $x + \Delta x$. Hence $mq_m(x)$, and thus
(x) is increased at least $m\Delta x$. Thus

$$\frac{\Delta Q}{\Delta x} \geq m.$$

Thus

$$\overline{R}P' \leq \overline{R}f' - m < 0, \text{ by } 2),$$

since x lies in $\mathfrak{E}_m$. Thus $\overline{R}P' \leq 0$ at any point of $\mathfrak{A}$. Thus P is a monotone decreasing function in $\mathfrak{A}$, by 508, 2. Hence

$$P(c) - P(d) \geq 0.$$

Hence
$$f(c) - f(d) - \{Q(c) - Q(d)\} \geq 0,$$

or using 1), 3)

$$p - q \leq 0,$$

which is not so, as p is $> q$.

2. (*Lebesgue.*) *Let $f(x)$, $g(x)$ be continuous in the interval $\mathfrak{A}$, and have a pair of corresponding derivates as $\overline{R}f'$, $\overline{R}g'$ which are finite at each point of $\mathfrak{A}$, and also equal, the equality holding except possibly at a null set. Then $f(x) - g(x) =$ constant in $\mathfrak{A}$.*

The proof is entirely similar to that of 515, 3, the enumerable set $\mathfrak{E}$ being here replaced by a null set. We then make use of 1.

518. *Let $f'(x)$ be continuous in some interval $\Delta = (u - \delta, u + \delta)$. Let $f''(x)$ exist, finite or infinite, in Δ, but be finite at the point $x = u$. Then*

$$f''(u) = \lim_{h=0} Qf, \tag{1}$$

where

$$Qf(u) = \frac{f(u+h) + f(u-h) - 2f(u)}{h^2}, \quad h > 0.$$

Let us *first suppose* that $f''(u) = 0$. We have for $0 < h \leq \eta < \delta$,

$$Qf = \frac{1}{h}\left\{\frac{f(u+h) - f(u)}{h} - \frac{f(u-h) - f(u)}{-h}\right\}$$

$$= \frac{1}{h}\{f'(x') - f'(x'')\}, \quad u < x' < u + h, \quad u - h < x'' < u$$

$$= \frac{1}{h}[(x' - u)\{f''(u) + \epsilon'\} - (x'' - u)\{f''(u) + \epsilon''\}],$$

where $|\epsilon'|$, $|\epsilon''|$ are $< \epsilon/2$ for η sufficiently small.

Now
$$\frac{x' - u}{h} \leq 1, \quad \frac{|x'' - u|}{h} \leq 1,$$

while
$$f''(u) = 0, \quad \text{by hypothesis.}$$

Hence
$$|Qf| < \epsilon, \quad \text{for } 0 < h \leq \eta,$$

and 1) holds in this case.

Suppose now that $f''(u) = a \neq 0$. Let

$$g(x) = f(x) - q(x) \quad , \quad \text{where } q(x) = \tfrac{1}{2}\, ax^2 + bx + c.$$

Since $$q''(u) = a \quad , \quad g''(u) = 0.$$

Thus we are in the preceding case, and $\lim\ Qg = 0$.

But $$Qg = Qf - Qq.$$

Hence $$\lim\ Qf = a.$$

Maxima and Minima

519. 1. In I, 466 and 476, we have defined the terms $f(x)$ as maximum or a minimum at a point. Let us extend these terms follows. Let $f(x_1 \cdots x_m)$ be defined over $\mathfrak{A}$, and let $x = a$ be an ner point of $\mathfrak{A}$.

We say f has a maximum at $x = a$ if 1°, $f(a) - f(x) \geq 0$, for any n some $V(a)$, and 2°, $f(a) - f(x) > 0$ for some x in any $V(a)$. the sign $\geq$ can be replaced by $>$ in 1°, we will say f has a *oper* maximum at a, when we wish to emphasize this fact; and hen $\geq$ cannot be replaced by $>$, we will say f has an *improper* aximum. A similar extension of the old definition holds for e minimum. A common term for maximum and minimum is *treme.*

2. If $f(x)$ is a constant in some segment $\mathfrak{B}$, lying in the inter- l $\mathfrak{A}$, $\mathfrak{B}$ is called a *segment of invariability*, or a *constant segment* f in $\mathfrak{A}$.

Example. Let $f(x)$ be continuous in $\mathfrak{A} = (0, 1^*)$.

Let $$x = \cdot\, a_1 a_2 a_3 \cdots \tag{1}$$

the expression of a point of $\mathfrak{A}$ in the normal form in the dyadic stem. Let

$$\xi = \cdot\, \alpha_1 \alpha_2 \alpha_3 \cdots \tag{2}$$

expressed in the triadic system, where $\alpha_n = a_n$, when $a_n = 0$, d $= 2$ when $a_n = 1$. The points $\mathfrak{C} = \{\xi\}$ form a Cantor set, 272. Let $\{\mathfrak{J}_n\}$ be the adjoint set of intervals. We associate

now the point 1) with the point 2), which we indicate as usual by $x \sim \xi$. We define now a function $g(x)$ as follows:

$$g(\xi) = f(x) \quad , \quad \text{when } x \sim \xi.$$

This defines g for all the points of $\mathfrak{C}$. In the interval $\mathfrak{J}_n$, let g have a constant value. Obviously g is continuous, and has a pantactic set of intervals in each of which g is constant.

3. We have given criteria for maxima and minima in I, 468 seq., to which we may add the following:

Let $f(x)$ be continuous in $(a-\delta, a+\delta)$. If $\underline{R}f'(a) > 0$ and $\overline{L}f'(a) < 0$, finite or infinite, $f(x)$ has a minimum at $x = a$.

If $\overline{R}f'(a) < 0$ and $\underline{L}f'(a) > 0$, finite or infinite, $f(x)$ has a maximum at $x = a$.

For on the 1° hypothesis, let us take α such that $\underline{R} - \alpha > 0$. Then there exists a $\delta' > 0$ such that

$$\frac{f(a+h) - f(a)}{h} > \underline{R} - \alpha > 0 \quad , \quad 0 < h \leq \delta'.$$

Hence

$$f(a+h) > f(a) \quad , \quad a+h \text{ in } (a^*, a+\delta').$$

Similarly if β is chosen so that $\overline{L} + \beta < 0$, there exists a $\delta'' > 0$, such that

$$\frac{f(a-h) - f(a)}{-h} < \overline{L} + \beta.$$

Hence

$$f(a-h) > f(a) \quad , \quad a+h \text{ in } (a - \delta'', a^*).$$

520. *Example 1.* Let $f(x)$ oscillate between the x-axis and the two lines $y = x$ and $y = -x$, similar to

$$y = \left| x \sin \frac{\pi}{x} \right|.$$

In any interval about the origin, y oscillates infinitely often, having an infinite number of proper maxima and minima. At the point $x = 0$, f has an improper minimum.

Example 2. Let us take two parabolas P_1, P_2 defined by $y = x^2$, $y = 2x^2$. Through the points $x = \pm \frac{1}{2}, \pm \frac{1}{3} \cdots$ let us erect ordinates, and join the points of intersection with P_1, P_2, alternately by straight lines, getting a broken line oscillating between the

rabolas P_1, P_2. The resulting graph defines a continuous func-
on $f(x)$ which has proper extremes at the points $\mathfrak{E} = \left\{ \pm \frac{1}{n} \right\}$.
owever, unlike Ex. 1, the limit point $x = 0$ of these extremes is
so a point at which $f(x)$ has a proper extreme.

Example 3. Let $\{\delta\}$ be a set of intervals which determine a
arnack set $\mathfrak{H}$ lying in $\mathfrak{A} = (0, 1)$. Over each interval $\delta = (\alpha, \beta)$
longing to the n^{th} stage, let us erect a curve, like a segment of
sine curve, of height $h_n \doteq 0$, as $n \doteq \infty$, and having horizontal
ngents at α, β, and at γ, the middle point of the interval δ. At
e points $\{\xi\}$ of $\mathfrak{A}$ not in any interval δ, let $f(x) = 0$. The func-
on f is now defined in $\mathfrak{A}$ and is obviously continuous. At the
ints $\{\gamma\}$, f has a proper maximum; at points of the type α, β,
f has an improper minimum. These latter points form the set
whose cardinal number is $\mathfrak{c}$. The function is increasing in each
terval (α, γ), and decreasing in each (γ, β). It oscillates in-
itely often in the vicinity of any point of $\mathfrak{H}$.

We note that while the points where f has a proper extreme
rm an enumerable set, the points of improper extreme may form
et whose cardinal number is $\mathfrak{c}$.

Example 4. We use the same set of intervals $\{\delta\}$ but change
e curve over δ, so that it has a constant segment $\eta = (\lambda, \mu)$ in its
ddle portion. As before $f = 0$, at the points ξ not in the
tervals δ.

The function $f(x)$ has now no proper extremes. At the points
$\mathfrak{H}$, f has an improper minimum; at the points of the type λ, μ, it
s an improper maximum.

Example 5. Weierstrass' Function. Let $\mathfrak{E}$ denote the points in
interval $\mathfrak{A}$ of the type

$$x = \frac{r}{b^s} \quad , \quad r, s, \text{ positive integers.}$$

r such an x we have, using the notation of 502,

$$b^m x = \iota_m + \xi_m = b^{m-s} r.$$

Hence
$$\xi_m = 0 \quad , \quad \text{for } m \geq s.$$

Thus
$$e_m = (-1)^{\iota_m + 1} = (-1)^{r+1}.$$

Hence $$\operatorname{sgn} \frac{\Delta F}{\Delta x} = \operatorname{sgn} Q = \operatorname{sgn} e_m \eta_m = \operatorname{sgn} (-1)^r h.$$

Thus $$\operatorname{sgn} Rf'(x) = +1 \quad , \quad \operatorname{sgn} Lf'(x) = -1,$$

if r is even, and reversed if r is odd. Thus at the points $\mathfrak{C}$, the curve has a vertical cusp. By 519, 3, F has a maximum at the points $\mathfrak{C}$, when r is odd, and a minimum when r is even. The points $\mathfrak{C}$ are pantactic in $\mathfrak{A}$.

Weierstrass' function has no constant segment δ, for then $f'(x) = 0$ in δ. But F' does not exist at any point.

521. 1. *Let $f(x_1 \cdots x_m)$ be continuous in the limited or unlimited set $\mathfrak{A}$. Let $\mathfrak{E}$ denote the points of $\mathfrak{A}$ where f has a proper extreme. Then $\mathfrak{E}$ is enumerable.*

Let us first suppose that $\mathfrak{A}$ is limited. Let $\delta > 0$ be a fixed positive number. There can be but a finite number of points α in $\mathfrak{A}$ such that

$$f(\alpha) > f(x) \quad , \quad \text{in } V_\delta^*(\alpha). \tag{1}$$

For if there were an infinity of such points, let β be a limiting point and $\eta < \frac{1}{2}\delta$. Then in $V_\eta(\beta)$ there exist points α', α'' such that $V_\delta(\alpha')$, $V_\delta(\alpha'')$ overlap. Thus in one case

$$f(\alpha') > f(\alpha''),$$

and in the other

$$f(\alpha') < f(\alpha''),$$

which contradicts the first.

Let now $\delta_1 > \delta_2 > \cdots \doteq 0$. There are but a finite number of points α for which 1) holds for $\delta = \delta_1$, only a finite number for $\delta = \delta_2$, etc. Hence $\mathfrak{E}$ is enumerable. The case that $\mathfrak{A}$ is unlimited follows now easily.

2. We have seen that Weierstrass' function has a pantactic set of proper extremes. However, according to 1, they must be enumerable. In Ex. 3, the function has a minimum at each point of the non-enumerable set $\mathfrak{H}$; but these minima are improper. On the other hand, the function has a proper maximum at the points $\{\gamma\}$, but these form an enumerable set.

522. 1. *Let $f(x)$ be continuous in the interval $\mathfrak{A}$. Let f have a proper maximum at $x = \alpha$, and $x = \beta$ in $\mathfrak{A}$. Then there is a point γ between α, β where f has a minimum, which need not however be a proper minimum.*

For say $\alpha < \beta$. In the vicinity of α, $f(x)$ is $< f(\alpha)$; also in the vicinity of β, $f(x)$ is $< f(\beta)$. Thus there are points $\mathfrak{B}$ in (α, β) where f is $<$ either $f(\alpha)$ or $f(\beta)$. Let μ be the minimum of the values of $f(x)$, as x ranges over $\mathfrak{B}$. There is a least value of x in (α, β) for which $f(x) = \mu$. We may take this as the point in question. Obviously γ is neither α nor β.

2. That at the point γ, f does not need to have a proper minimum is illustrated by Exs. 1, or 3.

3. *In $\mathfrak{A} = (a, b)$ let $f'(x)$ exist, finite or infinite. The points within $\mathfrak{A}$ at which f has an extreme proper or improper, lie among the zeros of $f'(x)$.*

This follows from the proof used in I, 468, 2, if we replace there < 0, by ≤ 0, and > 0, by ≥ 0.

4. *Let $f'(x)$ be continuous in the interval $\mathfrak{A}$, and let $f(x)$ have no constant segments in $\mathfrak{A}$. The points $\mathfrak{E}$ of $\mathfrak{A}$ where f has an extreme, form an apantactic set in $\mathfrak{A}$. Let $\mathfrak{Z}$ denote the zeros of $f'(x)$ in $\mathfrak{A}$. If $\mathfrak{B} = \{\mathfrak{b}_n\}$ is the border set of intervals lying in $\mathfrak{A}$ corresponding to $\mathfrak{Z}$, $f(x)$ is univariant in each $\mathfrak{b}_n$.*

For by 3, the points $\mathfrak{E}$ lie in $\mathfrak{Z}$. As $f'(x)$ is continuous, $\mathfrak{Z}$ is complete and determines the border set $\mathfrak{B}$. Within each $\mathfrak{b}_n$, $f'(x)$ has one sign. Hence $f(x)$ is univariant in $\mathfrak{b}_n$.

5. *Let $f(x)$ be a continuous function having no constant segment in the interval $\mathfrak{A}$. If the points $\mathfrak{E}$ where f has an extreme form a pantactic set in $\mathfrak{A}$, then the points $\mathfrak{B}$ where $f'(x)$ does not exist or is discontinuous, form also a pantactic set in $\mathfrak{A}$.*

For if $\mathfrak{B}$ is not pantactic in $\mathfrak{A}$, there is an interval $\mathfrak{C}$ in $\mathfrak{A}$ containing no point of $\mathfrak{B}$. Thus $f'(x)$ is continuous in $\mathfrak{C}$. But the points of $\mathfrak{E}$ in $\mathfrak{C}$ form an apantactic set in $\mathfrak{C}$ by 4. This, however, contradicts our hypothesis.

Example. Weierstrass' function satisfies the condition of theorem 5. Hence the points where $F'(x)$ does not exist or is

discontinuous form a pantactic set. This is indeed true, since F' exists at no point.

6. *Let $f(x)$ be continuous and have no constant segment in the interval $\mathfrak{A}$. Let $f'(x)$ exist, finite or infinite. The points where $f'(x)$ is finite and is $\neq 0$ form a pantactic set in $\mathfrak{A}$.*

For let $\alpha < \beta$ be any two points in $\mathfrak{A}$. If $f(\alpha) = f(\beta)$, there is a point $\alpha < \gamma < \beta$ such that $f(\alpha) \neq f(\gamma)$, since f has no constant segment in $\mathfrak{A}$. Then the Law of the Mean gives

$$f'(\xi) = \frac{f(\alpha) - f(\gamma)}{\alpha - \gamma} \neq 0.$$

Thus in the arbitrary interval (α, β) there is a point ξ, where $f'(x)$ exists and is $\neq 0$.

7. *Let $f'(x)$ be continuous in the interval $\mathfrak{A}$. Then any interval $\mathfrak{B}$ in $\mathfrak{A}$ which is not a constant segment contains a segment $\mathfrak{C}$ in which f is univariant.*

For since f is not constant in $\mathfrak{B}$, there are two points a, b in $\mathfrak{B}$ at which f has different values. Then by the Law of the Mean

$$f(a) - f(b) = (a - b) f'(c) \quad , \quad c \text{ in } \mathfrak{B}.$$

Hence $f'(c) \neq 0$. As $f'(x)$ is continuous, it keeps its sign in some interval $(c - \delta, c + \delta)$, and f is therefore univariant.

523. *Let $f(x)$ be continuous in the interval $\mathfrak{A}$, and have in any interval in $\mathfrak{A}$ a constant segment or a point at which f has an extreme. If $f'(x)$ exists, finite or infinite, it is discontinuous infinitely often in any interval in $\mathfrak{A}$, not a constant segment. At a point of continuity of the derivative, $f'(x) = 0$.*

For if $f'(x)$ were continuous in an interval $\mathfrak{B}$, not a constant segment, f would be univariant in some interval $\mathfrak{C} \leq \mathfrak{B}$, by 522, 7. But this contradicts the hypothesis, which requires that any interval as $\mathfrak{C}$ has a constant segment. Hence $f'(x)$ is discontinuous in any interval, however small.

Let now $x = c$ be a point of continuity. Then if c lies in a constant segment, $f'(c) = 0$ obviously. If not, there is a sequence of points $e_1, e_2 \cdots \doteq e$ such that $f(x)$ has an extreme at e_n. But then $f'(e_n) = 0$, by 522, 3. As $f'(x)$ is continuous at $x = c$, $f'(c) = 0$ also.

524. (*König.*) *Let $f(x)$ be continuous in $\mathfrak{A}$ and have a pantactic set of cuspidal points $\mathfrak{C}$. Then for any interval $\mathfrak{B}$ of $\mathfrak{A}$, there exists a β such that $f(x)=\beta$ at an infinite set of points in $\mathfrak{B}$. Moreover, there is a pantactic set of points $\{\xi\}$ in $\mathfrak{B}$, such that k being taken at pleasure,*

$$\underline{f}'(x) \leq k \leq \bar{f}'(x). \tag{1}$$

For among the points $\mathfrak{C}$ there is an infinite pantactic set $\mathfrak{c}$ of proper maxima, or of proper minima. To fix the ideas, suppose the former. Let $x=c$ be one of these points within $\mathfrak{B}$. Then there exists an interval $\mathfrak{b} \leq \mathfrak{B}$, containing c, such that

$$f(c) > f(x) \quad , \quad \text{for any } x \text{ in } \mathfrak{b}.$$

Let
$$\mu = \operatorname{Min} f(x), \quad \text{in } \mathfrak{b}.$$

Then there is a point $\bar{x}$ where f takes on this minimum value. The point c divides the interval $\mathfrak{b}$ into two intervals. Let $\mathfrak{l}$ be that one of these intervals which contains $\bar{x}$, the other interval we denote by $\mathfrak{m}$. Within $\mathfrak{m}$ let us take a point c_1 of $\mathfrak{c}$. Then in $\mathfrak{l}$ there is a point c_1' such that

$$f(c_1) = f(c_1').$$

The point c_1 determines an interval $\mathfrak{b}_1$, just as c determined $\mathfrak{b}$. Obviously $\mathfrak{b}_1 \leq \mathfrak{m}$, and $\mathfrak{b}_1$ falls into two segments $\mathfrak{l}_1$, $\mathfrak{m}_1$ as before $\mathfrak{b}$ did. Within $\mathfrak{m}_1$ we take a point of $\mathfrak{c}$. Then in $\mathfrak{l}$ there is a point c_2', and in $\mathfrak{l}_1$ a point c_2'', such that

$$f(c_2) = f(c_2') = f(c_2'').$$

In this way we may continue indefinitely. Let

$$c_1' \quad , \quad c_2' \quad , \quad c_3' \cdots$$

be the points obtained in this way which fall in $\mathfrak{l}$. Let c' be a limit point of this set. Let

$$c_1'' \quad , \quad c_2'' \quad , \quad c_3'' \cdots$$

be the points obtained above which fall in $\mathfrak{l}_1$, and let c'' be a limit point of this set. Continuing in this way we get a sequence of limiting points

$$c' \quad , \quad c'' \quad , \quad c''' \cdots \tag{2}$$

lying respectively in $\mathfrak{l}, \mathfrak{l}_1, \mathfrak{l}_2 \cdots$

Since f is continuous,

$$f(c') = f(c'') = f(c''') = \cdots \tag{3}$$

Thus if we set $f(c') = \beta$ we see that $f(x)$ takes on the value β at the infinite set of points 2), which lie in $\mathfrak{B}$.

Let $\gamma_1, \gamma_2 \cdots$ be a set of points in 2) which $\doteq \gamma$.

Then

$$\frac{f(\gamma)-f(\gamma_1)}{\gamma-\gamma_1} = \frac{f(\gamma)-f(\gamma_2)}{\gamma-\gamma_2} = \cdots = 0. \tag{4}$$

Thus if $f'(x)$ exists at $x = \gamma$, the equations 3) show that $f'(\gamma) = 0$. If f' does not exist at γ, they show that

$$\underline{f}' \leq 0 \leq \overline{f}' \quad , \quad \text{at } x = \gamma.$$

Let now k be taken at pleasure. Then

$$g(x) = f(x) - kx$$

is constituted as f, and

$$\overline{\underline{g}}'(x) = \overline{\underline{f}}'(x) - k.$$

This gives 1).

525. 1. *Lineo-Oscillating Functions.* The oscillations of a continuous function fall into two widely different classes, according as $f(x)$ becomes monotone on adding a linear function $l(x) = ax + b$, or does not.

The former are called lineo-oscillating functions. A continuous function which does not oscillate in $\mathfrak{A}$, or if it does is lineo-oscillating, we say is *at most* a lineo-oscillating function.

Example 1. Let

$$f(x) = \sin x \quad , \quad l(x) = x.$$

If we set

$$y = f(x) + l(x)$$

and plot the graph, we see at once that y is an increasing function. At the point $x = \pi$, the slope of the tangent to $f(x) = \sin x$ is greatest negatively, *i.e.* $\sin x$ is decreasing here fastest. But the angle that the tangent to $\sin x$ makes at this point is $-45°$, while the slope of the line $l(x)$ is constantly $45°$. Thus at $x = \pi$, y has a point of inflection with horizontal tangent.

If we take $l(x) = ax$, $a > 1$, y is an increasing function, increasing still faster than before.

All this can be verified by analysis. For setting

$$y = \sin x + ax \quad , \quad a > 1,$$

get

$$y' = a + \cos x,$$

$$y' > 0.$$

Thus y is a lineo-oscillating function in any interval.

Example 2.
$$f(x) = x^2 \sin \frac{1}{x} \quad , \quad x \neq 0$$
$$= 0 \quad , \quad x = 0.$$
$$l(x) = ax + b \quad , \quad y = f(x) + l(x).$$

Then

$$y' = 2x \sin \frac{1}{x} - \cos \frac{1}{x} + a \quad , \quad x \neq 0$$
$$= a \quad , \quad x = 0.$$

Hence, if $a > 1 + 2\pi$, y is an increasing function in $\mathfrak{A} = (-\pi, \pi)$. e function f oscillates infinitely often in $\mathfrak{A}$, but is a lineo-oscilng function.

Example 3.
$$f(x) = x \sin \frac{1}{x} \quad , \quad x \neq 0$$
$$= 0 \quad , \quad x = 0.$$
$$l(x) = ax + b \quad , \quad y = f(x) + l(x).$$

Here

$$y' = \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x} + a \quad , \quad x \neq 0.$$

For $x = 0$, y' does not exist, finitely or infinitely.

Obviously, however great a is taken, y has an infinity of oscillas in any interval about $x = 0$. Hence f is not a lineo-oscillat- function in such an interval.

. *If one of the four derivates of the continuous function $f(x)$ is ited in the interval $\mathfrak{A}$, $f(x)$ is at most lineo-oscillating in $\mathfrak{A}$.*

For say $\underline{R}f' > -\alpha$ in $\mathfrak{A}$. Let $0 < \alpha < \beta$,

$$g(x) = f(x) + \beta x.$$

Then
$$\underline{g}'(x)=\beta+\underline{f}'(x)>0.$$

Hence g is monotone increasing by 508, 1.

3. *Let $f(x)$ be at most lineo-oscillating in the interval $\mathfrak{A}$. If Uf' does not exist finitely at a point x in $\mathfrak{A}$, it is definitely infinite at the point. Moreover, the sign of the ∞ is the same throughout $\mathfrak{A}$.*

For if f is monotone in $\mathfrak{A}$, the theorem is obviously true. If not, let
$$g(x)=f(x)+ax$$
be monotone. Then
$$Uf'=Ug'-a,$$
and this case is reduced to the preceding.

Remark. This shows that no continuous function whose graph has a vertical cusp can be lineo-oscillating. All its vertical tangents correspond to points of inflection, as in
$$y=x^{\frac{1}{3}}.$$

Variation

526. 1. *Let $f(x)$ be continuous in the interval $\mathfrak{A}$, and have limited variation. Let D be a division of $\mathfrak{A}$ of norm d. Then using the notation of* 443,
$$\lim V_D f=Vf \quad , \quad \lim P_D f=Pf \quad , \quad \lim N_D f=Nf. \qquad (1$$

For there exists a division Δ such that
$$V-\frac{\epsilon}{2}<V_\Delta\leq V,$$

where for brevity we have dropped f after the symbol V. Let now Δ divide $\mathfrak{A}$ into ν segments whose minimum length call λ. Let D be a division of $\mathfrak{A}$ of norm $d\leq d_0<\lambda$. Then not more than one point of Δ, say a_κ, can lie in any interval as $(a_\iota, a_{\iota+1})$ of D. Let $E=D+\Delta$, the division obtained by superposing Δ on D. Then μ denoting some integer $\leq\nu$,

$$V_E-V_D=\sum_{\kappa=1}^{\mu}\{|f(a_\kappa)-f(a_\iota)|+|f(a_{\iota+1})-f(a_\kappa)|-|f(a_{\iota+1})-f(a_\iota)|\}.$$

If now d_0 is taken sufficiently small, Osc f in any interval of D is small as we choose, say $< \frac{\epsilon}{6\nu}$. Then

$$|V_E - V_D| \leq \frac{\mu\epsilon}{2\nu} \leq \frac{\epsilon}{2}.$$

But since E is got by superposing Δ on D,

$$V_\Delta < V_E \leq V.$$

Hence for any D of norm $< d_0$,

$$|V_D - V| < \epsilon,$$

which proves the first relation in 1. The other two follow at once now from 443.

527. *If $f(x)$ is continuous and has limited variation in the interval $\mathfrak{A} = (a < b)$, then*

$$P(x) \quad , \quad N(x) \quad , \quad V(x)$$

are also continuous functions of x in $\mathfrak{A}$.

Let us show that $V(x)$ is continuous; the rest of the theorem follows at once by 443.

By 526, there exists a d_0, such that for any division D of norm $< d_0$,

$$V(b) = V_D(b) + \epsilon' \quad , \quad 0 \leq \epsilon' < \epsilon/3.$$

Then *a fortiori*, for any $x < b$ in $\mathfrak{A}$,

$$V(x) = V_D(x) + \epsilon_1 \quad , \quad 0 \leq \epsilon_1 < \epsilon/3. \tag{1}$$

In the division D, we may take x as one of the end points of an interval, and $x + h$ as the other end point. Then

$$V(x+h) = V_D(x) + |f(x+h) - f(x)| + \epsilon_2 \quad , \quad 0 \leq \epsilon_2 < \epsilon/3. \tag{2}$$

On the other hand, if d_0 is taken sufficiently small,

$$|f(x+h) - f(x)| < \frac{\epsilon}{3} \quad , \quad \text{for } 0 < h < \delta. \tag{3}$$

From 1), 2), 3) we have

$$0 \leq V(x+h) - V(x) < \epsilon \quad , \quad \text{for any } 0 \leq h < \delta. \tag{4}$$

But in the division D, x is the right-hand end point of some interval as $(x-k, x)$. The same reasoning shows that

$$|V(x-k)-V(x)|<\epsilon \quad , \quad \text{for any } 0 \leq k < \delta. \tag{5}$$

From 4), 5) we see $V(x)$ is continuous.

528. 1. *If one of the derivates of the continuous function $f(x)$ is numerically $\leq M$ in the interval $\mathfrak{A}$, the variation V of f is $\leq M\overline{\mathfrak{A}}$.*

For by definition
$$V = \text{Max } V_D,$$

with respect to all divisions $D = \{d_\iota\}$ of $\mathfrak{A}$. Here

$$V_D = \Sigma \, | f(a_\iota) - f(a_{\iota+1}) \, |.$$

Now by 506, 1,
$$-M \leqq \frac{f(a_\iota) - f(a_{\iota+1})}{a_\iota - a_{\iota+1}} \leq M,$$

or
$$| f(a_\iota) - f(a_{\iota+1}) | \leq M d_\iota.$$

Hence
$$V_D \leq M \Sigma d_\iota \leq M\overline{\mathfrak{A}}.$$

2. *Let $f(x)$ be limited and R-integrable in $\mathfrak{A} = (a < b)$. Then*

$$F(x) = \int_a^x f dx \quad , \quad a \leq x \leq b$$

has limited variation in $\mathfrak{A}$.

For let D be a division of $\mathfrak{A}$ into the intervals $d_\iota = (a_\iota, a_{\iota+1})$.

Then
$$V_D \cdot F = \Sigma \, | F(a_{\iota+1}) - F(a_\iota) \, | = \Sigma \, | \int_{a_\iota}^{a_{\iota+1}} f dx \, |$$

$$\leq \Sigma \int_{a_\iota}^{a_{\iota+1}} | f | \, dx \leq M \Sigma \int_{a_\iota}^{a_{\iota+1}} dx = M \Sigma d_\iota = M\overline{\mathfrak{A}}.$$

Thus
$$\text{Max } V_D \cdot F \leq M\overline{\mathfrak{A}},$$

and F has limited variation.

529. 1. *If $f(x)$ has limited variation in the interval $\mathfrak{A}$, the points $\mathfrak{K}$ where* $\text{Osc} f \geq k$, *are finite in number.*

For suppose they were not. Then however large G is taken, we may take n so large that $nk > G$. There exists a division D

of $\mathfrak{A}$, such that there are at least n intervals, each containing a point of $\mathfrak{K}$ within it. Thus for the division D,

$$\Sigma \operatorname{Osc} f \geq nk > G.$$

Thus the variation of f is large at pleasure, and therefore is not limited.

2. *If f has limited variation in the interval $\mathfrak{A}$, its points of discontinuity form an enumerable set.*

This follows at once from 1.

530. 1. *Let $D_1, D_2 \cdots$ be a sequence of superposed divisions, of norms $d_n \doteq 0$, of the interval $\mathfrak{A}$. Let Ω_{D_n} be the sum of the oscillations of f in the intervals of D_n. If* $\operatorname*{Max}_n \Omega_{D_n}$ *is finite, $f(x)$ has limited variation in $\mathfrak{A}$.*

For suppose f does not have limited variation in $\mathfrak{A}$. Then there exists a sequence of divisions $E_1, E_2 \cdots$ such that if Ω_{E_n} is the sum of the oscillations of f in the intervals of E_n, then

$$\Omega_{E_1} < \Omega_{E_2} < \cdots \doteq +\infty. \qquad (1$$

Let us take ν so large that no interval of D_ν contains more than one interval of E_n or at most parts of two E_n intervals. Let $F_n = E_n + D_\nu$. Then an interval δ of D_ν is split up into at most two intervals δ', δ'' in F_n. Let $\omega, \omega', \omega''$ denote the oscillation of f in $\delta, \delta', \delta''$. Then the term ω in D_ν goes over into

$$\omega' + \omega'' \leq 2\,\omega$$

in Ω_{F_n}. Hence if $\operatorname{Max} \Omega_{D_n} = M$,

$$\Omega_{E_n} \leq 2\,\Omega_{D_\nu} \leq 2\,M,$$

which contradicts 1).

2. Let $V_{D_n} = \Sigma \left| f(a_\iota) - f(a_{\iota+1}) \right|$, the summation extended over the intervals $(a_\iota, a_{\iota+1})$ of the division D_n. If $\operatorname*{Max}_n V_{D_n}$ is finite with respect to a sequence of superposed divisions $\{D_n\}$, we cannot say that f has limited variation.

Example. For let $f(x) = 0$, at the rational points in the interval $\mathfrak{A} = (0, 1)$, and $= 1$, at the irrational points. Let D_n be

obtained by interpolating the points $\frac{2m+1}{2^n}$ in $\mathfrak{A}$. Then $f=0$ at the end points a_ι, $a_{\iota+1}$ of the intervals of D_n. Hence $V_{D_n}=0$. On the other hand, $f(x)$ has not limited variation in $\mathfrak{A}$ as is obvious.

531. *Let* $F(x)=\lim_{t=\tau} f(x,t)$, τ *finite or infinite, for* x *in the interval* $\mathfrak{A}$. *Let* $\operatorname{Var} f(x,t) \leq M$ *for each* t *near* τ.

Then $F(x)$ *has limited variation in* $\mathfrak{A}$.

To fix the ideas let τ be finite. Let

$$F=f(x,t)+g(x,t).$$

Then for a division D of $\mathfrak{A}$,

$$V_D F \leq V_D f + V_D g.$$

But

$$V_D g = \Sigma \mid g(a_m) - g(a_{m+1}) \mid,$$

where (a_m, a_{m+1}) are the intervals of D.

But for some $t=t'$ near τ, each

$$g(a_\iota, t') < \frac{\eta}{2s},$$

where s is the number of intervals in the division D.

Thus

$$V_D g < \eta.$$

Hence

$$V_D F < M + \eta,$$

and F has limited variation.

532. *Let* $f(x)$, $g(x)$ *have limited variation in the interval* $\mathfrak{A}$, *then their sum, difference, and product have limited variation.*

If also

$$|g| \geq \gamma > 0 \quad , \quad \text{in } \mathfrak{A}$$

then f/g *has limited variation.*

Let us show, for example, that $h=fg$ has limited variation. For let

$$\left.\begin{array}{ll} \operatorname{Min} f = m \quad , & \operatorname{Min} g = n \\ \operatorname{Osc} f = \omega \quad , & \operatorname{Osc} g = \tau \end{array}\right\} \text{in the interval } d_\iota.$$

Then
$$f = m + \alpha\omega \quad , \quad g = n + \beta\tau \quad , \quad \text{in } d_\iota,$$
$$0 < \alpha < 1 \quad , \quad 0 \leq \beta \leq 1.$$

Thus
$$fg = mn + m\beta\tau + n\alpha\omega + \alpha\beta\omega\tau.$$

Now
$$mn - |m|\tau - |n|\omega - \omega\tau < fg \leq mn + |m|\tau + |n|\omega + \omega\tau.$$

Hence
$$\eta = \text{Osc}\, h \leq 2\{\tau|m| + \omega|n| + \omega\tau\}.$$

But
$$|m|, |n|, \tau \leq \text{some } K.$$

Thus
$$V_D h \leq 4 K\Sigma\omega + 2 K\Sigma\tau,$$
$$< \text{some } G,$$

nd h has limited variation.

533. 1. Let us see what change will be introduced if we eplace the finite divisions D employed up to the present by visions E, which divide the interval $\mathfrak{A} = (a < b)$ into an infinite numerable set of intervals $(a_\iota, a_{\iota+1})$.

Let
$$W_E = \sum_1^\infty |f(a_m) - f(a_{m+1})|, \qquad (1$$

nd
$$W = \text{Max } V_E,$$

r the class of finite or infinite enumerable divisions $\{E\}$.

Obviously
$$W \geq V;$$

ence if W is finite, so is V.

We show that if V is finite, so is W. For suppose W were finite. Then for any $G > 0$, there exists a division E, and an such that the sum of the first n terms in 1) is $\geq G$, or
$$W_{E,n} \geq G. \qquad (2$$

et now D be the finite division determined by the points a_1, $\cdots a_{n+1}$ which figure in 2).

Then
$$V_D \geq G,$$

ence $V = \infty$, which is contrary to our hypothesis.

We show now that V and W are equal, when finite. For let E be so chosen that

$$W - \frac{\epsilon}{2} < W_E \leq W.$$

Now
$$W_E = W_{E,n} + \epsilon' \quad , \quad |\epsilon'| < \epsilon/2$$

if n is sufficiently large.

Let D correspond to the points $a_1\ a_2 \cdots$ in $W_{E,n}$. Then

$$V_D \geq W_{E,n},$$

and hence
$$V_D + \epsilon' \geq W_{E,n} + \epsilon' = W_E.$$

Hence
$$W - V_D < \epsilon.$$

We may therefore state the theorem :

2. *If f has limited variation in the interval $\mathfrak{A}$ with respect to the class of finite divisions D, it has with respect to the class of enumerable divisions E, and conversely. Moreover*

$$\text{Max } V_D = \text{Max } V_E.$$

534. Let us show that *Weierstrass' function F*, considered in 502, *does not have limited variation in any interval $\mathfrak{A} = (\alpha < \beta)$ when $ab > 1$.* Since F is periodic, we may suppose $\alpha > 0$. Let

$$\frac{k}{b^m}, \ \frac{k+1}{b^m}, \cdots \frac{k+\mu}{b^m}$$

be the fractions of denominator b^m which lie in $\mathfrak{A}$.

These points effect a division D_m of $\mathfrak{A}$, and

$$V_{D_m} = \left| F\left(\frac{k}{b^m}\right) - F(\alpha) \right| + \sum_{j=0}^{\mu-1} \left| F\left(\frac{k+j+1}{b^m}\right) - F\left(\frac{k+j}{b^m}\right) \right| + \left| F(\beta) - F\left(\frac{k+\mu}{b^m}\right) \right|.$$

If l is the minimum of the terms F_j under the Σ sign,

$$V_{D_m} \geq \mu l. \tag{1}$$

Now
$$\frac{k-1}{b^m} < \alpha \quad , \quad \frac{k+\mu+1}{b^m} > \beta.$$

Hence
$$\overline{\mathfrak{A}} < \frac{\mu+2}{b^m} \quad , \quad \mu > b^m \overline{\mathfrak{A}} - 2. \tag{2}$$

On the other hand, using the notation and results of 502,

$$b^m x = \iota_m + \xi_m \quad , \quad h = \frac{\eta_m - \xi_m}{b^m};$$

also

$$\left| \frac{F(x+h) - F(x)}{h} \right| \geq a^m b^m \left(\frac{2}{3} - \frac{\pi}{ab-1} \right). \tag{3}$$

Let us now take

$$\xi_m = 0, \qquad \eta_m = +1, \qquad \iota_m = k + j.$$

Then

$$x = \frac{k+j}{b^m} \quad , \quad h = \frac{1}{b^m}.$$

Hence from 3),

$$F_j \geq a^m \left(\frac{2}{3} - \frac{\pi}{ab-1} \right).$$

Thus

$$V_{D_m} \geq a^m \left(\frac{2}{3} - \frac{\pi}{ab-1} \right)(b^m \mathfrak{A} - 2) \quad , \quad \text{by 1), 2).}$$

As $a < 1$, and $ab > 1$, we see that

$$V_{D_m} \doteq +\infty, \text{ as } m \doteq \infty.$$

Non-intuitional Curves

535. 1. Let $f(x)$ be continuous in the interval $\mathfrak{A}$. The graph of f is a continuous curve C. If f has only a finite number of oscillations in $\mathfrak{A}$, and has a tangent at each point, we would call C an ordinary or intuitional curve. It might even have a finite number of angle points, *i.e.* points where the right-hand tangent is different from the left-hand one [cf. I, 366]. But if there were an infinity of such points, or an infinity of points in the vicinity of each of which f oscillates infinitely often, the curve grows less and less clear to the intuition as these singularities increase in number and complexity. Just where the dividing point lies between curves whose peculiarities can be clearly seen by the intuition, and those which cannot, is hard to say. Probably different persons would set this point at different places.

For example, one might ask: Is it possible for a continuous curve to have tangents at a pantactic set of points, and no tangent at another pantactic set? If one were asked to picture such a curve to the imagination, it would probably prove an impossibility.

Yet such curves exist, as Ex. 3 in 501 shows. Such curves might properly be called *non-intuitional*.

Again we might ask of our intuition: Is it possible for a continuous curve to have a tangent at every point of an interval $\mathfrak{A}$, which moreover turns abruptly at a pantactic set of points? Again the answer would not be forthcoming. Such curves exist, however, as was shown in Ex. 2 in 501.

We wish now to give other examples of non-intuitional curves. Since their singularity depends on their derivatives or the nature of their oscillations, they may be considered in this chapter.

Let us first show how to define curves, which, like Weierstrass' curve, have a pantactic set of cusps. To effect this we will extend the theorem of 500, 2, so as to allow $g(x)$ to have a cusp at $x = 0$.

536. *Let $\mathfrak{E} = \{e_n\}$ denote the rational points in the interval $\mathfrak{A} = (-a, a)$. Let $g(x)$ be continuous in $\mathfrak{B} = (-2a, 2a)$, and $= 0$, at $x = 0$. Let $\mathfrak{B}^*$ denote the interval $\mathfrak{B}$ after removing the point $x = 0$. Let g have a derivative in $\mathfrak{B}^*$, such that*

$$|g'(x)| \leq \frac{M}{|x|^\alpha} \quad , \quad \alpha > 0. \tag{1}$$

Then

$$F(x) = \sum \frac{1}{n^{2+\alpha+\beta}} g(x - e_n) = \Sigma a_n g(x - e_n) \quad , \quad \beta > 0$$

*is a continuous function in $\mathfrak{A}$, and $\frac{\Delta F}{\Delta x}$ behaves at $x = e_m$ essentially as $\frac{\Delta g}{\Delta x}$ does at the origin.**

To simplify matters, let us suppose that $\mathfrak{E}$ does not contain the origin. Having established this case, it is easy to dispose of the general case. We begin by ordering the e_n as in 233. Then obviously if

$$e_n = \frac{p}{q} \quad , \quad q > 0 \quad , \quad p \text{ positive or negative,}$$

we have

$$n \geq q.$$

Let

$$e_{mn} = e_m - e_n. \qquad \text{If } e_m = \frac{r}{s},$$

$$|e_{mn}| = \left|\frac{p}{q} - \frac{r}{s}\right| \geq \frac{1}{qs} \geq \frac{1}{mn}. \tag{2}$$

* Cf. Dini, *Theorie der Functionen, etc.*, p. 102 *seq.* Leipzig, 1892.

Let $E(x)$ be the F series after deleting the m^{th} term. Then

$$F(x) = a_m g(x - e_m) + E(x).$$

We show that E has a differential coefficient at $x = e_m$, obtained differentiating E termwise. To this end we show that as $h \doteq 0$,

$$D(h) = \sum a_n \frac{g(e_{mn} + h) - g(e_{mn})}{h} \quad , \quad m \neq n \tag{3}$$

verges to

$$G = \Sigma a_n g'(e_{mn}) \quad , \quad m \neq n. \tag{4}$$

That is, we show

$$\epsilon > 0 \quad , \quad \eta > 0 \quad , \quad |D(h) - G| < \epsilon \quad , \quad 0 < |h| < \eta. \tag{5}$$

Let us break up the sums 3), 4) which figure in 5), into three ts

$$\sum_1^\infty = \sum_1^r + \sum_{r+1}^s + \sum_{s+1}^\infty. \tag{6}$$

Thus

$$|D - G| \leq |D_r - G_r| + |D_{r,s} - G_{r,s}| + |\bar{D}_s - \bar{G}_s| \tag{7}$$

$$\leq A + B + C.$$

Since $g'(e_{mn})$ exists, the first term may be made as small as we ose for an arbitrary but fixed r; thus

$$A < \frac{\epsilon}{3}.$$

Let us now turn to B. We have

$$B \leq |D_{rs}| + |G_{rs}|,$$

$$\frac{g(e_{mn} + h) - g(e_{mn})}{h} = g'(e_{mn} + h') \quad , \quad |h'| < |h|$$

vided $g'(x)$ exists in the interval $(e_{mn}, e_{mn} + h)$.

But by 2),

$$|e_{mn} + h'| \geq |e_{mn}| - |h| \geq \frac{1}{2\,mn} \geq \frac{1}{2\,ms} \quad , \quad \text{for } r < n \leq s$$

$$\eta < \frac{1}{2\,ms}. \tag{8}$$

Thus by 1),

$$|g'(e_{mn} + h')| \leq 2^a M m^a n^a < M_1 n^a \quad , \quad M_1 \text{ a constant.}$$

Hence *a fortiori*, $$| g'(e_{mn}) | < M_1 n^{\alpha}. \tag{9}$$

Now the sum
$$H = \sum \frac{1}{n^{1+\mu}}$$

converges if $\mu > 0$. Hence $H_{p,q}$ and $\overline{H}_p$ may be made as small as we choose, by taking p sufficiently large. Let us note that by 91,
$$\overline{H}_p < \frac{1}{\mu}\frac{1}{p^{\mu}}. \tag{10}$$

Thus if $\mu = \text{Min}\ (\alpha, \beta)$,
$$B \leq | D_{rs} | + | G_{rs} | \leq 2 \sum_{r+1}^{s} \frac{M_1 n^{\alpha}}{n^{2+\alpha+\beta}} = 2\ M_1 H_{r,s} < \frac{\epsilon}{3},$$
for a sufficiently large r.

We consider finally C. We have
$$C \leq | \overline{D}_s | + | \overline{G}_s |$$
$$\leq \frac{1}{|h|}\sum_{s+1}^{\infty} a_n \,| g(e_{mn} + h) | + \frac{1}{|h|}\sum_{s+1}^{\infty} a_n \,| g(e_{mn}) | + | \overline{G}_s |$$
$$\leq C_1 + C_2 + C_3.$$

From 9) we see that
$$C_3 < M_1 \overline{H}_s < \frac{\epsilon}{6},$$

for s sufficiently large. Since $g(x)$ is continuous in $\mathfrak{B}$,
$$| g(x) | < N.$$

Hence
$$C_1 \text{ and } C_2 \leq \frac{1}{|h|}\sum_{s+1}^{\infty}\frac{N}{n^{2+\alpha+\beta}} \leq \frac{N}{|h|}\frac{1}{1+\alpha+\beta}\cdot\frac{1}{s^{1+\alpha+\beta}}$$
$$\leq \frac{N}{1+\alpha+\beta}\cdot\frac{1}{s^{\alpha+\beta}},$$

if $s \geq \frac{1}{|h|}$, on using 10).

Taking s still larger if necessary, we can make
$$C_1,\ C_2 < \frac{\epsilon}{6}.$$

Thus
$$C < \frac{\epsilon}{3}.$$

e reader now sees why we broke the sum 6) into three parts. $\doteq 0$, the middle term contains an increasing number of terms. whatever given value h has, s has a finite value.

us as A, B, C are each $< \epsilon/3$, the relation 5) is established. nce E has a differential coefficient at $x = e_m$, and as

$$\frac{\Delta F}{h} = a_m \frac{\Delta(0)}{h} + \frac{\Delta E}{h},$$

heorem is established.

Example 1. Let

$$g(x) = \sqrt[3]{x^2}.$$

n for $x \neq 0$, $g'(x) = \frac{2}{3}\frac{1}{\sqrt[3]{x}}$. Here $\alpha = \frac{1}{3}$.

$x = 0$, $\qquad Rg'(x) = +\infty \quad , \quad Lg'(x) = -\infty.$

us

$$F(x) = \sum \frac{\sqrt[3]{(x - e_n)^2}}{n^{\frac{1}{3}+\beta}} \quad , \quad \beta > 0$$

ntinuous function, and at the rational points e_m in the in- $\mathfrak{A}$,

$$RF'(x) = +\infty \quad , \quad LF'(x) = -\infty.$$

ce the graph of F has a pantactic set of cuspidal tangents The curve is not monotone in any interval of $\mathfrak{A}$, however

mple 2. Let

$$g(x) = x \sin\frac{1}{x} \quad , \quad x \neq 0$$

$$= 0 \quad , \quad x = 0.$$

n

$$g'(x) = \sin\frac{1}{x} - \frac{1}{x}\cos\frac{1}{x} \quad , \quad x \neq 0.$$

$\alpha = 1$. For $x = 0$,

$$\overline{g}'(x) = +1 \quad , \quad \underline{g}'(x) = -1.$$

Then

$$F(x) = \sum \frac{1}{n^{3+\beta}} (x - e_n) \sin \frac{1}{x - e_n} \quad , \quad \beta > 0$$

is a continuous function in $\mathfrak{A}$, and at the rational point e_m,

$$\overline{F}'(x) = \frac{1}{n^{3+\beta}} + E'(e_m)$$

$$\underline{F}'(x) = -\frac{1}{n^{3+\beta}} + E'(e_m),$$

where E is the series obtained from F by deleting the m^{th} term.

538. *Pompeiu Curves.** Let us now show the existence of curves which have a tangent at each point, and a pantactic set of vertical inflectional tangents.

We first prove the theorem (*Borel*):

Let
$$B(x) = \sum \frac{a_n}{x - e_n} = \sum \frac{a_n}{r_n} \quad , \quad a_n > 0,$$

where $\mathfrak{E} = \{e_n\}$ *is an enumerable set in the interval* $\mathfrak{A}$, and

$$A = \Sigma \sqrt{a_n}$$

is convergent. Then B *converges absolutely and uniformly in a set* $\mathfrak{B} < \mathfrak{A}$, *and* $\widehat{\mathfrak{B}}$ *is as near* $\widehat{\mathfrak{A}}$ *as we choose.*

The points $\mathfrak{D}$ *where adjoint* B *is divergent form a null set.*

For let us enclose each point e_n in an interval δ_n of length $\frac{2\sqrt{a_n}}{k}$, with e_n as center.

The sum of these intervals is

$$\leq \sum \frac{2\sqrt{a_n}}{k} = \frac{2A}{k} < \epsilon,$$

for $k > 0$ sufficiently large. Let now k be fixed. A point x of $\mathfrak{A}$ will not lie in any δ_n if

$$r_n = |x - e_n| > \frac{\sqrt{a_n}}{k}.$$

Then at such a point,

$$\text{Adjoint } B < \Sigma a_n \frac{k}{\sqrt{a_n}} = k \Sigma \sqrt{a_n} = kA.$$

* *Math. Annalen*, v. 63 (1907), p. 326.

$\bar{\mathfrak{B}} > \bar{\mathfrak{A}} - \epsilon$, the points $\mathfrak{D}$ where B does not converge ab-
ly form a null set.

. 1. We now consider the function

$$F(x) = \sum_1^\infty a_n(x - e_n)^{\frac{1}{3}} = \Sigma f_n(x) \tag{1}$$

$\mathfrak{E} = \{e_n\}$ is an enumerable pantactic set in an interval $\mathfrak{A}$, and

$$A = \Sigma a_n \tag{2}$$

onvergent positive term series.

en *F is a continuous function of x in $\mathfrak{A}$.* For $|x - e_n|^{\frac{1}{3}}$ is $<$
M in $\mathfrak{A}$.

us note that each $f_n(x)$ is an increasing function and the
corresponding to it has a vertical inflectional tangent at the
$x = e_n$.

next show that *$F(x)$ is an increasing function in $\mathfrak{A}$.* For let
$''$. Then

$$f_n(x') < f_n(x'').$$

$$F_n(x') < F_n(x'').$$

$$\bar{F}_n(x') \leq \bar{F}_n(x'').$$

$$F(x') < F(x'').$$

Let us now consider the convergence of

$$D(x) = \frac{1}{3}\sum \frac{a_n}{(x - e_n)^{\frac{2}{3}}} \tag{3}$$

ed by differentiating F termwise at the points of $\mathfrak{A} - \mathfrak{E}$.
denote the points in $\mathfrak{A}$ where

$$B = \sum \frac{a_n}{|x - e_n|} \tag{4}$$

ges. We have seen $\mathfrak{D}$ is a null set if

$$\Sigma \sqrt{a_n} \tag{5}$$

is convergent. Let $\mathfrak{A} = \mathfrak{D} + \mathfrak{C}$. Let x be a point of $\mathfrak{C}$, *i.e.* a point where 4) is convergent. We break 3) into two parts

$$D = D_1 + D_2,$$

such that in D_1, each $\xi_n < 1$. Then D_2 is obviously convergent, since each of its terms

$$\frac{a_n}{\xi_n^{\frac{2}{3}}} \leq a_n, \qquad \text{where } \xi_n = |x - e_n|,$$

and the series 2) is convergent.

The series D_1 is also convergent. For as $\xi_n < 1$, the term

$$\frac{a_n}{\xi_n^{\frac{2}{3}}} < \frac{a_n}{\xi_n}$$

and the series 4) converges by hypothesis, at a point x in $\mathfrak{C}$. *Hence $D(x)$ is convergent at any point in $\mathfrak{C}$, and $\bar{\mathfrak{C}} = \bar{\mathfrak{A}}$ when 5) is convergent.*

3. Let C denote the points in $\mathfrak{A}$ where 3) converges. Let $\mathfrak{A} = C + \Delta$.

We next show that $F'(x) = D(x)$, for x in C. For taking x at pleasure in C but fixed,

$$Q(h) = \frac{\Delta F}{\Delta x} = \Sigma\, a_n \frac{(x + h - e_n)^{\frac{1}{3}} - (x - e_n)^{\frac{1}{3}}}{h}, \quad \Delta x = h. \quad (6$$

We now apply 156, 2, showing that Q is uniformly convergent in $(0^*, \eta)$. By direct multiplication we find that

$$\frac{(a+b)^{\frac{1}{3}} - a^{\frac{1}{3}}}{b} = \frac{1}{(a+b)^{\frac{2}{3}} + a^{\frac{1}{3}}(a+b)^{\frac{1}{3}} + a^{\frac{2}{3}}}.$$

Thus 6) gives

$$Q(h) = \sum \frac{a_n}{(x + h - e_n)^{\frac{2}{3}} + (x + h - e_n)^{\frac{1}{3}}(x - e_n)^{\frac{1}{3}} + (x - e_n)^{\frac{2}{3}}}.$$

Let us set

$$h_n = \left(\frac{x + h - e_n}{x - e_n}\right)^{\frac{1}{3}}.$$

Then

$$Q(h) = \sum \frac{1}{1 + h_n + h_n^2} \cdot \frac{a_n}{(x - e_n)^{\frac{2}{3}}} \leq 2 \sum \frac{a_n}{\xi^{\frac{2}{3}}}, \qquad (7$$

for $0 < |h| \leq \eta$, η sufficiently small. As the series on the right is independent of h, Q converges uniformly in $(0^*, \eta)$. Thus by 156, 2

$$F' = D \quad , \quad \text{for any } x \text{ in } C.$$

4. *Let now x be a point of Δ, not in $\mathfrak{E}$. At such a point we show that*

$$F'(x) = +\infty, \tag{8}$$

and thus the curve F has a vertical inflectional tangent. For as D is divergent at x, there exists for each M an m, such that

$$D_m > 2M.$$

But the middle term in 7) shows that for $|h| <$ some η' each term in Q_m is $> \frac{1}{2}$ the corresponding term in D_m. Thus

$$Q_m(h) > M \quad , \quad 0 < |h| < \eta'.$$

Since each term of Q is > 0, as 7) shows,

$$Q(h) > M.$$

Hence 8) is established.

5. Let us finally consider the points $x = e_m$. If Φ denotes the series obtained from F by deleting the m^{th} term, we have

$$\frac{\Delta F}{\Delta x} = \frac{a_m}{h^{\frac{2}{3}}} + \frac{\Delta \Phi}{\Delta x} \quad , \quad \text{for } x = e_m.$$

As F is increasing, the last term is ≥ 0.

Hence

$$F'(x) = +\infty \quad , \quad \text{in } \mathfrak{E}.$$

As a result we see the curve F has at each point a tangent. At an enumerable pantactic set V, it has points of inflection with vertical tangents.

7. Let us now consider the *inverse of the function F*, which we denote by

$$x = G(t). \tag{9}$$

As x in 1) ranges over the interval $\mathfrak{A}$, $t = F(x)$ will range over an interval $\mathfrak{B}$, and by I, 381, the inverse function 9) is a one-valued continuous function of t in $\mathfrak{B}$ which has a tangent at each

point of $\mathfrak{B}$. If W are the points in $\mathfrak{B}$ which correspond to the points V in $\mathfrak{A}$, then the tangent is parallel to the t-axis at the points W, or $G'(t)=0$, at these points. The points W are pantactic in $\mathfrak{B}$.

Let Z denote the points of $\mathfrak{B}$ at which $G'(t)=0$. We show that Z is of the 2° category, and therefore

$$\text{Card } Z = \mathfrak{c}.$$

For $G'(t)$ being of class ≤ 1 in $\mathfrak{B}$, its points of discontinuity form a set of the 1° category, by 486, 2. On the other hand. the points of continuity of G' form precisely the set Z, since the points W are pantactic in $\mathfrak{B}$ and $G'=0$ in W. In passing let us note that the points Z in $\mathfrak{B}$ correspond 1–1 to a set of points $\mathfrak{Z}$ at which the series 3) diverges. For at these points the tangent to F is vertical. But at any point of convergence of 3), we saw in 2 that the tangent is not vertical.

Finally we observe that 3) shows that

$$\text{Min } D(x) > \frac{1}{3} \cdot \frac{1}{\mathfrak{A}^{\frac{2}{3}}} \Sigma a_n \quad , \quad \text{in } \mathfrak{A}.$$

Hence

$$\text{Max } G'(t) \leq \frac{3\,\mathfrak{A}^{\frac{2}{3}}}{\Sigma a_n}.$$

Summing up, we have this result:

8. *Let the positive term series* $\Sigma\sqrt{a_n}$ *converge. Let* $\mathfrak{E}=\{e_n\}$ *be an enumerable pantactic set in the interval* $\mathfrak{A}$. *The Pompeiu curve defined by*

$$F(x) = \Sigma a_n (x-e_n)^{\frac{1}{3}}$$

have a tangent at each point in $\mathfrak{A}$, *whose slope is given by*

$$F'(x) = \frac{1}{3}\sum \frac{a_n}{(x-e_n)^{\frac{2}{3}}},$$

when this series is convergent, i.e. for all x *in* $\mathfrak{A}$ *except a null set. At a point set* $\mathfrak{Z}$ *of the* 2° *category which embraces* $\mathfrak{E}$, *the tangents are vertical. The ordinates of the curve* F *increase with* x.

540. 1. *Faber Curves.** Let $F(x)$ be continuous in the interval $\mathfrak{A}=(0, 1)$. Its graph we denote by F. For simplicity let

* *Math. Annalen*, v. 66 (1908), p. 81.

$(0) = 0,\ F(1) = l_0$. We proceed to construct a sequence of
oken lines or polygons,

$$L_0,\ L_1,\ L_2 \cdots \qquad (1$$

hich converge to the curve F as follows:

As first line L_0 we take the segment joining the end points of
. Let us now divide $\mathfrak{A}$ into n_1 equal intervals

$$\delta_{11},\ \delta_{12} \cdots \delta_{1, n_1} \qquad (2$$

length

$$\delta_1 = \frac{1}{n_1},$$

d having

$$a_{11},\ a_{12},\ a_{13} \cdots \qquad (3$$

end points. As second line L_1 we take the broken line or
lygon joining the points on F whose abscissæ are the points 3).
e now divide each of the intervals 2) into n_2 equal intervals,
tting the $n_1 n_2$ intervals

$$\delta_{21},\ \delta_{22},\ \delta_{23} \cdots \qquad (4$$

length

$$\delta_2 = \frac{1}{n_1 n_2},$$

d having

$$a_{21},\ a_{22},\ a_{23} \cdots \qquad (5$$

end points. In this way we proceed on indefinitely. Let us
ll the points

$$A = \{a_{mn}\}$$

rminal points. The number of intervals in the r^{th} division is

$$\nu_r = n_1 \cdot n_2 \cdots n_r.$$

If $L_m(x)$ denote the one-valued continuous function in $\mathfrak{A}$ whose
lue is the ordinate of a point on L_m, we have

$$F(a_{mn}) = L_m(a_{mn}), \qquad (6$$

ce the vertices of L_m lie on the curve F.

2. *For each x in $\mathfrak{A}$,*

$$\lim_{m=\infty} L_m(x) = F(x). \qquad (7$$

For if x is a terminal point, 7) is true by 6).

If x is not a terminal point, it lies in a sequence of intervals

$$\delta_1 > \delta_2 > \cdots$$

belonging to the 1°, 2° ⋯ division of $\mathfrak{A}$.

Let

$$\delta_m = (a_{m,n}, a_{m,n+1}).$$

Since $F(x)$ is continuous, there exists an s, such that

$$|F(x) - F(a_{m,n})| \leq \frac{\epsilon}{2}, \quad m > s \tag{8}$$

for any x in δ_m. As $L_m(x)$ is monotone in δ_m,

$$\begin{aligned} |L_m(x) - L_m(a_{mn})| &\leq |L_m(a_{mn}) - L_m(a_{m,n+1})| \\ &\leq |F_m(a_{mn}) - F_m(a_{m,n+1})| \\ &\leq \frac{\epsilon}{2} \quad , \quad \text{by 8)}. \end{aligned}$$

Thus

$$|L_m(x) - F_m(a_{mn})| \leq \frac{\epsilon}{2}. \tag{9}$$

Hence from 8), 9),

$$|F(x) - L_m(x)| < \epsilon \quad , \quad m \geq s$$

which is 7).

3. *We can write* 7) *as a telescopic series.* For

$$L_1 = L_0 + (L_1 - L_0)$$

$$L_2 = L_1 + (L_2 - L_1) = L_0 + (L_1 - L_0) + (L_2 - L_1)$$

etc. Hence

$$F(x) = \lim L_n(x) = L_0(x) + \sum_1^\infty \{L_n(x) - L_{n-1}(x)\}.$$

If we set

$$f_0(x) = L_0(x) \quad , \quad f_n(x) = L_n(x) - L_{n-1}(x), \tag{10}$$

we have

$$F(x) = \sum_0^\infty f_n(x), \tag{11}$$

and

$$F_n(x) = \sum_0^n f_s(x) = L_n(x). \tag{12}$$

The function $f_n(x)$, as 10) shows, is the difference between the ordinates of two successive polygons L_{n-1}, L_n at the point x. It may be positive or negative. In any case its graph is a polygon

$_n$ which has a vertex on the x-axis at the end point of each interval δ_{n-1}. Let l_{ns} be the value of $f_n(x)$ at the point $x = a_{ns}$, that is, at a point corresponding to one of the vertices of f_n. We call l_{ns} the *vertex differences* of the polygon L_n.

Let
$$p_n = \operatorname*{Min}_s |l_{ns}| \quad , \quad q_n = \operatorname*{Max}_s |l_{ns}|.$$

Then
$$|f_n(x)| \le q_n \quad , \quad \text{in } \mathfrak{A}. \tag{13}$$

In the foregoing we have supposed $F(x)$ given. Obviously if the vertex differences were given, the polygons 1) could be constructed successively.

We now show:

If
$$\Sigma q_n \tag{14}$$
is convergent,
$$F(x) = \Sigma f_n(x)$$
is uniformly convergent in $\mathfrak{A}$, *and is a continuous function in* $\mathfrak{A}$.

For by 13), 14), F converges uniformly in $\mathfrak{A}$. As each $f_n(x)$ is continuous, F is continuous in $\mathfrak{A}$.

The functions so defined may be called *Faber functions.*

541. 1. We now investigate the *derivatives of Faber's functions*, and begin by proving the theorem:

If
$$\sum_s n_1 \cdots n_s q_s = \Sigma \nu_s q_s \tag{1}$$
converge, the unilateral derivatives of $F(x)$ *exist in* $\mathfrak{A} = (0, 1)$. *Moreover they are equal, except possibly at the terminal points* $A = \{a_{mn}\}$.

For let x be a point not in A. Let x', x'' lie in $V = V_\eta^*(x)$; let $x' - x = h'$, $x'' - x = h''$.

Let
$$Q = \frac{F(x') - F(x)}{h'} - \frac{F(x'') - F(x)}{h''}.$$

Then $F'(x)$ exists at x, if

$$\epsilon > 0 \quad , \quad \eta > 0 \quad , \quad |Q| < \epsilon \quad , \quad \text{for any } x', x'' \text{ in } V. \tag{2}$$

Now

$$|Q| \leq \left|\frac{F_m(x') - F_m(x)}{h'} - \frac{F_m(x'') - F_m(x)}{h''}\right| + \left|\frac{\overline{F}_m(x') - \overline{F}_m(x)}{h'}\right| + \left|\frac{\overline{F}_m(x'') - \overline{F}_m(x)}{h''}\right|$$

$$\leq Q_1 + Q_2 + Q_3.$$

But
$$\left|\frac{f_s(x') - f_s(x)}{x' - x}\right| \leq \frac{2\,q_s}{\delta_s}.$$

Hence

$$Q_2 \leq 2\sum_{s=m+1}^{\infty} \nu_s q_s < \frac{\epsilon}{2} \quad , \quad m \text{ sufficiently large.}$$

Similarly
$$Q_3 < \frac{\epsilon}{2}.$$

Finally, if η is taken sufficiently small, x, x', x'' will correspond to the side of the polygon L_m. Hence using 540, 12), we see that $Q_1 = 0$. Thus 2) holds, and $F'(x)$ exists at x.

If x is a terminal point a_{mn}, and the two points x', x'' are taken on the same side of a_{mn}, the same reasoning shows that the unilateral derivatives exist at a_{mn}. They may, however, be different.

2. *Let* $n_1 = n_2 = \cdots = 2$. *For the differential coefficient* $F'(x)$ *to exist at the terminal point* x, *it is necessary that*

$$\overline{\lim}\ 2^n q_n < \infty. \tag{3}$$

If

$$\overline{\lim}\ 2^n p_n = \infty, \tag{4}$$

the points where the differential coefficient does not exist form a pantactic set in $\mathfrak{A}$.

Let us *first* prove 3). Let $b < a < c$ be terminal points. Then they belong to every division after a certain stage. We will therefore suppose that b, c are consecutive points in the n^{th} division, and a is a point of the $n+1^{\text{st}}$ division falling in the interval $\delta_n = (b, c)$. If a differential coefficient is to exist at a,

$$\frac{F(a) - F(b)}{a - b} \text{ and } \frac{F(a) - F(c)}{a - c} \tag{5}$$

must be numerically less than some M, as $n \doteq \infty$, and hence their sum Q remains numerically $< 2\,M$.

Now

$$F(a) = L_{n+1}(a) \quad , \quad F(b) = L_n(b) \quad , \quad F(c) = L_n(c),$$

$$|a-b| = |a-c| = \delta_n = \frac{1}{2^{n+1}}.$$

Thus $$Q = 2^{n+1}\{2\,L_{n+1}(a) - [L_n(b) + L_n(c)]\}$$

$$= 2 \cdot 2^{n+1}\left\{L_{n+1}(a) - \frac{L_n(b)+L_n(c)}{2}\right\},$$

r $$|Q| = 4 \cdot 2^n l_{n,s} \quad , \quad \text{supposing } a = \alpha_{ns}.$$

Hence $$2^n q_n < M,$$

hich establishes 3).

Let us now consider 4). By hypothesis there exists a sequence $_1 < n_2 < \cdots \doteq \infty$, such that

$$2^{n_m} p_{n_m} > G \quad , \quad m = 1, 2 \cdots,$$

G being large at pleasure. Hence at least one of the difference uotients 5) belonging to this sequence of divisions is numerically rge at pleasure.

3. *If*
$$\lambda = \Sigma l_{ms} \tag{1}$$

absolutely convergent, the functions $F(x)$ have limited variation in .

For $f_m(x)$ is monotone in each interval δ_{ms}. Hence in δ_{ms},

$$\operatorname{Var} f_m = |\,l_{ms} - l_{m,s+1}| \leq |\,l_{ms}| + |\,l_{m,s+1}|.$$

Hence in $\mathfrak{A}$, $$\operatorname{Var} f_m(x) \leq 2\sum_s l_{ms}.$$

Hence
$$\operatorname{Var} F_n(x) \leq 2\sum_{m=1}^{n}\sum_s l_{ms} = 2\lambda \quad , \quad \text{in } \mathfrak{A}.$$

We apply now 531.

542. *Faber Functions without Finite or Infinite Derivatives.*

To simplify matters let us consider the following example. The method employed admits easy generalization and gives a *class* of functions of this type. We use the notation of the preceding sections.

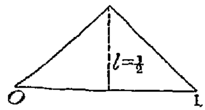


FIG. 1

Let $f_0(x)$ have as graph Fig. 1. We next divide $\mathfrak{A}=(0, 1)$ into 2^{1} equal parts δ_{11}, δ_{12} and take $f_1(x)$ as in Fig. 2. We now divide $\mathfrak{A}$ into 2^{2} equal parts δ_{21}, δ_{22}, δ_{23}, δ_{24} and take $f_2(x)$ as in Fig. 3. The height of the peaks is $l_2=\frac{l}{10^2}$. In the m^{th} division $\mathfrak{A}$ falls into 2^{m} equal parts

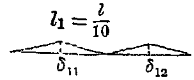


FIG. 2

$$\delta_{m1},\ \delta_{m2}\ \cdots$$

one of which may be denoted by

$$\delta_m=(a_{mn},\ a_{m,n+1}).$$

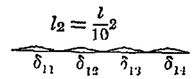


FIG. 3

Its length may be denoted by the same letter, thus

$$\delta_m=\frac{1}{2^{m}}.$$

In Fig. 4, δ_m is an interval of the $m-1^{\text{st}}$ division.

FIG. 4

The maximum ordinate of $f_m(x)$ is $l_m=\frac{l}{10^m}=\frac{1}{2}\cdot\frac{1}{10^m}$. The part of the curve whose points have an ordinate $\leq\frac{1}{2}\,l_m$ have been marked more heavily. The x of such points, form class 1. The other x's make up class 2. With each x in class 1, we associate the points $\alpha_m<\beta_m$ corresponding to the peaks of f_m adjacent to x. Thus $\alpha_m<x<\beta_m$. If x is in class 2, the points α_m, β_m are the adjacent valley points, where $f_m=0$.

Let now x be a point of class 1. The numerators in

$$\frac{f_m(\beta_m)-f(x)}{\beta_m-x}\qquad\frac{f_m(\alpha_m)-f(x)}{\alpha_m-x}\tag{1}$$

have like signs, while their denominators are of opposite sign. Thus the signs of the quotients 1) are different. Similarly if x belongs to class 2, the signs of 1) are opposite. Hence for any x,

e signs of 1) are opposite. It will be convenient to let e_m denote either α_m or β_m. We have

$$|x - e_m| < \delta_m,$$

$$|f_m(x) - f_m(e_m)| \geq \tfrac{1}{2} l_m. \qquad (2$$

Hence

$$\left|\frac{f_m(x) - f_m(e_m)}{x - e_m}\right| > \frac{1}{4}\frac{2^{m!}}{10^m}. \qquad (3$$

On the other hand, for any $x \neq x'$ in δ_m,

$$\left|\frac{f_m(x') - f_m(x)}{x' - x}\right| < \frac{2\, l_m}{\delta_m}.$$

Hence setting $x' = e_n$, and letting $n > m$,

$$|f_m(e_n) - f_m(x)| \leq \frac{l_m}{\delta_m} |e_n - x| < l_m \cdot \frac{\delta_n}{\delta_m}$$

$$\leq \frac{1}{10^m} \cdot \frac{2^{m!}}{2^{n!}} < \frac{1}{10^m} \cdot \frac{2^{n-1!}}{2^{n!}}$$

$$< \frac{1}{10^n} \cdot \frac{1}{10^m}. \qquad (4$$

For if $\log_2 a$ be the logarithm of a with the base 2,

$$n - 1! > \frac{n}{n-1} \log_2 10 \quad , \quad \text{for } n \text{ sufficiently large.}$$

Hence

$$(n-1)!(n-1) > \log_2 10^n.$$

Thus

$$\frac{2^{n!}}{2^{(n-1)!}} > 10^n \quad , \quad \text{or } \frac{2^{n-1!}}{2^{n!}} < \frac{1}{10^n},$$

and this establishes 4).

Let us now extend the definition of the functions $f_n(x)$ by giving them the period 1. The corresponding Faber function $F(x)$ defined by 540, 12) will admit 1 as period. We have now

$$F(e_n) - F(x) = \{f_n(e_n) - f_n(x)\} + \{F_{n-1}(e_n) - F_{n-1}(x)\}$$

$$+ \{\overline{F}_{n+1}(e_n) - \overline{F}_{n+1}(x)\} = T_1 + T_2 + T_3.$$

From 2) we have

$$T_1 \geq \tfrac{1}{2} l_n.$$

As to T_2, we have, using 4) and taking n sufficiently large,

$$| T_2 | < \frac{1}{10^n} \sum_{m=1}^{n-1} \frac{1}{10^m} < \frac{1}{10^n} \sum_{1}^{\infty} \frac{1}{10^m} = \frac{1}{9} \cdot \frac{1}{10^n}$$

$$< \tfrac{2}{9} l_n.$$

Similarly

$$| T_3 | \leq \sum_{m=n+1}^{\infty} | f_m(e_n) - f_m(x) | \leq \sum_{m=n+1}^{\infty} \{ f_m(e_n) + f_m(x) \}$$

$$\leq \sum_{n+1}^{\infty} 2 \, l_m = \frac{1}{9} \cdot \frac{1}{10^n}$$

$$\leq \tfrac{2}{9} l_n.$$

Thus finally

$$| F(e_n) - F(x) | > l_n (\tfrac{1}{2} - \tfrac{2}{9} - \tfrac{2}{9})$$

$$> \tfrac{1}{18} l_n.$$

As

$$| T_1 | > | T_2 | + | T_3 |$$

$$\operatorname{sgn} \frac{F(e_n) - F(x)}{e_n - x} = \operatorname{sgn} \frac{f_n(e_n) - f_n(x)}{e_n - x}.$$

Thus

$$\left| \frac{F(e_n) - F(x)}{e_n - x} \right| > \frac{1}{18} \frac{l_n}{\delta_n}$$

$$> \frac{1}{36} \frac{2^{n!}}{10^n} \doteq \infty.$$

As e_n may be at pleasure α_n or β_n, and as the signs of 1) are opposite, we see that

$$\overline{F}'(x) = +\infty \quad , \quad \underline{F}'(x) = -\infty \, ;$$

and $F(x)$ has neither a finite nor an infinite differential coefficient at any point.

CHAPTER XVI

SUB- AND INFRA-UNIFORM CONVERGENCE

Continuity

543. In many places in the preceding pages we have seen how nportant the notion of uniform convergence is when dealing ith iterated limits. We wish in this chapter to treat a kind of niform convergence first introduced by *Arzelà*, and which we ill call *subuniform.* By its aid we shall be able to give condi-ons for integrating and differentiating series termwise much ore general than those in Chapter V.

We refer the reader to Arzelà's two papers, "*Sulle Serie di unzioni,*" *R. Accad. di Bologna*, ser. V, vol. 8 (1899). Also a fundamental paper by *Osgood*, *Am. Journ. of Math.*, vol. 19 1897), and to another by *Hobson*, *Proc. Lond. Math. Soc.*, ser. 2, ol. 1 (1904).

544. 1. Let $f(x_1 \cdots x_m,\ t_1 \cdots t_n) = f(x,\ t)$ be a function of two ts of variables. Let $x = (x_1 \cdots x_m)$ range over $\mathfrak{X}$ in an m-way pace, and $t = (t_1 \cdots t_n)$ range over $\mathfrak{T}$ in an n-way space. As x nges over $\mathfrak{X}$ and t over $\mathfrak{T}$, the point $(x_1 \cdots t_1 \cdots) = (x,\ t)$ will nge over a set $\mathfrak{A}$ lying in a space $\mathfrak{R}_p$, $p = m + n$.

Let τ, finite or infinite, be a limiting point of $\mathfrak{T}$.

Let
$$\lim_{t=\tau} f(x_1 \cdots x_m,\ t_1 \cdots t_n) = \phi(x_1 \cdots x_m) \qquad \text{in } \mathfrak{X}.$$

Let the point x range over $\mathfrak{B} \leq \mathfrak{X}$, while t remains fixed, then e point $(x,\ t)$ will range over a *layer of ordinate* t, which we ill denote by $\mathfrak{L}_t$. We say x belongs to or is associated with this yer.

We say now that $f \doteq \phi$, *subuniformly* in $\mathfrak{X}$ when for each $\epsilon > 0$, > 0 :

1° There exists a finite number of layers $\mathfrak{L}_t$ whose ordinates t lie in $V_\eta^*(\tau)$.

2° Each point x of $\mathfrak{X}$ is associated with one or more of these layers. Moreover if $x=a$ belongs to the layer $\mathfrak{L}_t$, all the points x in some $V_\delta(a)$ also belong to $\mathfrak{L}_t$.

3° $$|f(x, t)-\phi(x)|<\epsilon$$

while (x, t) ranges over *any* one of the layers $\mathfrak{L}_t$. When $m=1$, that is when there is but a single variable x which ranges over an interval, the layers reduce to segments. For this reason Arzelà calls the convergence *uniform in segments*.

2. In case that subuniform convergence is applied to the series

$$F(x_1 \cdots x_m)=\Sigma f_n(x_1 \cdots x_m)$$

convergent in $\mathfrak{A}$, we may state the definition as follows:

F converges subuniformly in $\mathfrak{A}$ when

1° For each $\epsilon>0$, and for each ν there exists a finite set of layers of ordinates $\geq \nu$, call them

$$\mathfrak{L}_1,\ \mathfrak{L}_2 \cdots \qquad (2$$

such that each point x of $\mathfrak{A}$ belongs to one or more of them, and if $x=a$ belongs to $\mathfrak{L}_m$, then all the points of $\mathfrak{A}$ near a also belong to $\mathfrak{L}_m$.

2° $$|\overline{F}_n(x_1 \cdots x_m)|<\epsilon$$

as the point (x, n) ranges over any one of the layers 2).

545. *Example.* Let

$$F(x)=\sum_1^\infty\left\{\frac{nx}{1+n^2x^2}-\frac{(n-1)x}{1+(n-1)^2x^2}\right\} \qquad \text{in } \mathfrak{A}=(-1, 1).$$

Here

$$F_n(x)=\frac{nx}{1+n^2x^2}, \quad F(x)=0.$$

The series converges uniformly in $\mathfrak{A}$, except at $x=0$. The convergence is therefore not uniform in $\mathfrak{A}$; it is, however, subuniform. For

$$|\overline{F}_n(x)|=\frac{n|x|}{1+n^2x^2}.$$

Hence taking m at pleasure and fixed,

$$|\bar{F}_m| < \epsilon \quad , \quad x \text{ in } s_1 = (-\delta, \delta),$$

sufficiently small. On the other hand,

$$|\bar{F}_n| \leq \frac{n}{1 + \frac{1}{4}n^2\delta^2} \qquad \text{in } (-1, -\tfrac{1}{2}\delta) + (\tfrac{1}{2}\delta, 1) = s_2 + s_3.$$

Thus for n sufficiently large,

$$|\bar{F}_n| < \epsilon.$$

Hence we need only three segments s_1, s_2, s_3 to get subuniform convergence.

546. 1. *Let $f(x_1 \cdots x_m, t_1 \cdots t_n) \doteq \phi(x_1 \cdots x_m)$ in $\mathfrak{X}$, as $t \doteq \tau$, finite or infinite. Let $f(x, t)$ be continuous in $\mathfrak{X}$ for each t near τ. For ϕ to be continuous at the point $x = a$ in $\mathfrak{X}$, it is necessary that for each $\epsilon > 0$, there exists an $\eta > 0$, and a d_t for each t in $V_\eta^*(\tau)$ such that*

$$|f(x, t) - \phi(x)| < \epsilon \tag{1}$$

for each t in V_η and for any x in $V_{d_t}(a)$.

It is sufficient if there exists a single $t = \beta$ in $V_\eta^*(\tau)$ for which the inequality 1) holds for any x in some $V_\delta(a)$.

It is *necessary*. For since ϕ is continuous at $x = a$,

$$|\phi(x) - \phi(a)| < \frac{\epsilon}{3} \quad , \quad \text{for any } x \text{ in some } V_\delta(a).$$

Also since $f \doteq \phi$,

$$|f(a, t) - \phi(a)| < \frac{\epsilon}{3} \quad , \quad \text{for any } t \text{ in some } V_\eta^*(\tau).$$

Finally, since f is continuous in x for any t near τ,

$$|f(x, t) - f(a, t)| < \frac{\epsilon}{3} \quad , \quad \text{for any } x \text{ in some } V_{\delta_t}(a).$$

Adding these three inequalities we get 1), on taking

$$d_t < \delta, \delta_t.$$

It is sufficient. For by hypothesis

$$|f(x, \beta) - \phi(x)| > \frac{\epsilon}{3} \quad , \quad \text{for any } x \text{ in some } V_{\delta'}(a);$$

and hence in particular.

$$|f(a, \beta) - \phi(a)| < \frac{\epsilon}{3}.$$

Also since $f(x, \beta)$ is continuous in x,

$$|f(x, \beta) - f(a, \beta)| < \frac{\epsilon}{3} \quad , \quad \text{for any } x \text{ in some } V_{\delta''}(a).$$

Thus if $\delta < \delta', \delta''$, these unequalities hold simultaneously. Adding them we get

$$|\phi(x) - \phi(a)| < \epsilon \quad , \quad \text{for any } x \text{ in } V_\delta(a),$$

and thus ϕ is continuous at $x = a$.

2. As a corollary we get:

Let $$F(x) = \Sigma f_{\iota_1 \cdots \iota_m}(x_1 \cdots x_m)$$

converge in $\mathfrak{A}$, each term being continuous in $\mathfrak{A}$. For $F(x)$ to be continuous at the point $x = a$ in $\mathfrak{A}$, it is necessary that for each $\epsilon > 0$, and for any cell $R_\mu >$ some R_λ, there exists a δ_μ such that

$$|\overline{F}_\mu(x)| < \epsilon \quad , \quad \textit{for any } x \textit{ in } V_{\delta_\mu}(a).$$

It is sufficient if there exists an R_λ and a $\delta > 0$ such that

$$|\overline{F}_\lambda(x)| < \epsilon \quad , \quad \textit{for any } x \textit{ in } V_\delta(a).$$

547. 1. *Let $\lim_{x=\tau} f(x_1 \cdots x_m, t_1 \cdots t_n) = \phi(x_1 \cdots x_m)$ in $\mathfrak{X}$, τ finite or infinite. Let $f(x, t)$ be continuous in $\mathfrak{X}$ for each t near τ.*

1° *If $f \doteq \phi$ subuniformly in $\mathfrak{X}$, ϕ is continuous in $\mathfrak{X}$.*

2° *If $\mathfrak{X}$ is complete, and ϕ is continuous in $\mathfrak{X}$, $f \doteq \phi$ subuniformly in $\mathfrak{X}$.*

To prove 1°. Let $x = a$ be a point of $\mathfrak{X}$. Let $\epsilon > 0$ be taken at pleasure and fixed. Then there is a layer $\mathfrak{L}_\beta$ to which the point a belongs and such that

$$|f(x, t) - \phi(x)| < \epsilon, \tag{1}$$

hen (x, t) ranges over the points of $\mathfrak{L}_\beta$. But then 1) holds for $=\beta$ and x in some $V_\delta(a)$. Thus the condition of 546, 1 is satis-ed.

To prove 2°. Since ϕ is continuous at $x=a$, the relation 1) olds by 546, 1, for each t in $V_\eta^*(\tau)$ and for any x in $V_{d_t}(a)$. Vith the point a let us associate a cube $C_{a,t}$ lying in $D_{d_t}(a)$ and aving a as center. Then each point of $\mathfrak{X}$ lies within a cube. Hence by Borel's theorem there exists a finite number of these ubes C, such that each point of $\mathfrak{X}$ lies within one of them, say

$$C_{a_1 t_1} \quad , \quad C_{a_2 t_2} \cdots \tag{2}$$

ut the cubes 2) determine a set of layers

$$\mathfrak{L}_{t_1} \quad , \quad \mathfrak{L}_{t_2} \cdots \tag{3}$$

ich that 1) holds as (x, t) ranges over the points of $\mathfrak{A}$ in each yer of 3). Thus the convergence of f to ϕ is subuniform in $\mathfrak{X}$.

2. As a corollary we have the theorem:

Let
$$F(x_1 \cdots x_m) = \Sigma f_{\iota_1 \cdots \iota_n}(x_1 \cdots x_m)$$

nverge in $\mathfrak{X}$, each f_ι being continuous in $\mathfrak{X}$. If F converges sub-uniformly in $\mathfrak{X}$, F is continuous in $\mathfrak{X}$. If $\mathfrak{X}$ is complete and F is ntinuous in $\mathfrak{X}$, F converges subuniformly in $\mathfrak{X}$.

548. 1. *Let*
$$F(x) = \Sigma f_{\iota_1 \cdots \iota_n}(x_1 \cdots x_m)$$
nverge in $\mathfrak{A}$.

Let the convergence be uniform in $\mathfrak{A}$ except possibly for the points a complete discrete set $\mathfrak{B} = \{b\}$. For each b, let there exist a λ_0 ich that for any $\lambda \geq \lambda_0$,

$$\lim_{x=b} \overline{F}_\lambda(x) = 0.$$

Then F converges subuniformly in $\mathfrak{A}$.

For let D be a cubical division of norm d of the space $\mathfrak{R}_m$ in hich $\mathfrak{A}$ lies. We may take d so small that $\overline{\mathfrak{B}}_D$ is small at easure. Let B_D denote the cells of D containing points of $\mathfrak{A}$ ut none of $\mathfrak{B}$. Then by hypothesis F converges uniformly in B_D. hus there exists a μ_0 such that for any $\mu \geq \mu_0$,

$$|\overline{F}_\mu(x)| < \epsilon \quad , \quad \text{for any } x \text{ of } \mathfrak{A} \text{ in } B_D.$$

At a point b of $\mathfrak{B}$, there exists by hypothesis a $V_\delta(b)$ and a λ_0 such that for each $\lambda \geq \lambda_0$

$$|\bar{F}_\lambda(x)| < \epsilon \quad , \quad \text{for any } x \text{ in } V_\delta(b).$$

Let $C_{b,\lambda}$ be a cube lying in $D_\delta(b)$, having b as center. Since $\mathfrak{B}$ is complete there exists a finite number of these cubes

$$C_{b_1\lambda_1} \quad , \quad C_{b_2\lambda_2} \cdots \tag{1}$$

such that each point of $\mathfrak{B}$ lies within one of them.

Moreover

$$|\bar{F}_{\lambda_\kappa}(x)| < \epsilon,$$

for any x of $\mathfrak{A}$ lying in the κ^{th} cube of 1).

As B_D embraces but a finite number of cubes, and as the same is true of 1), there is a finite set of layers $\mathfrak{L}$ such that

$$|\bar{F}_\nu(x)| < \epsilon \quad , \quad \text{in each } \mathfrak{L}.$$

The convergence is thus subuniform, as λ, μ are arbitrarily large.

2. The reasoning of the preceding section gives us also the theorem:

Let

$$\lim_{t=\tau} f(x_1 \cdots x_m,\ t_1 \cdots t_n) = \phi(x_1 \cdots x_m)$$

in $\mathfrak{X}$, τ finite or infinite. Let the convergence be uniform in $\mathfrak{X}$ except possibly for the points of a complete discrete set $\mathfrak{E} = \{e\}$. For each point e, let there exist an η such that setting $\epsilon(x, t) = f(x, t) - \phi(x)$,

$$\lim_{x=e} \epsilon(x, t) = 0 \quad , \quad \textit{for any } t \textit{ in } V_\eta^*(\tau).$$

Then $f \doteq \phi$ subuniformly in $\mathfrak{X}$.

3. As a special case of 1 we have the theorem:

Let

$$F(x) = f_1(x) + f_2(x) + \cdots$$

converge in $\mathfrak{A}$, and converge uniformly in $\mathfrak{A}$, except at $x = \alpha_1, \cdots x = \alpha_s$. At $x = \alpha_\iota$ let there exist a ν_ι such that

$$\lim_{x=\alpha_\iota} \bar{F}_{n_\iota}(x) = 0 \quad , \quad n_\iota \geq \nu_\iota \quad , \quad \iota = 1, 2 \cdots s.$$

Then F converges subuniformly in $\mathfrak{A}$.

4. When
$$\lim_{t=\tau} f(x, t) = \phi(x)$$
e will often set
$$f(x, t) = \phi(x) + \epsilon(x, t),$$
nd call ϵ the *residual function*.

549. *Example 1.*

$$f(x, n) = \frac{n^\lambda x^\alpha}{e^{n^\mu x^\beta}} \doteq \phi(x) = 0 \quad , \quad \text{for } n \doteq \infty \text{ in } \mathfrak{A} = (0 < a),$$
$$\alpha, \beta, \lambda \geq 0 \quad , \quad \mu > 0.$$

The convergence is subuniform in $\mathfrak{A}$. For $x = 0$ is the only ossible point of non-uniform convergence, and for any m,

$$|\epsilon(x, m)| = \frac{m^\lambda x^\alpha}{e^{m^\mu x^\beta}} \doteq 0 \quad , \quad \text{as } x \doteq 0.$$

Example 2. $\quad f(x, n) = \dfrac{n^\lambda x^\alpha}{c + n^\mu x^\beta} \doteq \phi(x) = 0 \quad , \quad \text{as } n \doteq \infty,$

$$x \text{ in } \mathfrak{A} = (0 < a) \quad , \quad \alpha, \beta, \lambda, \mu > 0 \quad , \quad \mu > \lambda \quad , \quad c > 0.$$

The convergence is uniform in $\mathfrak{B} = (e < a)$, where $e > 0$. For

$$|\epsilon(x, n)| \leq \frac{n^\lambda a^\alpha}{c + n^\mu e^\beta} \quad , \quad \text{in } \mathfrak{B}$$
$$< \frac{a^\alpha}{e^\beta} \cdot \frac{n^\lambda}{n^\mu}$$
$$< \epsilon \quad , \quad \text{for } n > \text{some } m.$$

Thus the convergence is uniform in $\mathfrak{A}$, except possibly at $x = 0$. he convergence is subuniform in $\mathfrak{A}$. For obviously for a given n

$$\lim_{x=0} f(x, n) = 0.$$

550. 1. *Let* $\lim_{t=\tau} f(x_1 \cdots x_m t_1 \cdots t_n) = \phi(x_1 \cdots x_m)$ *in* $\mathfrak{X}$, τ *finite infinite.*

Let the convergence be uniform in $\mathfrak{X}$ *except at the points*

$$\mathfrak{B} = (b_1, b_2, \cdots b_p).$$

For the convergence to be sub-uniform in $\mathfrak{X}$, it is necessary that for each b in $\mathfrak{B}$, and for each $\epsilon > 0$, there exists a $t = \beta$ near τ, such that

$$\overline{\lim_{x=b}} \, | \epsilon(x, t) | \geq \epsilon. \tag{1}$$

For if the convergence is subuniform, there exists for each ϵ and $\eta > 0$ a finite set of layers $\mathfrak{L}_t$, t in $V_\eta^*(\tau)$ such that

$$| \epsilon(x, t) | < \epsilon \quad , \quad x \text{ in } \mathfrak{L}_t.$$

Now the point $x = b$ lies in one of these layers, say in $\mathfrak{L}_\beta$. Then

$$| \epsilon(x, \beta) | < \epsilon \quad , \quad \text{for all } x \text{ in some } V^*(b).$$

But then 1) holds.

2. *Example.* Let
$$F(x) = \sum_0^\infty x^n(1 - x).$$

This is the series considered in 140, Ex. 2.

F converges uniformly in $\mathfrak{A} = (-1, 1)$, except at $x = 1$.

As
$$\overline{F}_m(x) = -x^m,$$
we see that
$$\lim_{x=1} \overline{F}_m(x) = -1.$$

Hence F is not subuniformly convergent in $\mathfrak{A}$.

Integrability

551. 1. *Infra-uniform Convergence.* It often happens that

$$f(x_1 \cdots x_m t_1 \cdots t_n) \doteq \phi(x_1 \cdots x_m)$$

subuniformly in $\mathfrak{X}$ except possibly at certain points $\mathfrak{E} = \{e\}$ forming a discrete set. To be more specific, let Δ be a cubical division of $\mathfrak{R}_m$ in which $\mathfrak{X}$ lies, of norm δ. Let X_Δ denote those cells containing points of $\mathfrak{X}$, but none of $\mathfrak{E}$. Since $\mathfrak{E}$ is discrete, $\overline{X}_\Delta \doteq \overline{\mathfrak{X}}$. Suppose now $f \doteq \phi$ subuniformly in any X_Δ; we shall say the convergence is *infra-uniform* in $\mathfrak{X}$. When there are no exceptional points, infra-uniform convergence goes over into subuniform convergence.

This kind of convergence Arzelà calls uniform convergence by segments, in general.

2. We can make the above definition independent of the set $\mathfrak{E}$, and this is desirable at times.

Let $\mathfrak{X} = (X, \mathfrak{x})$ be an unmixed division of $\mathfrak{X}$ such that $\overline{\mathfrak{x}}$ may be taken small at pleasure. If $f \doteq \phi$ subuniformly in each X, we say the convergence is infra-uniform in $\mathfrak{X}$.

3. Then to each $\epsilon, \eta > 0$, and a given X, there exists a set of layers $\mathfrak{l}_1, \mathfrak{l}_2 \cdots, t$ in $V_\eta{}^*(\tau)$, such that the residual function $\epsilon(x, t)$ is numerically $< \epsilon$ for each of these layers. As the projections of these layers $\mathfrak{l}$ do not in general embrace all the points of $\mathfrak{X}$, we call them *deleted layers*.

4. The points $\mathfrak{x}$ we shall call the *residual points*.

5. *Example 1.*
$$F = \sum_0^\infty \frac{x^2}{(1+nx^2)(1+(n+1)x^2)}.$$

This series was studied in 150. We saw that it converges uniformly in $\mathfrak{A} = (0, 1)$, except at $x = 0$.

As
$$\overline{F}_n(x) = \frac{1}{1+nx^2} \quad , \quad x \neq 0$$

and as this $\doteq 1$ as $x \doteq 0$ for an arbitrary but fixed n, F does not converge subuniformly in $\mathfrak{A}$, by 550. The series converges infra-uniformly in $\mathfrak{A}$, obviously.

6. *Example 2.*
$$F = \sum_0^\infty x^n(1-x).$$

This series was considered in 550, 2. Although it does not converge subuniformly in an interval containing the point $x = 1$, the convergence is obviously infra-uniform.

552. 1. *Let* $\lim_{x=\tau} f(x_1 \cdots x_m t_1 \cdots t_n) = \phi(x_1 \cdots x_m)$ *be limited in* $\mathfrak{X}$, *finite or infinite. For each* t *near* τ, *let* f *be limited and R-integrable in* $\mathfrak{X}$. *For* ϕ *to be R-integrable in* $\mathfrak{X}$, *it is sufficient that* $f \doteq \phi$ *infra-uniformly in* $\mathfrak{X}$. *If* $\mathfrak{X}$ *is complete, this condition is necessary.*

It is sufficient. We show that for each $\epsilon, \omega > 0$ there exists a division D of $\mathfrak{R}_m$ such that the cells in which

$$\text{Osc } \phi \geq \omega \tag{1}$$

have a volume $< \sigma$. For setting as usual

$$f = \phi + \epsilon,$$

we have in any point set,

$$\text{Osc } \phi \leq \text{Osc} f + \text{Osc } \epsilon.$$

Using the notation of 551,

$$| \epsilon(x, t) | < \frac{\omega}{4}$$

in the finite set of deleted layers $\mathfrak{l}_1, \mathfrak{l}_2 \cdots$ corresponding to $t = t_1, t_2 \cdots$ For each of these ordinates $t_\iota, f(x, t_\iota)$ is integrable in $\mathfrak{X}$. There exists, therefore, a rectangular division D of $\mathfrak{R}_m$, such that those cells in which

$$\text{Osc} f(x, t_\iota) \geq \frac{\omega}{2}$$

have a content $< \frac{\sigma}{2}$, whichever ordinate t_ι is used. Let E be a division of $\mathfrak{R}_m$ such that the cells containing points of the residual set $\mathfrak{r}$ have a content $< \sigma/2$. Let $F = D + E$. Then those cells of F in which

$$\text{Osc} f(x, t_\iota) \geq \frac{\omega}{2}, \quad \text{or} \quad \text{Osc} | \epsilon(x, t_\iota) | \geq \frac{\omega}{2}$$

$\iota = 1, 2 \cdots$ have a content $< \sigma$. Hence those cells in which 1) holds have a content $< \sigma$.

It is necessary, if $\mathfrak{X}$ is complete. For let

$$t_1, t_2 \cdots \doteq \tau.$$

Since ϕ and $f(x, t_n)$ are integrable, the points of discontinuity of $\phi(x)$ and of $f(x, t_n)$ are null sets by 462, 6. Hence if $\mathfrak{C}, \mathfrak{C}_\iota$ denote the points of continuity of $\phi(x)$ and $f(x, t)$ in $\mathfrak{X}$,

$$\widehat{\mathfrak{C}} = \widehat{\mathfrak{C}}_\iota = \widehat{\mathfrak{X}},$$

since $\mathfrak{X}$ is measurable, as it is complete.

Let $$\mathfrak{G} = \text{Qdv}\,\{\mathfrak{C}_t\},$$

then $$\hat{\hat{\mathfrak{G}}} = \hat{\hat{\mathfrak{X}}}$$

by 410, 6.

Let $$\mathfrak{D} = Dv(\mathfrak{C}, \mathfrak{G}),$$

then $$\hat{\hat{\mathfrak{D}}} = \hat{\hat{\mathfrak{X}}}, \tag{1}$$

as we proceed to show. For if $G = \mathfrak{X} - \mathfrak{G}$,

$$\mathfrak{C} = Dv(\mathfrak{C}, \mathfrak{G}) + Dv(\mathfrak{C}, G) = \mathfrak{D} + Dv(\mathfrak{C}, G).$$

But G is a null set. Hence Meas $Dv(\mathfrak{C}, G) = 0$, and thus $\hat{\hat{\mathfrak{C}}} = \hat{\hat{\mathfrak{X}}} = \hat{\hat{\mathfrak{D}}}$, which is 1).

Let now ξ be a point of $\mathfrak{D}$, let it lie in $\mathfrak{C}_{t_1}, \mathfrak{C}_{t_2} \cdots$ where $t_1, t_2 \cdots$ form a monotone sequence $\doteq \tau$. Then since

$$f(\xi, t_n) \doteq \phi(\xi),$$

there is an m such that

$$|\,\epsilon(\xi, t_n)\,| < \frac{\epsilon}{3} \quad , \quad \text{for any } n > m. \tag{2}$$

But ξ lying in $\mathfrak{D}$, it lies in $\mathfrak{C}$ and $\mathfrak{C}_{t_n}$.

Thus $$|\,\phi(x) - \phi(\xi)\,| < \frac{\epsilon}{3},$$

$$|\,f(x, t_n) - f(\xi, t_n)\,| < \frac{\epsilon}{3},$$

for any x in $V_\delta(\xi)$. Hence

$$|\,\epsilon(x, t_n) - \epsilon(\xi, t_n)\,| < \frac{2\,\epsilon}{3} \quad , \quad x \text{ in } V_\delta(\xi). \tag{3}$$

Now
$$\epsilon(x, t_n) = \epsilon(x, t_n) - \epsilon(\xi, t_n) + \epsilon(\xi, t_n).$$

Hence from 2), 3),

$$|\,\epsilon(x, t_n)\,| < \epsilon \quad , \quad \text{for any } x \text{ in } V_\delta(\xi).$$

Thus associated with the point ξ, there is a cube Γ lying in $D_\delta(\xi)$, having ξ as center. As $D = \mathfrak{X} - \mathfrak{D}$ is a null set, each of its points can be enclosed within cubes C, such that the resulting enclosure

$\mathfrak{E}$ has a measure $< \sigma$, small at pleasure. Thus each point of $\mathfrak{X}$ lies within a cube. By Borel's theorem there exists a finite set of these cubes

$$\Gamma_1, \Gamma_2 \cdots \Gamma_r \quad ; \quad C_1, C_2 \cdots C_s,$$

such that each point of $\mathfrak{X}$ lies within one of them. But corresponding to the Γ's, are layers

$$\mathfrak{L}_1, \mathfrak{L}_2, \cdots \mathfrak{L}_r$$

such that in each of them

$$| \epsilon(x, t) | < \epsilon.$$

Thus $f \doteq \phi$ subuniformly in $X = (\Gamma_1, \Gamma_2 \cdots \Gamma_r)$. Let $\mathfrak{x}$ be the residual set. Obviously $\bar{\mathfrak{x}} < \sigma$. Thus the convergence is infra-uniform.

2. As a corollary we have:

Let $$F(x) = \Sigma f_{\iota_1 \cdots \iota_n}(x_1 \cdots x_m)$$

converge in $\mathfrak{A}$. Let F be limited, and each f_ι be limited and R-integrable in $\mathfrak{A}$. For F to be R-integrable in $\mathfrak{A}$, it is sufficient that F converges infra-uniformly in $\mathfrak{A}$.

If $\mathfrak{A}$ is complete, this condition is necessary.

553. *Infinite Peaks.* 1. Let $\lim_{t=\tau} f(x_1 \cdots x_m t_1 \cdots t_n) = \phi(x)$ in $\mathfrak{X}$, τ finite or infinite. Although $f(x, t)$ is limited in $\mathfrak{X}$ for each t near τ, and although $\phi(x)$ is also limited in $\mathfrak{X}$, we cannot say that

$$|f(x, t)| < \text{some } M \tag{1}$$

for any x in $\mathfrak{X}$ and any t near τ, as is shown by the following

Example. Let $f(x, t) = \dfrac{tx}{e^{tx^2}} \doteq \phi(x) = 0$, as $t \doteq \infty$ for x in $\mathfrak{X} = (-\infty, \infty)$.

It is easy to see that the peak of f becomes infinitely high as $n \doteq \infty$.

In fact, for $x = \dfrac{1}{\sqrt{t}}$, $f = \dfrac{\sqrt{t}}{e}$. Thus the peak is at least as high as $\dfrac{\sqrt{t}}{e}$, which $\doteq \infty$.

The origin is thus a point in whose vicinity the peaks of the amily of curves $f(x, t)$ are infinitely high. In general, if the eaks of

$$f(x_1 \cdots x_m\ t_1 \cdots t_n)$$

n the vicinity V_δ of $x = \xi$ become infinitely high as $t \doteq \tau$, however mall δ is taken, we say *ξ is a point with infinite peaks.*

On the other hand, if the relation 1) holds for all x and t involved, we shall say $f(x, t)$ is *uniformly limited.*

2. *If* $\lim\limits_{t=\tau} f(x_1 \cdots x_m\ \ t_1 \cdots t_n) = \phi(x_1 \cdots x_m)$, *and if* $f(x, t)$ *is niformly limited in* $\mathfrak{X}$, *then* ϕ *is limited in* $\mathfrak{X}$.

For x being taken at pleasure in $\mathfrak{X}$ and fixed, $\phi(x)$ is a limit oint of the points $f(x, t)$ as $t \doteq \tau$. But all these points lie in ome interval $(-G, G)$ independent of x. Hence ϕ lies in this nterval.

3. *If $\mathfrak{X}$ is complete, the points $\mathfrak{K}$ in $\mathfrak{X}$ with infinite peaks also form complete set. If these points $\mathfrak{K}$ are enumerable, they are discrete.*

That $\mathfrak{K}$ is complete is obvious. But then $\overline{\mathfrak{K}} = \widehat{\mathfrak{K}} = 0$, as $\mathfrak{K}$ is numerable.

554. 1. *Let* $\lim\limits_{t=\tau} f(x_1 \cdots x_m\, t_1 \cdots t_n) = \phi(x_1 \cdots x_m)$ *in* $\mathfrak{X}$, *metric or omplete. Let* $f(x, t)$ *be uniformly limited in* $\mathfrak{X}$, *and R-integrable or each t near τ. For the relation*

$$\lim_{t=\tau} \int_{\mathfrak{X}} f(x, t) = \int_{\mathfrak{X}} \phi(x)$$

hold, it is sufficient that $f \doteq \phi$ infra-uniformly in $\mathfrak{X}$. If $\mathfrak{X}$ is r each t complete, this condition is necessary.

For by 552, ϕ is R-integrable if $f \doteq \phi$ infra-uniformly, and when is complete, this condition is necessary. By 424, 4, each $f(x, t)$ measurable. Thus we may apply 381, 2 and 413, 2.

2. As a corollary we have the theorem:

Let
$$F(x) = \Sigma f_{\iota_1 \cdots \iota_n}(x_1 \cdots x_m)$$

nverge in the complete or metric field $\mathfrak{A}$. Let the partial sums F_λ be niformly limited in $\mathfrak{A}$. Let each term f_ι be limited and R-integrable $\mathfrak{A}$. Then for the relation

$$\int_{\mathfrak{A}} F = \Sigma \int_{\mathfrak{A}} f_\iota$$

to hold it is sufficient that F is infra-uniformly convergent in $\mathfrak{A}$. If $\mathfrak{A}$ is complete, this condition is necessary.

555. *Example 1.* Let us reconsider the example of 150,

$$F(x) = \sum_0^\infty \frac{x^2}{(1+nx^2)(1+(n+1)x^2)}.$$

We saw that we may integrate termwise in $\mathfrak{A} = (0, 1)$, although F does not converge uniformly in $\mathfrak{A}$. The only point of non-uniform convergence is $x = 0$. In 551, 5, we saw that it converges, however, infra-uniformly in $\mathfrak{A}$. As

$$|F_n(x)| \leq 1 \quad , \quad \text{for any } x \text{ in } \mathfrak{A}, \text{ and for every } n,$$

all the conditions of 554 are satisfied and we can integrate the series termwise, in accordance with the result already obtained in 150.

Example 2. Let $F(x) = \sum_1^\infty \left\{ \dfrac{nx}{e^{nx^2}} - \dfrac{(n-1)x}{e^{(n-1)x^2}} \right\} = 0.$

Then
$$F_n(x) = \frac{nx}{e^{nx^2}}.$$

We considered this series in 152, 1. We saw there that this series cannot be integrated termwise in $\mathfrak{A} = (0 < a)$. It is, however, subuniformly convergent in $\mathfrak{A}$ as we saw in 549, Ex. 1. We cannot apply 554, however, as F_n is not uniformly limited. In fact we saw in 152, 1, that $x = 0$ is a point with an infinite peak.

Example 3.
$$F(x) = \sum_0^\infty x^n(1-x).$$

We saw in 551, 6, that F converges infra-uniformly in $\mathfrak{A} = (0, 1)$. Here

$$F_n(x) | = | 1 - x^n | < \text{some } M,$$

for any x in $\mathfrak{A} = (0 \leq u)$, $u \leq 1$, and any n. Thus the F_n are uniformly limited in $\mathfrak{A}$.

We may therefore integrate termwise by 554, 2. We may verify this at once. For

$$F(x) = 1 \quad , \quad 0 \leq x < 1$$
$$= 0 \quad , \quad x = 0.$$

Hence $$\int_0^u F(x)dx = u. \qquad (1$$
On the other hand,
$$\int_0^u F_n dx = u - \frac{u^{n+1}}{n+1} \doteq u \quad , \quad \text{as } n \doteq \infty. \qquad (2$$
From 1), 2) we have
$$G(u) = u = \sum_0^\infty \left\{ \frac{u^{n+1}}{n+1} - \frac{u^{n+2}}{n+2} \right\}.$$

556. 1. *If* 1° $f(x_1 \cdots x_m\, t_1 \cdots t_n) \doteq \phi(x_1 \cdots x_m)$ *infra-uniformly the metric or complete field* $\mathfrak{X}$, *as* $t \doteq \tau$, τ *finite or infinite;*

2° $f(x, t)$ *is uniformly limited in* $\mathfrak{X}$ *and* R-*integrable for each* t *ar* τ*;*

Then
$$\lim_{t=\tau} \int_{\mathfrak{A}} f(x, t) = \int_{\mathfrak{A}} \phi,$$
iformly with respect to the set of measurable fields $\mathfrak{A}$ *in* $\mathfrak{X}$.

If $\mathfrak{X}$ *is complete, condition* 1° *may be replaced by* 3° $\phi(x)$ *is -integrable in* $\mathfrak{X}$.

For by 552, 1, when 3° holds, 1° holds; and when 1° holds, ϕ R-integrable in $\mathfrak{X}$.

Now the points $\mathfrak{E}_t$ where
$$|\epsilon(x, t_n)| > \epsilon$$
e such that
$$\lim_{t=\tau} \widehat{\mathfrak{E}}_t = 0 \quad , \quad \text{by 412.}$$
Let $\mathfrak{X} = \mathfrak{E}_t + \mathfrak{X}_t$. Then
$$\int_{\mathfrak{X}} \epsilon(x, t) = \int_{\mathfrak{E}_t} \epsilon(x, t) + \int_{\mathfrak{X}_t} \epsilon(x, t),$$
$$\left| \int_{\mathfrak{A}} \epsilon \right| \leq \int_{\mathfrak{X}} |\epsilon| \leq 2M\widehat{\mathfrak{E}}_t + \epsilon\widehat{\mathfrak{X}}.$$
But
$$\lim_{t=\tau} \widehat{\mathfrak{E}}_t = 0,$$
ich establishes the theorem.

2. As a corollary we have:

If 1° $F(x) = \Sigma f_{\iota_1 \cdots \iota_n}(x_1 \cdots x_m)$ *converges infra-uniformly, and ch of its terms* f_ι *is* R-*integrable in the metric or complete field* $\mathfrak{A}$*;*

2° $F_\lambda(x)$ *is uniformly limited in* $\mathfrak{A}$;

Then

$$\int_{\mathfrak{B}} F(x) = \Sigma \int_{\mathfrak{B}} f_\iota,$$

and the series on the right converges uniformly with respect to all measurable $\mathfrak{B} \leq \mathfrak{A}$.

3. *If* 1° $\lim_{t=\tau} f(x, t_1 \cdots t_n) = \phi(x)$ *is R-integrable in the interval* $\mathfrak{A} = (a < b)$, τ *finite or infinite;*

2° $f(x, t)$ *is uniformly limited, and R-integrable for each t near* τ;

Then

$$\lim_{t=\tau} \int_a^x f(x, t)dx = \int_a^x \phi(x)dx = \Phi(x),$$

uniformly in $\mathfrak{A}$, *and* $\Phi(x)$ *is continuous in* $\mathfrak{A}$.

4. *If* 1°

$$F(x) = \Sigma f_{\iota_1 \cdots \iota_n}(x)$$

and also each term f_ι *are R-integrable in the interval* $\mathfrak{A} = (a < b)$;

2° $F_\lambda(x)$ *is uniformly limited in* $\mathfrak{A}$;

Then

$$G(x) = \Sigma \int_a^x f_\iota(x)dx \quad , \quad x \text{ in } \mathfrak{A}$$

is continuous.

For G is a uniformly convergent series in $\mathfrak{A}$, each of whose terms

$$\int_a^x f_\iota dx$$

is a continuous function of x.

Differentiability

557. 1. *If* 1° $\lim_{t=\tau} f(x, t_1 \cdots t_n) = \phi(x)$ *in* $\mathfrak{A} = (a < b)$, τ *finite or infinite;*

2° $f'_x(x, t)$ *is R-integrable for each t near* τ, *and uniformly limited in* $\mathfrak{A}$;

3° $f'_x(x, t) \doteq \psi(x)$ *infra-uniformly in* $\mathfrak{A}$, *as* $t \doteq \tau$;

Then at a point x of continuity of ψ *in* $\mathfrak{A}$

$$\phi'(x) = \psi(x), \tag{1}$$

or what is the same

$$\frac{d}{dx} \lim_{t=\tau} f(x, t) = \lim_{t=\tau} \frac{d}{dx} f(x, t). \tag{2}$$

For by 554,

$$\lim_{t=\tau}\int_a^x f_x'(x, t)dx = \int_a^x \psi(x)dx \tag{3}$$

$$= \lim_{t=\tau}\left[f(x, t) - f(a, t)\right] \quad , \quad \text{by I, 538}$$

$$= \phi(x) - \phi(a) \quad , \quad \text{by } 1°.$$

Now by I, 537, at a point of continuity of ψ,

$$\frac{d}{dx}\int_a^x \psi(x)dx = \psi(x). \tag{4}$$

From 3), 4), we have 1), or what is the same 2).

2. *In the interval* $\mathfrak{A}$, *if*

1° $F(x) = \Sigma f_{\iota_1 \cdots \iota_n}(x)$ *converges;* (1

2° *Each* $f_\iota'(x)$ *is limited and R-integrable;*

3° $F_\lambda'(x)$ *is uniformly limited;*

4° $G(x) = \Sigma f_\iota'$ *is infra-uniformly convergent;*

Then at a point of continuity of $G(x)$ *in* $\mathfrak{A}$, *we may differentiate* *e series* 1) *termwise, or* $F'(x) = G(x)$.

3. *In the interval* $\mathfrak{A}$, *if*

1° $f(x, t_1 \cdots t_n) \doteq \phi(x)$ *as* $t \doteq \tau$, τ *finite or infinite;*

2° $f(x, t)$ *is uniformly limited, and a continuous function of* x*;*

3° $\psi(x) = \lim_{t=\tau} f_x'(x, t)$ *is continuous;*

Then

$$\phi'(x) = \psi(x), \tag{1}$$

what is the same

$$\frac{d}{dx}\lim_{t=\tau} f(x, t) = \lim_{t=\tau}\frac{d}{dx}f(x, t). \tag{2}$$

For by 547, 1, condition 3° requires that $f' \doteq \psi$ subuniformly $\mathfrak{A}$. But then the conditions of 1 are satisfied and 1) and 2) old.

4. *In the interval* $\mathfrak{A}$ *let us suppose that*

1° $F(x) = \Sigma f_{\iota_1 \cdots \iota_n}(x)$ *converges;* (1

2° *Each term f_ι is continuous;*

3° *$F_\lambda'(x)$ is uniformly limited;*

4° *$G(x) = \Sigma f_\iota'(x)$ is continuous;*

Then we may differentiate 1) termwise, or $F'(x) = G(x)$.

558. *Example 1.* We saw in 555, Ex. 3 that

$$F(x) = x = \sum_0^\infty \left\{ \frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2} \right\} \quad , \quad \text{in } \mathfrak{A} = (0, 1). \tag{1}$$

The series got by differentiating termwise is

$$\begin{aligned} G(x) = \sum_0^\infty x^n(1-x) &= 1 \quad , \quad 0 \le x < 1 \\ &= 0 \quad , \quad x = 0. \end{aligned} \tag{2}$$

Thus by 557, 4, $\quad G(x) = F'(x)$ in $(0^*, 1) = \mathfrak{A}^*$. $\quad$ (3

The relation 3) does not hold for $x = 0$.

Example 2.

$$F(x) = \sum_1^\infty \left\{ \frac{\text{arctg}\, x\sqrt{n}}{\sqrt{n}} - \frac{\text{arctg}\, x\sqrt{n+1}}{\sqrt{n+1}} \right\} = \Sigma f_n(x)$$

$$G(x) = \sum_1^\infty \left\{ \frac{1}{1+nx^2} - \frac{1}{1+(n+1)x^2} \right\} = \Sigma f_n'(x).$$

Here

$$F(x) = \text{arctg}\, x, \qquad \text{for any } x. \tag{1}$$

$$\begin{aligned} G(x) &= \frac{1}{1+x^2} \quad , \quad x \ne 0 \\ &= 0 \quad , \quad x = 0. \end{aligned} \tag{2}$$

Hence $G(x)$ is continuous in any interval $\mathfrak{A}$, not containing $x = 0$. Thus we should have by 557, 4,

$$F'(x) = G(x), \qquad x \text{ in } \mathfrak{A}. \tag{3}$$

This relation is verified by 1), 2). The relation 3) does not hold for $x = 0$, since

$$F'(0) = 1, \quad , \quad G(0) = 0.$$

Example 3.

$$F(x)=\sum_1^{\infty}\left\{\frac{1}{2n}\log(1+n^2x^2)-\frac{1}{2(n+1)}\log(1+(n+1)^2x^2)\right\}$$

$$=\Sigma f_n(x) \tag{1}$$

$$=\tfrac{1}{2}\log(1+x^2) \quad , \quad \text{for any } x.$$

$$G(x)=\sum f_n'(x)=\sum\left\{\frac{nx}{1+n^2x^2}-\frac{(n+1)x}{1+(n+1)^2x^2}\right\} \tag{2}$$

$$=\frac{x}{1+x^2} \quad , \quad \text{for any } x.$$

In any interval $\mathfrak{A}$, all the conditions of 557, 4, hold.

Hence

$$F'(x)=G(x) \quad , \quad \text{for any } x \text{ in } \mathfrak{A}. \tag{3}$$

In case we did not know the value of the sums 1), 2) we could still assert that 3) holds. For by 545, G is subuniformly convergent in $\mathfrak{A}$, and hence is continuous.

Example 4.

$$F(x)=\sum_1^{\infty}\left\{\frac{1+nx}{ne^{nx}}-\frac{1+(n+1)x}{(n+1)e^{(n+1)x}}\right\}=\frac{1+x}{e^x}. \tag{1}$$

Here

$$F'(x)=-\frac{x}{e^x}. \tag{2}$$

The series obtained by differentiating F termwise is

$$G(x)=\sum_1^{\infty}\left\{\frac{(n+1)x}{e^{(n+1)x}}-\frac{nx}{e^{nx}}\right\}=-\frac{x}{e^x}, \tag{3}$$

and hence

$$G_{n-1}(x)=-\frac{x}{e^x}+\frac{nx}{e^{nx}}.$$

The peaks of the residual function

$$\epsilon(x,n)=\frac{nx}{e^{nx}},$$

are of height $=1/e$. The convergence of G is not uniform at $x=0$. The conditions of 557, 4, are satisfied and we can differentiate 1) termwise. This is verified by 2), 3).

559. 1. *If* 1° $\lim_{t=\tau} f(x, t_1 \cdots t_n) = \phi(x)$ *is limited and R-integrable in the interval* $\mathfrak{A} = (a < b)$;

2° $f(x, t)$ *is limited, and R-integrable in* $\mathfrak{A}$, *for each* t *near* τ;

3° $$\psi(x) = \lim_{t=\tau} \int_a^x f(x, t) = \lim_{t=\tau} g(x, t)$$

is a continuous function in $\mathfrak{A}$;

4° *The points* $\mathfrak{E}$ *in* $\mathfrak{A}$ *in whose vicinity the peaks of* $f(x, t)$ *as* $t \doteq \tau$ *are infinitely high form an enumerable set;*

Then
$$\theta(x) = \int_a^x \phi(x) = \lim_{t=\tau} \int_a^x f(x, t)dx = \psi(x), \qquad (1$$

or
$$\lim_{t=\tau} \int_a^x f(x, t)dx = \int_a^x \lim_{t=\tau} f(x, t)dx,$$

and the set $\mathfrak{E}$ *is complete and discrete.*

For $\mathfrak{E}$ is discrete by 553, 3.

Let α be a point of $A = \mathfrak{A} - \mathfrak{E}$. Then in an interval $\mathfrak{a}$ about α,
$$|f(x, t)| < \text{some } M \quad , \quad x \text{ in } \mathfrak{a}, \text{ any } t \text{ near } \tau. \qquad (2$$

Now by 556, 3, taking $\epsilon > 0$ small at pleasure, there exists an $\eta > 0$ such that
$$\psi(x) - \psi(\alpha) = \int_\alpha^x f(x, t) + \epsilon' \quad , \quad |\epsilon'| < \epsilon$$

for *any* x in $\mathfrak{a}$, and t in $V_\eta^*(\tau)$. If we set $x = \alpha + h$, we have
$$\frac{\Delta\psi}{\Delta x} = \frac{\psi(x) - \psi(\alpha)}{h} = \frac{1}{h}\int_\alpha^x f(x, t)dx + \frac{\epsilon'}{h}. \qquad (3$$

Also by 556, 3, we have
$$\int_\alpha^x f(x, t)dx = \int_\alpha^x \phi(x)dx + \epsilon'' \quad , \quad |\epsilon''| < \epsilon$$

for *any* x in $\mathfrak{a}$, and t in $V_\eta^*(\tau)$. Thus
$$\frac{1}{h}\int_\alpha^x f(x, t)dx = \frac{\theta(x) - \theta(\alpha)}{h} + \frac{\epsilon''}{h} = \frac{\Delta\theta}{\Delta x} + \frac{\epsilon''}{h}. \qquad (4$$

From 3), 4) we have
$$\frac{\Delta\psi}{\Delta x} - \frac{\epsilon'}{h} = \frac{\Delta\theta}{\Delta x} + \frac{\epsilon''}{h}, \qquad |\epsilon'|, |\epsilon''| < \epsilon.$$

Now ϵ may be made small at pleasure, and that independent of h. Thus the last relation gives

$$\frac{\Delta\psi}{\Delta x} = \frac{\Delta\theta}{\Delta x} \quad , \quad \text{for } x \text{ in } A.$$

As this holds however small $h = \Delta x$ is taken, we have

$$\frac{d\psi}{dx} = \frac{d\theta}{dx} \quad , \quad \text{for } x \text{ in } A.$$

Hence by 515, 3,

$$\psi(x) = \theta(x) + \text{const} \quad , \quad \text{in } \mathfrak{A}.$$

For $x = a$, $\qquad \psi(a) = \theta(a) = 0$;

and thus

$$\psi(x) = \theta(x) \quad , \quad \text{in } \mathfrak{A}.$$

2. *As a corollary we have:*

If 1° $F(x) = \Sigma f_{\iota_1 \cdots \iota_n}(x)$ *is limited and R integrable in the interval* $\mathfrak{A} = (a < b)$;

2° $F'_\lambda(x)$ *is limited and each term* f_ι *is R-integrable;*

3° $G(x) = \sum \int_a^x f_\iota$ *is continuous;*

4° *The points* $\mathfrak{E}$ *in* $\mathfrak{A}$ *in whose vicinity the peaks of* $F_\lambda(x)$ *are infinitely high form an enumerable set;*

Then

$$\int_a^x F(x) = \sum \int_a^x f_\iota,$$

or we may integrate the F series termwise.

560. 1. *If* 1° $\lim\limits_{t=\tau} f(x, t_1 \cdots t_n) = \phi(x)$ *in* $\mathfrak{A} = (a < b)$, τ *finite or infinite;*

2° $f'_x(x, t)$ *is limited and R-integrable for each t near* τ;

3° *The points* $\mathfrak{E}$ *of* $\mathfrak{A}$ *in whose vicinity* $f'_x(x, t)$ *has infinite peaks as* $t \doteq \tau$ *form an enumerable set;*

4° $\phi(x)$ *is continuous at the points* $\mathfrak{E}$;

5° $\psi(x) = \lim\limits_{t=\tau} f'_x(x, t)$ *is limited and R-integrable in* $\mathfrak{A}$;

Then at a point of continuity of $\psi(x)$ *in* $\mathfrak{A}$

$$\phi'(x) = \psi(x), \tag{1}$$

or what is the same

$$\frac{d}{dx}\lim_{t=\tau} f(x, t) = \lim_{t=\tau} \frac{d}{dx} f(x, t).$$

For let $\delta = (\alpha < \beta)$ be an interval in $\mathfrak{A}$ containing no point of $\mathfrak{E}$. Then for any x in δ

$$\int_\alpha^x f_x'(x, t)dx = f(x, t) - f(\alpha, t) \quad , \quad \text{by } 2°.$$

Hence

$$\lim_{t=\tau}\int_\alpha^x f_x'(x, t)dx = \lim_{t=\tau}\{f(x, t) - f(\alpha, t)\}$$

$$= \phi(x) - \phi(\alpha) \quad , \quad \text{by } 1°. \tag{2}$$

By 556, 3, $\phi(x)$ is continuous in δ. Thus $\phi(x)$ is continuous at any point not in $\mathfrak{E}$. Hence by 4° it is continuous in $\mathfrak{A}$.

We may thus apply 559, 1, replacing therein $f(x, t)$ by $f_x'(x, t)$. We get

$$\lim_{t=\tau}\int_a^x f_x'(x, t)dx = \int_a^x \lim f_x'(x, t)dx = \int_a^x \psi(x)dx. \tag{3}$$

Since 2) obviously holds when we replace α by a, this relation with 3) gives

$$\int_a^x \psi(x)dx = \phi(x) - \phi(a).$$

At a point of continuity, this gives 1) on differentiating.

2. *If* 1° $F(x) = \Sigma f_{\iota_1 \dots \iota_n}(x)$ *converges in the interval* $\mathfrak{A}$*;*

2° $G(x) = \Sigma f_\iota'(x)$ *and each of its terms are limited and R-integrable in* $\mathfrak{A}$*;*

3° *The points of* $\mathfrak{A}$ *in whose vicinity* $G_\lambda(x)$ *has infinite peaks as* $\lambda \doteq \infty$, *form an enumerable set at which* $F(x)$ *is continuous;*

Then at a point of continuity of $G(x)$ *we have*

$$F'(x) = G(x),$$

or what is the same

$$\frac{d}{dx}\Sigma f_\iota(x) = \sum \frac{df_\iota(x)}{dx}.$$

561. *Example.*
$$F(x) = \sum_1^\infty \left\{ \frac{nx^2}{e^{nx^2}} - \frac{(n+1)x^2}{e^{(n+1)x^2}} \right\} = \frac{x^2}{e^{x^2}}.$$

Hence
$$F'(x) = \frac{2x}{e^{x^2}} - \frac{2x^3}{e^{x^2}}. \tag{1}$$

The series obtained by differentiating F termwise is

$$G(x) = \sum_1^\infty \left\{ \frac{2nx}{e^{nx^2}} - \frac{2n^2x^3}{e^{nx^2}} - \frac{2(n+1)}{e^{(n+1)x^2}} + \frac{2(n+1)^2x^3}{e^{(n+1)x^2}} \right\}.$$

Here
$$G_{n-1}(x) = \frac{2x}{e^{x^2}} - \frac{2x^3}{e^{x}} - \left\{ \frac{2nx}{e^{nx^2}} - \frac{2n^2x^3}{e^{nx^2}} \right\}.$$

Hence
$$G(x) = \frac{2x}{e^{x^2}} - \frac{2x^3}{e^{x^2}} \tag{2}$$

a continuous function of x.

The convergence of the G series is not uniform at $x = 0$. For $a_n = 1/n$. Then

$$G_{n-1}(a_n) = G(a_n) - \left\{ \frac{2}{e^{\frac{1}{n}}} - \frac{\frac{2}{n}}{e^{\frac{1}{n}}} \right\} \doteq -2.$$

To get the peaks of the residual function we consider the nts of extreme of

$$y = \frac{nx(1 - nx^2)}{e^{nx^2}}. \tag{3}$$

We find
$$y' = \frac{n(1 - 5nx^2 + 2n^2x^4)}{e^{nx^2}}.$$

Thus $y' = 0$ when
$$2n^2x^4 - 5nx^2 + 1 = 0,$$

when
$$x = \frac{a}{\sqrt{n}} \text{ or } \frac{\alpha}{\sqrt{n}}, \quad a, \alpha \text{ constants.}$$

Putting these values in 3), we find that y has the form

$$y = c\sqrt{n}.$$

Hence $x = 0$ is the only point where the residual function has infinite peak. Thus the conditions of 560, 2, are satisfied, and should have $F'(x) = G(x)$ for any x. This is indeed so, as 1), show.

CHAPTER XVII

GEOMETRIC NOTIONS

Plane Curves

562. In this chapter we propose to examine the notions of curve and surface together with other allied geometric concepts. Like most of our notions, we shall see that they are vague and uncertain as soon as we pass the confines of our daily experience. In studying some of their complexities and even paradoxical properties, the reader will see how impossible it is to rely on his unschooled intuition. He will also learn that the demonstration of a theorem in analysis which rests on the evidence of our geometric intuition cannot be regarded as binding until the geometric notions employed have been clarified and placed on a sound basis.

Let us begin by investigating our ideas of a plane curve.

563. Without attempting to define a curve we would say on looking over those curves most familiar to us that a plane curve has the following properties:

1° It can be generated by the motion of a point.

2° It is formed by the intersection of two surfaces.

3° It is continuous.

4° It has a tangent at each point.

5° The arc between any two of its points has a length.

6° A curve is not superficial.

7° Its equations can be written in any one of the forms

$$y = f(x), \tag{1}$$

$$x = \phi(t) \quad , \quad y = \psi(t), \tag{2}$$

$$F(x, y) = 0\ ; \tag{3}$$

and conversely such equations define curves.

8° When closed it forms the complete boundary of a region.

9° This region has an area.

Of all these properties the first is the most conspicuous and characteristic to the naïve intuition. Indeed many employ this as the definition of a curve. Let us therefore look at our ideas of motion.

564. *Motion.* In this notion, two properties seem to be essential. 1° motion is continuous, 2° it takes place at each instant in a definite direction and with a definite speed. The direction of motion, we agree, shall be given by dy/dx, its speed by ds/dt. We see that the notion of motion involves properties 4°, 5°, and 7°. Waiving this point, let us notice a few peculiarities which may arise.

Suppose the curve along which the motion takes place has an angle point or a cusp as in I, 366. What is the direction of motion at such a point? Evidently we must say that motion is impossible along such a curve, or admit that the ordinary idea of motion is imperfect and must be extended in accordance with the notion of right-hand and left-hand derivatives.

Similarly ds/dt may also give two speeds, a posterior and an anterior speed, at a point where the two derivatives of $s = \phi(t)$ are different.

Again we will admit that at any point of the path of motion, motion may begin and take place in either direction. Consider what happens for a path defined by the continuous function in I, 367. This curve has no tangent at the origin. We ask how does the point move as it passes this point, or to make the question still more embarassing, suppose the point at the origin. In what direction does it start to move? We will admit that no such motion is possible, or at least it is not the motion given us by our intuition. Still more complicated paths of this nature are given in I, 369, 371, and in Chapter XV of the present volume.

It thus appears that to define a curve as the path of a moving point, is to define an unknown term by another unknown term, equally if not more obscure.

565. 2° *Property. Intersection of Two Surfaces.* This property has also been used as the definition of a curve. As the notion

of a surface is vastly more complicated than that of a curve, it hardly seems advisable to define a complicated notion by one still more complicated and vague.

566. 3° *Property.* *Continuity.* Over this knotty concept philosophers have quarreled since the days of Democritus and Aristotle. As far as our senses go, we say a magnitude is continuous when it can pass from one state to another by imperceptible gradations. The minute hand of a clock appears to move continuously, although in reality it moves by little jerks corresponding to the beats of the pendulum. Its velocity to our senses appears to be continuous.

We not only say that the magnitude shall pass from one state to another by gradations imperceptible to our senses, but we also demand that between any two states another state exists and so without end. Is such a magnitude continuous? No less a mathematician than Bolzano admitted this in his philosophical tract *Paradoxien des Unendlichen.* No one admits it, however, to-day. The different states of such a magnitude are pantactic, but their ensemble is not a continuum.

But we are not so much interested in what constitutes a continuum in the abstract, as in what constitutes a continuous curve or even a continuous straight line or segment. The answer we have adopted to these questions is giv[illegible] theory of irrational numbers created by Cantor and [illegible] [see Vol. I, Chap. II], and in the notion of a co[illegible]ion due to Cauchy and Weierstrass [see Vol. I, C[illegible]

These definitions of continuity are [illegible]th them we can reason with the utmost precision an[illegible]consequences we deduce from them are sufficiently i[illegible] our intuition to justify their employment. We ca[illegible]rely analytic methods that a continuous function f [illegible]n its extreme values [I, 354], that if such a function [illegible]ue a at the point P, and the value b at the point [illegible] all intermediary values between a, b, as x ra[illegible] Q [I, 357]. We can also show that a closed curve [illegible] point does form the boundary of a complete regio[illegible]

567. 4° *Property.* *Tangents.* To be[illegible] a tangent? Euclid defines a tangent to a circle as [illegible]which meets

ne circle and being produced does not cut it again. In commenting on this definition Casey says, "In modern geometry a curve is made up of an infinite number of points which are placed in order along the curve, and then the secant through two consecutive points is a tangent." If the points on a curve were like beads on a string, we might speak of consecutive points. As, however, there are always an infinite number of points between any two points on a continuous curve, this definition is quite illusory.

The definition we have chosen is given in I, 365. That property 4° does not hold at each point of a continuous curve was brought out in the discussion of property 1°. Not only is it not necessary that a curve has a tangent at each of its points, but a curve does not need to have a tangent at a pantactic set of points, as we saw in Chapter XV.

For a long time it was supposed that every curve has a tangent at each point, or if not at each point, at least in general. Analytically, this property would go over into the following: *every continuous function has a derivative.* A celebrated attempt to prove this was made by Ampère.

Mathematicians were greatly surprised when Weierstrass exhibited the function we have studied in 502 and which has no derivative.

Weierstrass* himself remarks: "Bis auf die neueste Zeit hat man allgemein angenommen, dass eine eindeutige und continuirliche Function einer reellen Veränderlichen auch stets eine erste Ableitung habe, deren Werth nur an einzelnen Stellen unbestimmt oder unendlich gross werden könne. Selbst in den Schriften von Gauss, Cauchy, Dirichlet findet sich meines Wissens keine Äusserung, aus der unzweifelhaft hervorginge, dass diese Mathematiker, welche in ihrer Wissenschaft die strengste Kritik überall zu üben gewohnt waren, anderer Ansicht gewesen seien."

568. *Property* 5°. *Length.* We think of a curve as having length. Indeed we read as the definition of a curve in Euclid's Elements: a line is length without breadth. When we see two simple curves we can often compare one with the other in regard to length without consciously having established a way to measure

* *Werke*, vol. 2, p. 71.

them. Perhaps we unconsciously suppose them described at a uniform rate and estimate the time it takes. It may be that we regard them as inextensible strings whose length is got by straightening them out. A less obvious way to measure their lengths would be to roll a straightedge over them and measure the distance on the edge between the initial and final points of contact.

We ask how shall we formulate arithmetically our intuitional ideas regarding the length of a curve? The intuitionist says, a curve or the arc of a curve *has* length. This length is expressed by a number L which is obtained by taking a number of points $P_1, P_2, P_3 \cdots$ on the curve between the end points P, P', and forming the sum

$$\Sigma P_\iota P_{\iota+1}. \tag{1}$$

The limit of this sum as the points became pantactic is the length L of the arc PP'.

Our point of view is different. We would say: Whatever arithmetic formulation we choose we have no *a priori* assurance that it adequately represents our intuitional ideas of length. With the intuitionist we will, however, form the sum 1) and see if it has a limit, however the points P_ι are chosen. If it has, we will investigate this number used as a definition of length and see if it leads to consequences which are in harmony with our intuition.

This we now proceed to do.

569. 1. Let

$$x = \phi(t) \quad , \quad y = \psi(t) \tag{1}$$

be one-valued continuous functions of t in the interval $\mathfrak{A} = (a < b)$. As t ranges over $\mathfrak{A}$ the point x, y will describe a *curve* or *an arc of a curve* C. We might agree to call such curves *analytic*, in distinction to those given by our intuition. The interval $\mathfrak{A}$ is the *interval corresponding to* C.

Let D be a finite division of $\mathfrak{A}$ of norm d, defined by

$$a < t_1 < t_2 < \cdots < b.$$

To these values of t will correspond points

$$P, P_1, P_2 \cdots Q \tag{2}$$

on C, which may be used to define a polygon P_D whose vertices are 2).

Let $(m, m+1)$ denote the side $P_m P_{m+1}$, as well as its length. If we denote the length of P_D by the same letter, we have

$$P_D = \Sigma(m, m+1) = \Sigma\sqrt{\Delta x_m^2 + \Delta y_m^2}.$$

If

$$\lim_{d=0} P_D \tag{3}$$

exists, it is called the *length of the arc* C, and C is *rectifiable.*

2. (*Jordan.*) *For the arc* PQ *to be rectifiable, it is necessary and sufficient that the functions* ϕ, ψ *in* 1) *have limited variation in* $\mathfrak{A}$.

For

$$\sqrt{\Delta x^2 + \Delta y^2} \geq |\Delta x|.$$

Hence

$$P_D \geq \Sigma |\Delta x|.$$

But the sum on the right is the variation of ϕ for the division D. If now ϕ does not have limited variation in $\mathfrak{A}$, the limit 3) does not exist. The same holds for ψ. Hence limited variation is a necessary condition.

The condition is *sufficient.* For

$$P_D \leq \Sigma |\Delta x| + \Sigma |\Delta y| = \operatorname*{Var}_D \phi + \operatorname*{Var}_D \psi.$$

As ϕ, ψ have limited variation, this shows that

$$P_0 = \operatorname*{Max}_D P_D$$

is finite. *We show now that*

$$\lim_{d=0} P_D = P_0. \tag{4}$$

For there exists a division Δ such that

$$P_0 - \frac{\epsilon}{2} < P_\Delta \leq P_0. \tag{5}$$

Let Δ cause $\mathfrak{A}$ to fall into ν intervals, the smallest of which has the length λ. Let D be a division of $\mathfrak{A}$ of norm $d \leq d_0 < \lambda$. Then no interval of D contains more than one point of Δ. Let $E = D + \Delta$.

Obviously

$$P_E \geq P_D \quad \text{or} \quad P_\Delta.$$

Suppose that the point t_κ of Δ falls in the interval $(t_\iota, t_{\iota+1})$ of D. Then the chord $(\iota, \iota+1)$ in P_D is replaced by the two chords (ι, κ), $(\kappa, \iota+1)$ in P_E. Hence

$$P_E - P_D = \Sigma G_\kappa \quad , \quad \kappa = 1, 2 \cdots \mu \leq \nu$$

where

$$G_\kappa = (\iota, \kappa) + (\kappa, \iota+1) - (\iota, \iota+1).$$

Obviously as ϕ, ψ are continuous we may take d_0 so small that each

$$G_\kappa < \frac{\epsilon}{2\nu} \quad , \quad \text{for any } d \leq d_0.$$

Hence

$$P_E - P_D < \frac{\epsilon}{2}. \tag{6}$$

From 5), 6) we have

$$P_0 - P_D < \epsilon \quad , \quad \text{for any } d \leq d_0,$$

which gives 4).

3. *If the arc PQ is rectifiable, any arc contained in PQ is also rectifiable.*

For ϕ, ψ having limited variation in interval $\mathfrak{A}$, have *a fortiori* limited variation in any segment of $\mathfrak{A}$.

4. *Let the rectifiable arc C fall into two arcs C_1, C_2. If s, s_1, s_2 are the lengths of C, C_1, C_2, then*

$$s = s_1 + s_2. \tag{7}$$

For we saw that C_1, C_2 are rectifiable since C is. Let $\mathfrak{A}_1$, $\mathfrak{A}_2$ be the intervals in $\mathfrak{A}$ corresponding to C_1, C_2. Let D_1, D_2 be divisions of $\mathfrak{A}_1$, $\mathfrak{A}_2$ of norm d. Then

$$s_1 = \lim_{d=0} P_{D_1} \quad , \quad s_2 = \lim_{d=0} P_{D_2}.$$

But D_1, D_2 effect a division of $\mathfrak{A}$, and since

$$s = \lim_{e=0} P_E \tag{8}$$

with respect to the class of all divisions of $\mathfrak{A}$, the limit 8) is the same when E is restricted to range over divisions of the type of D. Now

$$P_D = P_{D_1} + P_{D_2}.$$

Passing to the limit, we get 7).

The preceding reasoning also shows that *if C_1, C_2 are rectifiable* [cu]*rves, then C is, and* 7) *holds again.*

5. *If* 1) *define a rectifiable curve, its length s is a continuous func*[tio]*n $s(t)$ of t.*

For ϕ, ψ having limited variation,

$$\phi = \phi_1 - \phi_2 \quad , \quad \psi = \psi_1 - \psi_2,$$

[wh]ere the functions on the right are continuous monotone *increas*[in]*g* functions of t in the interval $\mathfrak{A} = (a < b)$.

For a division D of norm d of the interval $\Delta\mathfrak{A} = (t, t + h)$ we [ha]ve

$$\begin{aligned} P_D &= \Sigma\sqrt{\Delta x^2 + \Delta y^2} \\ &\leq \Sigma \mid \Delta x \mid + \Sigma \mid \Delta y \mid \\ &\leq \Sigma\Delta\phi_1 + \Sigma\Delta\phi_2 + \Sigma\Delta\psi_1 + \Sigma\Delta\psi_2 \\ &\leq \delta\phi_1 + \delta\phi_2 + \delta\psi_1 + \delta\psi_2, \end{aligned}$$

[wh]ere $\delta\phi_1 = \phi_1(t + h) - \phi(t)$, and similarly for the other func[tio]ns. As ϕ_1 is continuous, $\delta\phi_1 \doteq 0$, etc., as $h \doteq 0$. We may [th]erefore take $\eta > 0$ so small that $\delta\phi_1, \delta\phi_2, \delta\psi_1, \delta\psi_2 < \epsilon/4$, if $h < \eta$.

Hence $\Delta s = s(t + h) - s(t) \leq \text{Max } P_D < \epsilon \quad , \quad \text{if } 0 < h > \eta.$

Thus s is continuous.

6. *The length s of the rectifiable arc C corresponding to the inter*[va]*l $(a < t)$ is a monotone increasing function of t.*

This follows from 4.

7. *If x, y do not have simultaneous intervals of invariability, $s(t)$* [is] *an increasing function of t. The inverse function is one-valued* [an]*d increasing and the coördinates x, y are one-valued functions of s.*

That the inverse function $t(s)$ is one-valued follows from I, 214. [W]e can thus express t in terms of s, and so eliminate t in 1).

570. 1. *If ϕ', ψ' are continuous in the interval $\mathfrak{A}$,*

$$s = \int_{\mathfrak{A}} dt\sqrt{\phi'^2 + \psi'^2}. \tag{1}$$

For

$$s = \lim_{d=0} \Sigma\sqrt{\Delta\phi_\kappa^2 + \Delta\psi_\kappa^2}. \tag{2}$$

Now
$$\Delta\phi_\kappa = \phi'(t'_\kappa)\Delta t_\kappa \quad , \quad \Delta\psi_\kappa = \psi'(t''_\kappa)\Delta t_\kappa \tag{3}$$
where t'_κ, t''_κ lie in the interval Δt_κ.

As ϕ', ψ' are continuous they are uniformly continuous. Hence for any division D of norm $\leq$ some d_0,
$$\phi'(t'_\kappa) = \phi'(t_\kappa) + \alpha_\kappa \quad , \quad \psi'(t''_\kappa) = \psi'(t_\kappa) + \beta_\kappa$$
where $|\alpha_\kappa|$, $|\beta_\kappa| <$ some η, small at pleasure, for any κ. Thus
$$\sqrt{\Delta\phi_\kappa^2 + \Delta\psi_\kappa^2} = \Delta t_\kappa \sqrt{\phi'(t_\kappa)^2 + \psi'(t_\kappa)^2} + \epsilon_\kappa \Delta t_\kappa,$$
and we may take
$$|\epsilon_\kappa| < \epsilon/\mathfrak{A} \quad , \quad \kappa = 1, 2 \cdots$$

Thus
$$s = \lim_{d=0} \Sigma \Delta t_\kappa \sqrt{\phi'(t_\kappa)^2 + \psi'(t_\kappa)^2} + \lim \Sigma \epsilon_\kappa \Delta t_\kappa.$$
Hence
$$\left| s - \int_{\mathfrak{A}} \right| < \epsilon,$$
which establishes 1).

For simplicity we have assumed ϕ', ψ' to be continuous in $\mathfrak{A}$. This is not necessary, as the following shows.

2. Let $a_1, \cdots a_n, b_1, \cdots b_n \geq 0$ but not all $= 0$.

Then
$$\left| \sqrt{a_1^2 + \cdots + a_n^2} - \sqrt{b_1^2 + \cdots + b_n^2} \right| \leq \sum_m |a_m - b_m|,$$
$$m = 1, 2 \cdots n. \tag{4}$$

For
$$(\sqrt{a_1^2 + \cdots} - \sqrt{b_1^2 + \cdots})(\sqrt{a_1^2 + \cdots} + \sqrt{b_1^2 + \cdots})$$
$$= (a_1^2 + \cdots + a_n^2) - (b_1^2 + \cdots + b_n^2)$$
$$= (a_1^2 - b_1^2) + \cdots + (a_n^2 - b_n^2)$$
$$= (a_1 - b_1)(a_1 + b_1) + \cdots + (a_n - b_n)(a_n + b_n).$$

Hence
$$\sqrt{a_1^2 + \cdots} - \sqrt{b_1^2 + \cdots} = \sum_{m=1}^{n} (a_m - b_m) \frac{a_m + b_m}{\sqrt{a_1^2 + \cdots} + \sqrt{b_1^2 + \cdots}}. \tag{5}$$

But
$$\left| \frac{a_m + b_m}{\sqrt{a_1^2 + \cdots} + \sqrt{b_1^2 + \cdots}} \right| \leq 1.$$

This in 5) gives 4).

Let us apply 4) to prove the following theorem, more general than 1.

3. (*Baire.*) *If ϕ', ψ' are limited and R-integrable, then*

$$s = \int_{\mathfrak{A}} \sqrt{\phi'^2 + \psi'^2}\, dt. \qquad (1$$

For by 4),

$$| \sqrt{\phi'(t_\kappa')^2 + \psi'(t_\kappa'')^2} - \sqrt{\phi'(t_\kappa)^2 + \psi'(t_\kappa)^2} \,| \leq | \phi'(t_\kappa') - \phi'(t_\kappa) |$$
$$+ | \psi'(t_\kappa'') - \psi'(t_\kappa) | ;$$

or

$$\Phi_\kappa - \Psi_\kappa = \eta_\kappa' \operatorname{Osc} \phi'(t) + \eta_\kappa'' \operatorname{Osc} \psi'(t) \quad , \quad \text{in } \delta_\kappa = \Delta t_\kappa,$$

where η_κ', η_κ'' are numerically ≤ 1. Thus

$$| \Sigma\delta_\kappa \Phi_\kappa - \Sigma \delta_\kappa \Psi_\kappa | = \Sigma \delta_\kappa \eta_\kappa' \operatorname{Osc} \phi' + \Sigma \delta_\kappa \eta_\kappa'' \operatorname{Osc} \psi'. \qquad (6$$

As ϕ', ψ' are integrable, the right side $\doteq 0$, as $d \doteq 0$. Now

$$\lim_{d=0} \Sigma \delta_\kappa \Psi_\kappa = \int_{\mathfrak{A}} \sqrt{\phi'^2 + \psi'^2}\, dt.$$

Thus passing to the limit in 6), we have

$$\lim \Sigma \Delta t_\kappa \sqrt{\phi'(t_\kappa')^2 + \psi'(t_\kappa'')^2} = \int_{\mathfrak{A}}.$$

This with 2), 3) gives 1) at once.

571. *Volterra's Curve.* It is interesting to note that *there are rectifiable curves for which $\phi'(t)$, $\psi'(t)$ are not both R-integrable.* Such a curve is Volterra's curve, discussed in 503. Let its equation be $y = f(x)$. Then $f'(x)$ behaves as

$$2x \sin \frac{1}{x} - \cos \frac{1}{x}$$

in the vicinity of a non null set in $\mathfrak{A} = (0, 1)$. Hence $f'(x)$ is not R-integrable in $\mathfrak{A}$. But then it is easy to show that

$$\int_0^1 \sqrt{1 + f'(x)^2}\, dx$$

does not exist. For suppose that

$$g = \sqrt{1 + f'(x)^2}$$

were R-integrable. Then $g^2 = 1 + f'(x)^2$ is R-integrable, and hence $f'(x)^2$ also. But the points of discontinuity of f'^2 in $\mathfrak{A}$ do not form a null set. Hence f'^2 is not R-integrable.

On the other hand, Volterra's curve is rectifiable by 569, 2, and 528, 1.

572. Taking the definition of length given in 569, 1, we saw that the coördinates

$$x = \phi(t) \quad , \quad y = \psi(t)$$

must have limited variation for the curve to be rectifiable. But we have had many examples of functions not having limited variation in an interval $\mathfrak{A}$. Thus the curve defined by

$$\begin{aligned} y &= x \sin\frac{1}{x} \quad , \quad x \neq 0 \\ &= 0 \quad , \quad x = 0 \end{aligned} \tag{4}$$

does not have a length in $\mathfrak{A} = (-1, 1)$; while

$$\begin{aligned} y &= x^2 \sin\frac{1}{x} \quad , \quad x \neq 0 \\ &= 0 \quad , \quad x = 0 \end{aligned} \tag{5}$$

does.

It certainly astonishes the naïve intuition to learn that the curve 4) has no length in any interval δ about the origin however small, or if we like, that this length is infinite, however small δ is taken. For the same reason we see that

No arc of Weierstrass' curve has a length (or its length is infinite) however near the end points are taken to each other, when $ab > 1$.

573. 1. 6° *Property. Space-filling Curves.* We wish now to exhibit a curve which passes through every point of a square, *i.e.* which completely fills a square. Having seen how to define one such curve, it is easy to construct such curves in great variety, not only for the plane but for space. The first to show how this may be done was *Peano* in 1890. The curve we wish now to define is due to *Hilbert*.

We start with a unit interval $\mathfrak{A} = (0, 1)$ over which t ranges, and a unit square $\mathfrak{B}$ over which the point x, y ranges. We define

$$x = \phi(t) \quad , \quad y = \psi(t) \tag{1}$$

s one-valued continuous functions of t in $\mathfrak{A}$ so that xy ranges over as t ranges over $\mathfrak{A}$. The analytic curve C defined by 1) thus ompletely fills the square $\mathfrak{B}$.

We do this as follows. We effect a division of $\mathfrak{A}$ into four qual segments $\delta'_1, \delta'_2, \delta'_3, \delta'_4$, and of $\mathfrak{B}$ into equal squares η'_1, η'_2, $_3, \eta'_4$, as in Fig. 1.

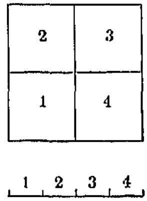


FIG. 1.

We call this the first division or D_1. The corre-pondence between $\mathfrak{A}$ and $\mathfrak{B}$ is given in *first pproximation* by saying that to each point P in shall correspond some point Q in η'_ι.

We now effect a second division D_2 by dividing ach interval and square of D_1 into four equal arts.

We number them as in Fig. 2,

$$\delta''_1 \quad , \quad \delta''_2 \quad \cdots \quad \delta''_{16}$$

$$\eta''_1 \quad , \quad \eta''_2 \quad \cdots \quad \eta''_{16}$$

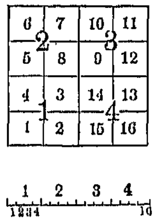


FIG. 2.

As to the numbering of the η's we observe the llowing two principles: 1° we may pass over the quares 1 to 16 continuously without passing the ame square twice, and 2° in doing this we pass ver the squares of D_1 in the same order as in ig. 1. The correspondence between $\mathfrak{A}$ and $\mathfrak{B}$ is iven in *second approximation* by saying that to each point P in ' shall correspond some point Q in η''_ι. In this way we continue ndefinitely.

To find the point Q in $\mathfrak{B}$ corresponding to P in $\mathfrak{A}$ we observe hat P lies in a sequence of intervals

$$\delta' > \delta'' > \delta''' > \cdots \doteq 0, \tag{2}$$

which correspond uniquely a sequence of squares

$$\eta' > \eta'' > \eta''' > \cdots \doteq 0. \tag{3}$$

The sequence 3) determines *uniquely* a point whose coördinates re one-valued functions of t, viz. the functions given in 1).

The functions 1) *are continuous in* $\mathfrak{A}$.

For let t' be a point near t; it either lies in the same interval as in D_n or in the adjacent interval. Thus the point Q' corre-

sponding to t' either lies in the same square of D_n as the point Q corresponding to t, or in an adjacent square. But the diagonal of the squares $\doteq 0$, as $n \doteq \infty$. Thus

$$\text{Dist}\,(Q'Q) \doteq 0 \quad , \quad \text{as } n \doteq \infty.$$

Thus $$\phi(t') - \phi(t) \quad , \quad \text{and } \psi(t') - \psi(t)$$

both $\doteq 0$, as $t' \doteq t$.

As t ranges over $\mathfrak{A}$, the point x, y ranges over every point in the square $\mathfrak{B}$.

For let Q be a given point of $\mathfrak{B}$. It lies in a sequence of squares as 3). If Q lies on a side or at a vertex of one of the η squares, there is more than one such sequence. But having taken such a sequence, the corresponding sequence 2) is uniquely determined. Thus to each Q corresponds at least one P. A more careful analysis shows that to a given Q never more than four points P can correspond.

2. The method we have used here may obviously be extended to space. By passing median planes through a unit cube we divide it into 2^3 equal cubes. Thus to get our correspondence each division D_n should divide each interval and cube of the preceding division D_{n-1} into 2^3 equal parts. The cubes of each division should be numbered according to the 1° and 2° principles of enumeration mentioned in 1.

By this process we define

$$x = \phi_1(t) \quad , \quad y = \phi_2(t) \quad , \quad z = \phi_3(t)$$

as one-valued continuous functions of t such that as t ranges over the unit interval $(0, 1)$, the point x, y, z ranges over the unit cube.

574. 1. *Hilbert's Curve.* We wish now to study in detail the correspondence between the unit interval $\mathfrak{A}$ and the unit square $\mathfrak{B}$ afforded by Hilbert's curve defined in 573. A number of interesting facts will reward our labor. We begin by seeking the points P in $\mathfrak{A}$ which correspond to a given Q in $\mathfrak{B}$.

To this end let us note *how P enters and leaves an η square.* Let B be a square of D_n. In the next division B falls into four

ares $B_1 \cdots B_4$ and in the $n+2^d$ division in 16 squares $B_{\iota j}$. these last, four lie at the vertices of B; we call them *vertex ares*. The other 12 are *median squares*. A simple considera- shows that the η squares of D_{n+2} are so numbered that we ays enter a square B belonging to D_n, and also leave it by a ex square.

ince this is true of every division, we see on passing to the t that the point Q enters and leaves any η square at the ver- s of η. We call this the *vertex law*.

et us now *classify the points* P, Q.

P is an end point of some division $D_n >$ we call it a *terminal t*, otherwise an *inner point*, because it lies within a sequence intervals $\delta' > \delta'' > \cdots \doteq 0$.

he points Q we divide into four classes:

' vertex points, when Q is a vertex of some division.

' inner points, when Q lies within a sequence of squares

$$\eta' > \eta'' > \cdots \doteq 0.$$

' lateral points, when Q lies on a side of some η square but er at a vertex.

' points lying on the edge of the original square $\mathfrak{B}$. Points his class also lie in 1°, 3°.

Ve now seek the points P corresponding to a Q lying in one of e four classes.

lass 1°. *Q a Vertex Point.* Let D_n be the first division such Q is at a vertex. Then Q lies in four squares η_ι, η_j, η_κ, η_l of

here are 5 cases:

) $\iota\, j\, k\, l$ are consecutive.

) $\iota\, j\, k$ are consecutive, but not l.

) $\iota\, j$ are consecutive, but not $k\, l$.

) $\iota\, j$, also $k\, l$, are consecutive.

) no two are consecutive.

simple analysis shows that α), β) are not permanent in the owing divisions; γ), δ) may or may not be permanent; ϵ) is nanent.

Now, whenever a case is permanent, we can enclose Q in a sequence of η squares whose sides $\doteq 0$. To this sequence corresponds uniquely a sequence of δ intervals of lengths $\doteq 0$. Thus to two consecutive squares will correspond two consecutive intervals which converge to a single point P in $\mathfrak{A}$. If the squares are not consecutive, the corresponding intervals converge to two distinct points in $\mathfrak{A}$. Thus we see that when γ) is permanent, to Q correspond three points P. When δ) is permanent, to Q correspond two points P. While when Q belongs to ϵ), four points P correspond to it.

Class 2°. *Q an Inner Point.* Obviously to each Q corresponds one point P and only one.

Class 3°. *Q a Lateral Point.* To fix the ideas let Q lie on a vertical side of one of the η's. Let it lie between η_ι, η_j of D_n. There are two cases:

α) $$j = \iota + 1.$$

β) $$j > \iota + 1.$$

We see easily that α) is not permanent, while of course β) is. Thus to each Q in class 3°, there correspond two points P.

Class 4°. *Q lies on the edge of* $\mathfrak{B}$. If Q is a vertex point, to it may correspond one or two points P. If Q is not a vertex point, only one point P corresponds to it.

To sum up we may say:

To each inner point Q corresponds one inner point P.

To each lateral point Q correspond two points P.

To each edge point Q correspond one or two points P.

To each vertex point Q, correspond two, three, or four points P.

2. As a result of the preceding investigation we show easily that:

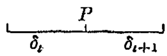
To the points on a line parallel to one of the sides of $\mathfrak{B}$ correspond in $\mathfrak{A}$ an apantactic perfect set.

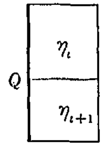
3. Let us now consider the tangents to Hilbert's curve which we denote by H.

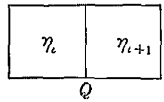
Let Q be a vertex point. We saw there were three permanent cases γ), δ), ϵ).

In cases γ), δ) we saw that to two consecutive δ intervals correspond permanently two contiguous vertical or horizontal squares.

Thus as t ranges over $\delta_\iota, \delta_{\iota+1}$, the point x, y ranges over these squares, and the secant line joining Q and this variable point x, y oscillates through 180°. There is thus no tangent at Q. In case ϵ) we see similarly that the secant line ranges through 90°. Again there is no tangent at Q.







In the same way we may treat the three other classes. We find that the secant line never converges to a fixed position, and may oscillate through 360°, viz. when Q is an inner point. As a result we see that *Hilbert's curve has at no point a tangent, nor even a unilateral tangent.*

4. Associated with Hilbert's curve H are two other curves,

$$x = \phi(t) \quad , \quad \text{and } y = \psi(t).$$

The functions ϕ, ψ being one-valued and continuous in $\mathfrak{A}$, these curves are continuous and they do not have a multiple point. A very simple consideration shows that *they do not have even a unilateral tangent at a pantactic set of points in $\mathfrak{A}$.*

575. *Property* 7°. *Equations of a Curve.* As already remarked, it is commonly thought that the equation of a curve may be written in any one of the three forms

$$y = f(x), \tag{1}$$

$$\Phi(x, y) = 0, \tag{2}$$

$$x = \phi(t) \quad , \quad y = \psi(t), \tag{3}$$

and if these functions are continuous, these equations define continuous curves.

Let us look at the Hilbert curve H. We saw its equation could be expressed in the form 3). H cuts an ordinate at every point of it for which $0 \le y \le 1$. Thus if we tried to define H by

an equation of the type 1), $f(x)$ would have to take on every value between 0 and 1 for each value of x in $\mathfrak{A} = (0, 1)$. No such functions are considered in analysis.

Again, we saw that to any value $x = a$ in $\mathfrak{A}$ corresponds a perfect apantactic set of values $\{t_a\}$ having the cardinal number $\mathfrak{c}$. Thus the inverse function of $x = \phi(t)$ is a many-valued function of x whose different values form a set whose cardinal number is $\mathfrak{c}$. Such functions have not yet been studied in analysis.

How is it possible in the light of such facts to say that we may pass from 3) to 1) or 2) by *eliminating* t from 3). And if we cannot, how can we say a curve can be represented equally well by any of the above three equations, or if the curve is given by one of these three equations, we may suppose it replaced by one of the other two whenever convenient. Yet this is often done.

In this connection we may call attention to the loose way *elimination* is treated. Suppose we have a set of equations

$$f_1(x_1 \cdots x_m\ t_1 \cdots t_n) = 0,$$

$$f_{n+1}(x_1 \cdots x_m\ t_1 \cdots t_n) = 0.$$

We often see it stated that one can eliminate $t_1 \cdots t_n$ and obtain a relation involving the x's alone. Any reasoning based on such a procedure must be regarded as highly unsatisfactory, in view of what we have just seen, until this elimination process has been established.

576. *Property* 8°. *Closed Curves.* A circle, a rectangle, an ellipse are examples of closed curves. Our intuition tells us that it is impossible to pass from the inside to the outside without crossing the curve itself. If we adopt the definition of a closed curve without multiple point given in I, 362, we find it no easy matter to establish this property which is so obvious for the simple closed curves of our daily experience. The first to effect the demonstration was *Jordan* in 1892. We give here* a proof due to *de la Vallée-Poussin.*†

Let us call for brevity a continuous curve without double point

* The reader is referred to a second proof due to Brouwer and given in 508 seq.

† *Cours d'Analyse*, Paris, 1903, Vol. 1, p. 307.

Jordan curve. A continuous closed curve without double point will then be a *closed Jordan curve*. Cf. I, 362.

577. *Let C be a closed Jordan curve. However small $\sigma > 0$ is taken, there exists a polygonal ring R containing C and such that*

1° *Each point of R is at a distance $< \sigma$ from C.*

2° *Each point of C is at a distance $< \sigma$ from the edges of R.*

For let
$$x = \phi(t) \quad , \quad y = \psi(t) \tag{1}$$
be continuous one-valued functions of t in $T = (a < b)$ defining C. Let $D = (a, a_1, a_2 \cdots b)$ be a division of T of norm d. Let $\alpha, \alpha_1, \alpha_2 \cdots$ be points of C corresponding to $a, a_1 \cdots$ If d is sufficiently small, the distance between two points on the arc $C_\iota = (\alpha_{\iota-1}, \alpha_\iota)$ is $< \epsilon'$, small at pleasure. Let Δ be a quadrate division of the x, y plane of norm δ. Let us shade all cells containing a point of C_ι. These form a connected domain since C_ι is continuous. We can thus go around its outer edge without a break.* If this shaded domain contains unshaded cells, let us shade these too. We call the result a link A_ι. It has only one edge E_ι, and the distance between any two points of E_ι is obviously $< \epsilon' + 2\sqrt{2}\,\delta$. We can choose d, δ so small that
$$\epsilon' + 2\sqrt{2}\,\delta < \sigma, \qquad \text{arbitrarily small.} \tag{1}$$
Then the distance between any two points of A_ι is $< \sigma$. Let ϵ'' be the least distance between non-consecutive arcs C_ι. We take δ so small that we also have
$$\sqrt{2}\,\delta < \frac{\epsilon''}{2}. \tag{2}$$
Then two non-consecutive links A_ι, A_j have no point in common. For then their edges would have a common point P. As P lies on E_ι its distance from C_ι is $\leq \sqrt{2}\,\delta$. Its distance from C_j is also $\leq \sqrt{2}\,\delta$. Thus there is a point P_ι on C_ι, and a point P_j on C_j such that
$$\eta = \overline{P_\iota P_j} \leq 2\sqrt{2}\,\delta.$$

* Here and in the following, intuitional properties of polygons are assumed as known.

But by hypothesis $\epsilon'' < \eta$. Hence

$$\epsilon'' \leq 2\sqrt{2}\,\delta,$$

which contradicts 2).

Thus the union of these links form a ring R whose edges are polygons without double point. One of the edges, say G_ι, lies within the other, which we call G_e. The curve C lies within R. The inner polygon G_ι must exist, since non-consecutive links have no point in common.

578. 1. *Interior and Exterior Points.* Let $\sigma_1 > \sigma_2 > \cdots \doteq 0$. Let $R_1, R_2 \cdots$ be the corresponding rings, and let

$$G'_\iota \quad , \quad G''_\iota \cdots$$

$$G'_e \quad , \quad G''_e \cdots$$

be their inner and outer edges. A point P of the plane not on C which lies inside some G_ι we call an *interior or inner point of* C. If P lies outside some G_e, we call it an *exterior or outer point of* C.

Each point P not on C must belong to one of these two classes. For let $\rho = \text{Dist}(P, C)$; then ρ is $>$ some σ_n. It therefore lies within $G^{(n)}_\iota$ or without $G^{(n)}_e$, and is thus an inner or an outer point. Obviously this definition is independent of the sequence of rings $\{R_n\}$ employed. The points of the curve C are interior to each $G^{(n)}_e$ and exterior to each $G^{(n)}_\iota$.

Inner points must exist, since the inner polygons exist as already observed. Let us denote the inner points by $\mathfrak{J}$ and the outer points by $\mathfrak{O}$. Then the frontiers of $\mathfrak{J}$ and $\mathfrak{O}$ are the curve C.

2. We show now that

1° *Two inner points can be joined by a broken line L_ι lying in $\mathfrak{J}$.*

2° *Two outer points can be joined by a broken line L_e lying in $\mathfrak{O}$.*

3° *Any continuous curve $\mathfrak{K}$ joining an inner point i and an outer point e has a point in common with C.*

To prove 3°, let

$$x = f(t) \quad , \quad y = g(t)$$

be the equations of $\mathfrak{K}$, the variable t ranging over an interval $\tau = (\alpha < \beta)$, $t = \alpha$ corresponding to i and $t = \beta$ to e. Let t' be

such that $\alpha \leq t < t'$ gives inner points, while $t = t'$ does not give an inner point. Thus the point corresponding to $t = t'$ is a frontier point of $\mathfrak{J}$ and hence a point of C.

To prove 1°. If A, B are inner points, they lie within some G_ι. We may join A, B, G_ι by broken lines L_a, L_b meeting G_ι at the points A', B', say. Let G_{ab} be the part of G_ι lying between A', B'. Then

$$L_a + G_{ab} + L_b$$

is a broken line joining A to B.

The proof of 2° is similar.

579. 1. Let P', P'' correspond to $t = t'$, $t = t''$, on the curve C defined by 577, 1). If $t' < t''$, we say P' *precedes* P'' and write $P' < P''$.

Any set of points on C corresponding to an increasing set of values of t is called an *increasing set.*

As t ranges from a to b, the point P ranges over C in a *direct sense.*

We may thus consider a Jordan curve as an ordered set, in the sense of 265.

2. (*De la Vallée-Poussin.*) *On each arc C_ι of the curve C, there exists at least one point P_j such that*

$$P_1 < P_2 < P_3 < \cdots \tag{1}$$

may be regarded as the vertices of a closed polygon without double point and whose sides are all $< \epsilon$.

For in the first place we may take $\delta > 0$ so small that no square of Δ contains a point lying on non-consecutive arcs C_ι of C. Let us also take Δ so that the point α corresponding to $t = a$ lies *within* a square, call it S_1, of Δ. As t increases from $t = a$, there is a last point P_1 on C where the curve leaves S_1. The point P_1 lies in another square of Δ, call it S_2, containing other points of C. Let P_2 be the last point of C in S_2. In this way we may continue, getting a sequence 1).

There exists at least one point of 1) on each arc C_ι. For otherwise a square of Δ would contain points lying on non-consecutive arcs C_κ. The polygon determined by 1) cannot have a double

point, since each side of it lies in one square. The sides are $< \epsilon$, provided we take $\delta\sqrt{2} < \epsilon$, since the diagonal is the longest line we can draw in a square of side δ.

580. *Existence of Inner Points.* To show that the links form a ring with inner points, *Schönfliess* * has given a proof which may be rendered as follows:

Let us take the number of links to be even, and call them L_1, $L_2, \cdots L_{2n}$. Then L_1, L_3, $L_5 \cdots$ lie entirely outside each other. Since L_1, L_2 overlap, let P be an inner common point. Similarly let Q be an inner common point of L_2, L_3. Then P, Q lying within L_2 may be joined by a finite broken line b lying within L_2. Let b_2 be that part of it lying between the last point of leaving L_1 and the following point of meeting L_3. In this way the pairs of links

$$L_1 L_3 \quad ; \quad L_3 L_5 \quad ; \cdots$$

define finite broken lines

$$b_2 \quad , \quad b_4 \quad , \quad \cdots b_{2n}.$$

No two of these can have a common point, since they lie in non-consecutive links. The union of the points in the sets

$$L_1 \quad , \quad b_2 \quad , \quad L_3 \quad , \quad b_4 \cdots L_{2n-1} \quad , \quad b_{2n}$$

we call a *ring*, and denote it by $\mathfrak{R}$. The points of the plane not in $\mathfrak{R}$ fall into two parts, *separated* by $\mathfrak{R}$. Let $\mathfrak{T}$ denote the part which is limited, together with its frontier. We call $\mathfrak{T}$ the *interior* of $\mathfrak{R}$. That $\mathfrak{T}$ has *inner* points is regarded as obvious since it is defined by the links

$$L_1 \quad , \quad L_3 \quad , \quad L_5 \cdots$$

which pairwise have no point in common, and by the broken lines

$$b_2 \quad , \quad b_4 \quad , \quad b_6 \cdots$$

each of which latter lies entirely *within* a link.

Let $\qquad \mathfrak{L}_{2m} = Dv(L_{2m}, \mathfrak{T}) \quad , \qquad m = 1, 2, \cdots$

* *Die Entwickelung der Lehre von den Punktmannigfaltigkeiten.* Leipzig, 1908, Part 2, p. 170.

Then these $\mathfrak{L}$ have pairwise no point in common since the L_{2m} have not.

Let
$$\mathfrak{T} = \mathfrak{L}_2 + \mathfrak{L}_4 + \cdots + \mathfrak{L}_{2n} + \mathfrak{K}.$$

Then $\mathfrak{K} > 0$. For let us adjoin L_2 to $\mathfrak{R}$, getting a ring $\mathfrak{R}_2$ whose interior call $\mathfrak{T}_2$. That $\mathfrak{T}_2$ has inner points follows from the fact that it contains $\mathfrak{L}_4$, $\mathfrak{L}_6 \cdots$ Let us continue adjoining the links L_4, $L_6 \cdots$ Finally we reach L_{2n}, to which corresponds the ring $\mathfrak{R}_{2n}$, whose interior, if it exists, is $\mathfrak{T}_{2n}$. If $\mathfrak{T}_{2n}$ does not exist, $\mathfrak{T}_{2n-2}$ contains only $\mathfrak{L}_{2n}$. This is not so, for on the edge of L_1 bounding $\mathfrak{T}$, is a point P, such that some $D_\rho(P)$ contains points of no L except L_1. In fact there is a point P on the edge of L_1 not in either L_2 or L_{2n}, as otherwise these would have a point in common. Now, if however small $\rho > 0$ is taken, $D_\rho(P)$ contains points of some L other than L_1, the point P must lie in L_κ which is absurd, since L_1 has only points in common with L_2, L_{2n}, and P is not in either of these. Thus the adjunction of L_2, L_4, $\cdots$ L_{2n} produces a ring $\mathfrak{R}_{2n}$ whose interior $\mathfrak{T}_{2n}$ does not reduce to 0; it has inner points.

581. *Property* 9°. *Area.* That a figure defined by a closed curve without double point, *i.e.* the interior of a Jordan curve, has an area, has long been an accepted fact in intuitional geometry. Thus Lindemann, *Vorlesungen über Geometrie*, vol. 2, p. 557, says "einer allseitig umgrenzten Figur kommt ein bestimmter Flächeninhalt zu." The truth of such a statement rests of course on the definition of the term area. In I, 487, 702 we have given a definition of area for any limited plane point set $\mathfrak{A}$ which reduces to the ordinary definition when $\mathfrak{A}$ becomes an ordinary plane figure. In our language $\mathfrak{A}$ has an area when its frontier points form a discrete set. Let
$$x = \phi(t) \quad , \quad y = \psi(t)$$
define a Jordan curve $\mathfrak{C}$, as t ranges over $T = (a < b)$. The figure $\mathfrak{A}$ defined by this curve has the curve as frontier. In I, 708, 710, we gave various cases in which $\mathfrak{C}$ is discrete. The reasoning of I, 710, gives us also this important case:

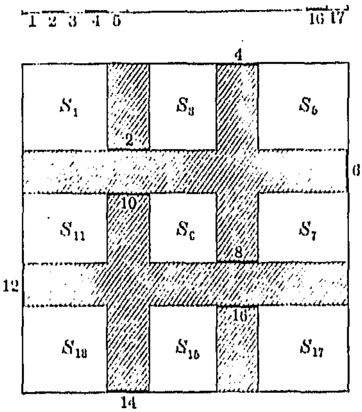
If one of the continuous functions ϕ, ψ defining the Jordan curve $\mathfrak{C}$, has limited variation in T, then $\mathfrak{C}$ is discrete.

It was not known whether $\mathfrak{C}$ would remain discrete if the condition of limited variation was removed from both coördinates, until Osgood * exhibited a Jordan curve which is not discrete. This we will now discuss.

582. 1. *Osgood's Curve.* We start with a unit segment $T = (0, 1)$ on the t axis, and a unit square S in the xy plane.

We divide T into 17 equal parts

$$T_1, T_2, \cdots T_{17}, \qquad (1$$



and the square S into 9 equal squares

$$S_1, S_3, S_5 \cdots S_{17}, \qquad (2$$

by drawing 4 bands B_1 which are shaded in the figure. On these bands we take 8 segments,

$$s_2, s_4, s_6 \cdots s_{16}, \qquad (3$$

marked heavy in the figure.

Then as t is ranging from left to right over the even or black intervals $T_2, T_4, \cdots T_{16}$ marked heavy in the figure, the point x, y on Osgood's curve, call it $\mathfrak{O}$, shall range univariantly over the segments 3).

While t is ranging over the odd or white intervals $T_1, T_3 \cdots T_{17}$ the point xy on $\mathfrak{O}$ shall range over the squares 2) as determined below.

Each of the odd intervals 1) we will now divide into 17 equal intervals T_{ij} and in each of the squares 2) we will construct horizontal and vertical bands B_2 as we did in the original square S. Thus each square 2) gives rise to 8 new segments on $\mathfrak{O}$ corresponding to the new black intervals in T, and 9 new squares S_{ij} corresponding to the white intervals. In this way we may continue indefinitely.

The points which finally get in a black interval call β, the others are limit points of the β's and we call them λ. The point

* *Trans. Am. Math. Soc.*, vol. 4 (1903), p. 107.

on $\mathfrak{D}$ corresponding to a β point has been defined. The point of $\mathfrak{D}$ corresponding to a point λ is defined to be the point lying in the sequence of squares, one inside the other, corresponding to the sequence of white intervals, one inside the other, in which λ falls, in the successive divisions of T.

Thus to each t in T corresponds a single point x, y in S. The aggregate of these points constitutes Osgood's curve. Obviously the x, y of one of its points are one-valued functions of t in T, say

$$x = \phi(t) \quad , \quad y = \psi(t) . \tag{4}$$

The curve $\mathfrak{D}$ has no double point. This is obvious for points of $\mathfrak{D}$ lying in black segments. Any other point falls in a sequence of squares

$$S_\iota > S_{\iota j} > S_{\iota j \kappa} \cdots$$

to which correspond intervals

$$T_\iota > T_{\iota j} > T_{\iota j \kappa} \cdots$$

in which the corresponding t's lie. But only one point t is thus determined.

The functions 4) *are continuous.* This is obvious for points β lying within the black intervals of T. It is true for the points λ. For λ lies within a sequence of white intervals, and while t ranges over one of these, the point on $\mathfrak{D}$ ranges in a square. But these squares shut down to a point as the intervals do. Thus ϕ, ψ are continuous at $t = \lambda$. In a similar manner we show they are continuous at the end points of the black intervals.

We note that to $t = 0$ corresponds the upper left-hand corner of S, and to $t = 1$, the diagonally opposite point.

2. Up to the present we have said nothing as to the width of the shaded bands

$$B_1 \quad , \quad B_2 \cdots$$

introduced in the successive steps. Let

$$A = a_1 + a_2 + \cdots$$

be a convergent positive term series whose sum $A \leq 1$. We choose B_1 so that its area is a_1, B_2 so that its area is a_2, etc. Then

$$\underline{\mathfrak{D}} = 0 \quad , \quad \overline{\mathfrak{D}} = 1 - A, \tag{5}$$

as we now show. For $\mathfrak{D}$ has obviously only frontier points; hence $\underline{\mathfrak{D}} = 0$. Since $\mathfrak{D}$ is complete, it is measurable and

$$\widehat{\mathfrak{D}} = \bar{\mathfrak{D}}.$$

Let $O = S - \mathfrak{D}$, and $B = \{B_n\}$. Then $O < B$. For any point which does not lie in some B_n lies in a sequence of convergent squares $S_\iota > S_{\iota j} > \cdots$ which converge to a point of $\mathfrak{D}$. Now

$$\widehat{B} = \widehat{B}_1 + \widehat{B}_2 + \cdots = A.$$

On the other hand, B contains a null set of points of $\mathfrak{D}$, viz. the black segments. Thus

$$\widehat{O} = \widehat{B} = A \quad , \quad \text{and hence } \widehat{\mathfrak{D}} = 1 - A$$

and 5) is established.

Thus *Osgood's curve is continuous, has no double point, and its upper content is* $1 - A$.

3. To get a continuous closed curve C without double point we have merely to join the two end points α, β of Osgood's curve by a broken line which does not cut itself or have a point in common with the square S except of course the end points α, β. Then C bounds a figure $\mathfrak{F}$ whose frontier is not discrete, and $\mathfrak{F}$ does not have an area. Let us call such curves *closed* Osgood curves.

Thus we see that there exist regions bounded by Jordan curves which do not have area in the sense current since the Greek geometers down to the present day.

Suppose, however, we discard this traditional definition, and employ as definition of area its measure. Then we can say:

A figure $\mathfrak{F}$ formed of a closed Jordan curve J and its interior $\mathfrak{J}$ has an area, viz. Meas $\mathfrak{F}$.

For Front $\mathfrak{F} = J$. Hence $\mathfrak{F}$ is complete, and is therefore measureable.

We note that

$$\widehat{\mathfrak{F}} = \mathcal{J} + \widehat{\mathfrak{J}}.$$

We have seen there are Jordan curves such that

$$\mathcal{J} > 0.$$

We now have a definition of area which is in accordance with the promptings of our geometric intuition. It must be remembered, however, that this definition has been only recently discovered, and that the definition which for centuries has been accepted leads to results which flatly contradict our intuition, which leads us to say that a figure bounded by a continuous closed curve has an area.

583. At this point we will break off our discussion of the relation between our intuitional notion of a curve, and the configuration determined by the equations

$$x=\phi(t) \quad , \quad y=\psi(t) \tag{1}$$

where ϕ, ψ are one-valued continuous functions of t in an interval T. Let us look back at the list of properties of an intuitional curve drawn up in 563. We have seen that the analytic curve 1) does not need to have tangents at a pantactic set of points on it; no arc on it needs have a finite length; it may completely fill the interior of a square; its equations cannot always be brought in the forms $y=f(x)$ or $F(xy)=0$, if we restrict ourselves to functions f or F employed in analysis up to the present; it does not need to have an area as that term is ordinarily understood.

On the other hand, it is continuous, and when closed and without double point it forms the complete boundary of a region.

Enough in any case has been said to justify the thesis that geometric reasoning in analysis must be used with the greatest circumspection.

Detached and Connected Sets

584. In the foregoing sections we have studied in detail some of the properties of curves defined by the equations

$$x=\phi(t) \quad , \quad y=\psi(t).$$

Now the notion of a curve, like many other geometric notions, is independent of an analytic representation. We wish in the following sections to consider some of these notions from this point of view.

585. 1. Let $\mathfrak{A}$, $\mathfrak{B}$ be point sets in m-way space $\mathfrak{R}_m$. If

$$\text{Dist}(\mathfrak{A}, \mathfrak{B}) > 0,$$

we say $\mathfrak{A}$, $\mathfrak{B}$ are *detached*. If $\mathfrak{A}$ cannot be split up into two parts $\mathfrak{B}$, $\mathfrak{C}$ such that they are detached, we say $\mathfrak{A}$ has *no detached parts*. If $\mathfrak{A} = \mathfrak{B} + \mathfrak{C}$ and $\text{Dist}(\mathfrak{B}, \mathfrak{C}) > 0$, we say $\mathfrak{B}$, $\mathfrak{C}$ are *detached parts* of $\mathfrak{A}$.

Let the set of points, finite or infinite,

$$a, a_1, a_2, \cdots b \tag{1}$$

be such that the distance between two successive ones is $\leq \epsilon$. We call 1) an ϵ-sequence between a, b; or a sequence with segments $(a_\iota, a_{\iota+1})$ of length $\leq \epsilon$. We suppose the segments ordered so that we can pass continuously from a to b over the segments without retracing. If 1) is a finite set, the sequence is *finite*, otherwise *infinite*.

2. *Let $\mathfrak{A}$ have no detached parts. Let a, b be two of its points. For each $\epsilon > 0$, there exists a finite ϵ-sequence between a, b, and lying in $\mathfrak{A}$.*

For about a describe a sphere of radius ϵ. About each point of $\mathfrak{A}$ in this sphere describe a sphere of radius ϵ. About each point of $\mathfrak{A}$ in each of these spheres describe a sphere of radius ϵ. Let this process be repeated indefinitely. Let $\mathfrak{B}$ denote the points of $\mathfrak{A}$ made use of in this procedure. If $\mathfrak{B} < \mathfrak{A}$, let $\mathfrak{C} = \mathfrak{A} - \mathfrak{B}$. Then $\text{Dist}(\mathfrak{B}, \mathfrak{C}) > \epsilon$, and $\mathfrak{A}$ has detached parts, which is contrary to hypothesis. Thus there are sets of ϵ-spheres in $\mathfrak{A}$ joining a and b.

Among these sets there are finite ones. For let $\mathfrak{F}$ denote the set of points in $\mathfrak{A}$ which may be joined to a by finite sequences; let $\mathfrak{G} = \mathfrak{A} - \mathfrak{F}$. Then $\text{Dist}(\mathfrak{F}, \mathfrak{G}) \geq \epsilon$. For if $< \epsilon$, there is a point f in $\mathfrak{F}$, and a point g in $\mathfrak{G}$ whose distance is $< \epsilon$. Then a and g can be joined by a finite ϵ-sequence, which is contrary to hypothesis.

3. *If $\mathfrak{A}$ has no detached parts, it is dense.*

For if not dense, it must have at least one isolated point a. But then a, and $\mathfrak{A} - a$ are detached parts of $\mathfrak{A}$, which contradicts the hypothesis.

4. *Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ be complete and $\mathfrak{A} = (\mathfrak{B}, \mathfrak{C})$. If $\mathfrak{A}$ has no detached parts, $\mathfrak{B}$, $\mathfrak{C}$ have at least one common point.*

or if $\mathfrak{B}$, $\mathfrak{C}$ have no common point, $\delta = \text{Dist}(\mathfrak{B}, \mathfrak{C})$ is > 0. δ cannot > 0, since $\mathfrak{B}$, $\mathfrak{C}$ would then be detached parts of $\mathfrak{A}$. ce $\delta = 0$ and since $\mathfrak{B}$, $\mathfrak{C}$ are complete, they have a point in mon.

. *If $\mathfrak{A}$ is such that any two of its points may be joined by an quence lying in $\mathfrak{A}$, where $\epsilon > 0$ is small at pleasure, $\mathfrak{A}$ has no ched parts.*

or if $\mathfrak{A}$ had $\mathfrak{B}$, $\mathfrak{C}$ as detached parts, let $\text{Dist}(\mathfrak{B}, \mathfrak{C}) = \delta$. Then 0. Hence there is no sequence joining a point of $\mathfrak{B}$ with a it of $\mathfrak{C}$ with segments $< \delta$.

If $\mathfrak{A}$ is complete and has no detached parts, it is said to be *ected.* We also call $\mathfrak{A}$ a *connex.*

s a special case, a point may be regarded as a connex.

If $\mathfrak{A}$ is connected, it is perfect.

or by 3 it is dense, and by definition it is complete.

If $\mathfrak{A}$ is a rectilinear connex, it has a first point α and a last t β, and contains every point in the interval (α, β).

or being limited and complete its minimum and maximum n $\mathfrak{A}$ and these are respectively α and β. Let now

$$\epsilon_1 > \epsilon_2 > \cdots \doteq 0.$$

re exists an ϵ_1-sequence C_1 between α, β. Each segment has ϵ_2-sequence C_2. Each segment of C_2 has an ϵ_3-sequence C_3, Let C be the union of all these sequences. It is pantactic $\alpha, \beta)$. As $\mathfrak{A}$ is complete,

$$\mathfrak{A} = (\alpha, \beta).$$

Images

36. Let $\quad x_1 = f_1(t_1 \cdots t_m) \quad \cdots \quad x_n = f_n(t_1 \cdots t_m)$ (1

ne-valued functions of t in the point set $\mathfrak{T}$. As t ranges over he point $x = (x_1 \cdots x_n)$ will range over a set $\mathfrak{A}$ in an n-way e $\mathfrak{R}_n$. We have called $\mathfrak{A}$ the image of $\mathfrak{T}$. Cf. I, 238, 3. ne functions f are not one-valued, to a point t may correspond ral images x', x'' $\cdots$ finite or infinite in number. Conversely

to the point x may correspond several values of t. If to each point t correspond in general r values of x, and to each x in general s values of t, we say the correspondence between $\mathfrak{T}$, $\mathfrak{A}$ is r to s. If $r = s = 1$ the correspondence is 1 to 1 or *unifold*; if $r > 1$, it is *manifold*. If $r = 1$, $\mathfrak{A}$ is a *simple* image of $\mathfrak{T}$, otherwise it is a *multiple* image. If the functions 1) are one-valued and continuous in $\mathfrak{T}$, we say $\mathfrak{A}$ is a *continuous image* of $\mathfrak{T}$.

587. *Transformations of the Plane.* *Example 1.* Let

$$u = x \sin y \quad , \quad v = x \cos y. \tag{1}$$

We have in the first place

$$u^2 + v^2 = x^2.$$

This shows that the image of a line $x = a$, $a \neq 0$, parallel to the y-axis is a circle whose center is the origin in the u, v plane, and whose radius is a. To the y-axis in the x, y plane corresponds the origin in the u, v plane.

From 1) we have, secondly,

$$\frac{u}{v} = \tan y.$$

This shows that the image of a line $y = b$, is a line through the origin in the u, v plane.

From 1) we have finally that u, v are periodic in y, having the period 2π. Thus as x, y ranges in the band B, formed by the two parallels $y = \pm\pi$, or $-\pi < y \leq \pi$, the point u, v ranges over the entire u, v plane once and once only.

The correspondence between B and the u, v plane is unifold, except, as is obvious, to the origin in the u, v plane corresponds the points on the y-axis.

Let us apply the theorem of I, 441, on implicit functions. The determinant Δ is here

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \sin y, & \cos y \\ x\cos y, & -x\sin y \end{vmatrix} = -x.$$

As this is $\neq 0$ when x, y is not on the y-axis, we see that the correspondence between the *domain* of any such point and its image is 1 to 1. This accords with what we have found above.

is, however, a much more restricted result than we have found; r we have seen that the correspondence between any limited int set $\mathfrak{A}$ in B which does not contain a point of the y-axis and image is unifold.

588. *Example 2.* Let

$$u = \frac{y}{\sqrt{x^2 + y^2}} \quad , \quad v = \sqrt{x^2 + y^2}, \tag{1}$$

e radical having the positive sign. Let us find the image of the st quadrant Q in the x, y plane.

From 1) we have at once

$$0 \leq u \leq 1 \quad , \quad v \geq 0.$$

Hence the image of Q is a band B parallel to the v-axis.

From 1) we get secondly

$$y = uv \quad , \quad x = v\sqrt{1 - u^2}. \tag{2}$$

Hence

$$x^2 + y^2 = v^2.$$

Thus the image of a circle in Q whose center is the origin and iose radius is a is a segment of a right line $v = a$.

When $x = y = 0$, the equations 1) do not define the correspondg point in the u, v plane. If we use 2) to define the correondence, we may say that to the line $v = 0$ in B corresponds the igin in the x, y plane. With this exception the correspondence tween Q and B is uniform, as 1), 2) show.

The determinant Δ of 1) is, setting

$$r = \sqrt{x^2 + y^2},$$

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \dfrac{-xy}{r^3} & \dfrac{x^2}{r^3} \\ \dfrac{x}{r} & \dfrac{y}{r} \end{vmatrix} = \frac{-x}{x^2 + y^2}$$

r any point x, y different from the origin.

589. *Example 3.* *Reciprocal Radii.* Let O be the origin in the y plane and Ω the origin in the u, v plane. To any point $= (x, y)$ in the x, y plane different from the origin shall corspond a point $Q = (u, v)$ in the u, v plane such that ΩQ has

the same direction as OP and such that $OP \cdot \Omega Q = 1$. Analytically we have

$$x = \lambda y \quad , \quad u = \lambda v \quad , \quad \lambda > 0,$$

and

$$(u^2 + v^2)(x^2 + y^2) = 1.$$

From these equations we get

$$x = \frac{u}{u^2 + v^2} \quad , \quad y = \frac{v}{u^2 + v^2} \tag{1}$$

and also

$$u = \frac{x}{x^2 + y^2} \quad , \quad v = \frac{y}{x^2 + y^2}.$$

The correspondence between the two planes is obviously unifold except that no point in either plane corresponds to the origin in the other plane. We find for any point x, y different from the origin that

$$\Delta = \frac{\partial (u, v)}{\partial (x, y)} = - \frac{1}{(x^2 + y^2)^2}.$$

Obviously from the definition, to a line through the origin in the x, y plane corresponds a similar line in the u, v plane. As xy moves toward the origin, u, v moves toward infinity.

Let x, y move on the line $x = a \neq 0$. Then 1) shows that u, v moves along the circle

$$a(u^2 + v^2) - u = 0$$

which passes through the origin. A similar remark holds when x, y moves along the line $y = b \neq 0$.

590. Such relations between two point sets $\mathfrak{A}$, $\mathfrak{B}$ as defined in 586 may be formulated independently of the functions f. In fact with each point a of $\mathfrak{A}$ we may associate one or more points $b_1, b_2 \cdots$ of $\mathfrak{B}$ according to some law. Then $\mathfrak{B}$ may be regarded as the image of $\mathfrak{A}$. We may now define the terms simple, manifold, etc., as in 586. When b corresponds to a we may write $b \sim a$.

We shall call $\mathfrak{B}$ a *continuous image of* $\mathfrak{A}$ when the following conditions are satisfied. 1° To each a in $\mathfrak{A}$ shall correspond but one b in $\mathfrak{B}$, that is, $\mathfrak{B}$ is a simple image of $\mathfrak{A}$. 2° Let $b \sim a$, let $a_1, a_2 \cdots$ be any sequence of points in $\mathfrak{A}$ which $\doteq a$. Let $b_n \sim a_n$. Then b_n must $\doteq b$ however the sequence $\{a_n\}$ is chosen.

When $\mathfrak{B}$ is a simple image of $\mathfrak{A}$, the law which determines which b of $\mathfrak{B}$ is associated with a point a of $\mathfrak{A}$ determines obviously n one-valued functions as in 586, 1), where $t_1 \cdots t_m$ are the m coördinates of a, and $x_1 \cdots x_n$ are the n coördinates of b. We call these functions 1) the *associated functions*. Obviously when $\mathfrak{B}$ is a continuous image, the associated functions are continuous in $\mathfrak{A}$.

591. 1. *Let $\mathfrak{B}$ be a simple continuous image of the limited complete set $\mathfrak{A}$. Then* 1° *$\mathfrak{B}$ is limited and complete. If* 2° *$\mathfrak{A}$ is perfect and only a finite number of points of $\mathfrak{A}$ correspond to any point of $\mathfrak{B}$, then $\mathfrak{B}$ is perfect. If* 3° *$\mathfrak{A}$ is a connex, so is $\mathfrak{B}$.*

To prove 1°. The case that $\mathfrak{B}$ is finite requires no proof. Let $b_1, b_2 \cdots$ be points of $\mathfrak{B}$ which $\doteq \beta$. We wish to show that β lies in $\mathfrak{B}$. To each b_n will correspond one or more points in $\mathfrak{A}$; call the union of all these points $\mathfrak{a}$. Since $\mathfrak{B}$ is a simple image, $\mathfrak{a}$ is an infinite set. Let $a_1, a_2 \cdots$ be a set of points in $\mathfrak{a}$ which $\doteq \alpha$, a limiting point of $\mathfrak{A}$. As $\mathfrak{A}$ is complete, α lies in $\mathfrak{A}$. Let $b \sim \alpha$. Let $b_{\iota_n} \sim a_n$. As $a_n \doteq \alpha$, $b_{\iota_n} \doteq \beta$. But $\mathfrak{B}$ being continuous, b_{ι_n} must $\doteq b$. Thus β lies in $\mathfrak{B}$. That $\mathfrak{B}$ is limited follows from the fact that the associated functions are continuous in the limited complete set $\mathfrak{A}$. *To prove* 2°. Suppose that $\mathfrak{B}$ had an isolated point b. Let $b \sim a$. Since $\mathfrak{A}$ is perfect, let $a_1, a_2 \cdots \doteq a$. Let $b_n \sim a_n$. Then as $\mathfrak{B}$ is continuous, $b_n \doteq b$, and b is not an isolated point. *To prove* 3°. We have only to show that there exists an ϵ-sequence between any two points α, β of $\mathfrak{B}$, ϵ small at pleasure. Let $\alpha \sim a$, $\beta \sim b$. Since $\mathfrak{A}$ is connected there exists an η-sequence between a, b. Also the associated functions are uniformly continuous in $\mathfrak{A}$, and hence η may be taken so small that each segment of the corresponding sequence in $\mathfrak{B}$ is $\geq \epsilon$.

2. *Let $f(t_1 \cdots t_m)$ be one-valued and continuous in the connex $\mathfrak{A}$, then the image of $\mathfrak{A}$ is an interval including its end points.*

This follows from the above and from 585, 8.

3. *Let the correspondence between $\mathfrak{A}$, $\mathfrak{B}$ be unifold. If $\mathfrak{B}$ is a continuous image of $\mathfrak{A}$, then $\mathfrak{A}$ is a continuous image of $\mathfrak{B}$.*

For let $\{b_n\}$ be a set of points in $\mathfrak{B}$ which $\doteq b$. Let $a_n \sim b_n$, $a \sim b$. We have only to show that $a_n \doteq a$. For suppose that it does not, suppose in fact that there is a sequence $a_{\iota_1}, a_{\iota_2} \cdots$ which

$\doteq \alpha \neq a$. Let $\beta \sim \alpha$. Then $b_{\iota_1}, b_{\iota_2} \cdots \doteq \beta$. But any partial sequence of $\{b_n\}$ must $\doteq b$. Thus $b = \beta$, hence $a = \alpha$, hence $a_n \doteq a$.

4. *A Jordan curve J is a unifold continuous image of an interval T. Conversely if J is a unifold continuous image of an interval T, there exist two one-valued continuous functions*

$$x = \phi(t) \quad , \quad y = \psi(t)$$

such that as t ranges over T, the point x, y ranges over J. In case J is closed it may be regarded as the image of a circle Γ.

All but the last part of the theorem has been already established. To prove the last sentence we have only to remark that if we set

$$x = r \cos t \quad , \quad y = r \sin t$$

we have a unifold continuous correspondence between the points of the interval $(0, 2\pi^*)$ and the points of a circle.

5. The first part of 4 may be regarded as a *geometrical definition* of a Jordan curve. The image of a segment of the interval T or of the circle Γ, will be called an *arc of J*.

592. *Side Lights on Jordan Curves.* These curves have been defined by means of the equations

$$x = \phi(t), \qquad y = \psi(t). \tag{1}$$

As t ranges over the interval $T = (a < b)$, the point $P = (x, y)$ ranges over the curve J. This curve is a certain point set in the x, y plane. We may now propose this problem: We have given a point set $\mathfrak{C}$ in the x, y plane; may it be regarded as a Jordan curve? That is, do there exist two continuous one-valued functions 1) such that as t ranges over some interval T, the point P ranges over the given set $\mathfrak{C}$ without returning on itself, except possibly for $t = a$, $t = b$, when the curve would be closed?

Let us look at a number of point sets from this point of view.

593. *Example 1.*

1. Let $$y = \sin \frac{1}{x} \quad , \quad x \text{ in the interval } \mathfrak{A} = (-1, 1), \text{ but } \neq 0$$

$$= 0 \quad , \quad \text{for } x = 0.$$

Is this point set $\mathfrak{C}$ a Jordan curve? The answer is, No. For a ordan curve is a continuous image of an interval $\mathfrak{A}$. By 591, 1, t is complete. But $\mathfrak{C}$ is not complete, as all the points on the axis, $-1 \leq y \leq 1$ are limiting points of $\mathfrak{C}$, and only one of them elongs to $\mathfrak{C}$, viz. the origin.

2. Let us modify $\mathfrak{C}$ by adjoining to it all these missing limiting oints, and call the resulting point set C. Is C a Jordan curve? 'he answer is again, No. For if it were, we can divide the interal T into intervals δ so small that the oscillation of ϕ, ψ in any ne of them is $< \omega$. To the intervals δ_ι will correspond arcs C_ι on he curve, and two non-consecutive arcs C_ι are distant from each ther by an amount $>$ some ϵ, small at pleasure. This shows that ne of these arcs, say C_κ, must contain the segment on the y-axis $-1 \leq y \leq 1$. But then $\text{Osc}\, \psi = 2$ as t ranges over the correspondig δ_κ interval. Thus the oscillation of ψ cannot be made $< \epsilon$, owever small δ_κ is taken.

3. Let us return to the set $\mathfrak{C}$ defined in 1. Let A, B be the wo end points corresponding to $x = -1$, $x = 1$. Let us join them y an ordinary curve, a polygon if we please, which does not cut self or $\mathfrak{C}$. The resulting point set $\mathfrak{K}$ divides all the other points f the plane into two parts which cannot be joined by a continous curve without crossing $\mathfrak{K}$. For this point of view $\mathfrak{K}$ must be egarded as a *closed* configuration. Yet $\mathfrak{K}$ is obviously not complete.

On the other hand, let us look at the curve formed by removing ne points on a circle between two given points a, b on it. The emaining arc $\mathfrak{L}$ including the end points a, b is a *complete* set, but s it does not divide the other points of the plane into two sepaated parts, we cannot say $\mathfrak{L}$ is a closed configuration.

We mention this circumstance because many English writers se the term closed set where we have used the term complete. Cantor, who first introduced this notion, called such sets *abgechlossen*, which is quite different from *geschlossen* = closed.

594. *Example 2.* Let $\rho = e^{-\frac{1}{\theta}}$, for θ in the interval $\mathfrak{A} = (0, 1)$ xcept $\theta = 0$, where $\rho = 0$. These polar coördinates may easily be eplaced by Cartesian coördinates

$$x = \phi(\theta) = e^{-\frac{1}{\theta}} \cos\theta \quad , \quad y = e^{-\frac{1}{\theta}} \sin\theta \quad , \quad \text{in } \mathfrak{A},$$

except $\theta = 0$, when x, y both $= 0$. The curve thus defined is a Jordan curve.

Let us take a second Jordan curve

$$\rho = e^{-\left(\pi + \frac{1}{\theta}\right)},$$

with $\rho = 0$ for $\theta = 0$. If we join the two end points on these curves corresponding to $\theta = 1$ by a straight line, we get a closed Jordan curve J, which has an interior $\mathfrak{J}$, and an exterior $\mathfrak{O}$.

The peculiarity of this curve J is the fact that one point of it, viz. the origin $x = y = 0$, cannot be joined to an arbitrary point of $\mathfrak{J}$ by a finite broken line lying entirely in $\mathfrak{J}$; nor can it be joined to an arbitrary point in $\mathfrak{O}$ by such a line lying in $\mathfrak{O}$.

595. 1. It will be convenient to introduce the following terms. Let $\mathfrak{A}$ be a limited or unlimited point set in the plane. A set of distinct points in $\mathfrak{A}$

$$a_1 \quad , \quad a_2 \quad , \quad a_3 \cdots \tag{1}$$

determine a broken line. In case 1) is an infinite sequence, let a_n converge to a fixed point. If this line has no double point, we call it a *chain*, and the segments of the line *links*. In case not only the points 1) but also the links lie in $\mathfrak{A}$, we call the chain a *path*. If the chain or path has but a finite number of links, it is called *finite*.

Let us call a *precinct* a region, *i.e.* a set all of whose points are inner points, limited or unlimited, such than any two of its points may be joined by a finite path.

2. Using the results of 578, we may say that,—

A closed Jordan curve J divides the other points of the plane into two precincts, an inner $\mathfrak{J}$ and an outer $\mathfrak{O}$. Moreover, they have a common frontier which is J.

3. The closed Jordan curve considered in 594 shows that not every point of such a closed Jordan curve can always be joined to an arbitrary point of $\mathfrak{J}$ or $\mathfrak{O}$ by a finite path.

Obviously it can by an infinite path. For about this point, call it P, we can describe a sequence of circles of radii $r \doteq 0$. Between any two of these circles there lie points of $\mathfrak{J}$ and of $\mathfrak{O}$, if r is suf-

ciently small. In this way we may get a sequence of points in $\mathfrak{J}$, *iz.* $I_1, I_2 \cdots \doteq P$. Any two of these I_m, I_{m+1} may be joined by a ath which does not cut the path joining I_1 to I_m. For if a loop vere formed, it could be omitted.

4. *Any arc $\mathfrak{K}$ of a closed Jordan curve J can be joined by a path o an arbitrary point of the interior or exterior, which call $\mathfrak{A}$.*

For let $J = \mathfrak{K} + \mathfrak{L}$. Let k be a point of $\mathfrak{K}$ not an end point. et $\delta = \text{Dist}(k, \mathfrak{L})$, let a be a point of $\mathfrak{A}$ such that $\text{Dist}(a, k) < \frac{1}{2}\delta$. Then

$$\eta = \text{Dist}(a, \mathfrak{L}) > \tfrac{1}{2}\delta.$$

Hence the link $l = (a, k)$ has no point in common with $\mathfrak{L}$. Let be the first point of l in common with $\mathfrak{K}$. Then the link $n = (a, b)$ lies in $\mathfrak{A}$. If now α is any point of $\mathfrak{A}$, it may be joined o a by a path p. Then $p + m$ is a path in $\mathfrak{A}$ joining the arbi-rary point α to a point b on the arc $\mathfrak{K}$.

596. *Example 3.* For θ in $\mathfrak{A} = (0^*, 1)$ let

$$\rho = a(1 + e^{-\frac{1}{\theta}}),$$

nd

$$\rho = a(1 + e^{-\left(\pi + \frac{1}{\theta}\right)}).$$

These equations in polar coördinates define two non-intersecting pirals S_1, S_2 which coil about $\rho = a$ as an asymptotic circle Γ. et us join the end points of the spirals corresponding to $\theta = 1$ y a straight line L. Let $\mathfrak{C}$ denote the figure formed by the pirals S_1, S_2, the segment L and the asymptotic circle Γ. Is $\mathfrak{C}$ closed Jordan curve? The answer is, No. This may be seen n many ways. For example, $\mathfrak{C}$ does not divide the other points nto two precincts, but into three, one of which is formed of points ithin Γ.

Another way is to employ the reasoning of 593, 2. Here the rcle Γ takes the place of the segment on the y-axis which figures here.

Still another way is to observe that no point on Γ can be joined o a point within $\mathfrak{C}$ by a path.

597. *Example 4.* Let $\mathfrak{C}$ be formed of the edge $\mathfrak{E}$ of a unit quare, together with the ordinates $\mathfrak{o}$ erected at the points

$x=\frac{m}{2^n}$, of length $\frac{1}{2^n}$, $n=1, 2 \cdots$ Although $\mathfrak{C}$ divides the other points of the plane into two precincts $\mathfrak{J}$ and $\mathfrak{O}$, we can say that $\mathfrak{C}$ is not a closed Jordan curve.

For if it were, $\mathfrak{J}$ and $\mathfrak{O}$ would have to have $\mathfrak{C}$ as a common frontier. But the frontier of $\mathfrak{O}$ is $\mathfrak{E}$, while that of $\mathfrak{J}$ is $\mathfrak{C}$.

That $\mathfrak{C}$ is not a Jordan curve is seen in other ways. For example, let γ be an inner segment of one of the ordinates o. Obviously it cannot be reached by a path in $\mathfrak{O}$.

Brouwer's Proof of Jordan's Theorem

598. We have already given one proof of this theorem in 577 seq., based on the fact that the coördinates of the closed curve are expressed as one-valued continuous functions

$$x=\phi(t) \quad , \quad y=\psi(t).$$

Brouwer's proof* is entirely geometrical in nature and rests on the definition of a closed Jordan curve as the unifold continuous image of a circle, cf. 591, 5.

If $\mathfrak{A}, \mathfrak{B}, \cdots$ are point sets in the plane, it will be convenient to denote their frontiers by $\mathfrak{F}_{\mathfrak{A}}, \mathfrak{F}_{\mathfrak{B}} \cdots$ so that

$$\mathfrak{F}_{\mathfrak{A}} = \text{Front } \mathfrak{A} \quad , \quad \text{etc.}$$

We admit that any closed polygon $\mathfrak{P}$ having a finite number of sides, without double point, divides the other points of the plane into an inner and an outer precinct $\mathfrak{P}_\iota$, $\mathfrak{P}_o$ respectively. In the following sections we shall call such a polygon *simple*, and usually denote it by $\mathfrak{P}$.

We shall denote the whole plane by $\mathfrak{E}$.

Then
$$\mathfrak{E} = \mathfrak{P} + \mathfrak{P}_o + \mathfrak{P}_\iota.$$

Let $\mathfrak{A}$ be complete. The complementary set A is formed, as we saw in 328, of an enumerable set of precincts, say $A = \{A_n\}$.

* *Math. Annalen*, vol. 69 (1910), p. 169.

599. 1. *If a precinct* $\mathfrak{A}$ *and its complement** A *each contain a oint of the connex* $\mathfrak{C}$, *then* $\mathfrak{F}_{\mathfrak{A}}$ *contains a point of* $\mathfrak{C}$.

For in the contrary case $\mathfrak{c} = Dv(\mathfrak{A}, \mathfrak{C})$ is complete. In fact $\mathfrak{B} = \mathfrak{A} + \mathfrak{F}_{\mathfrak{A}}$ is complete. As $\mathfrak{C}$ is complete, $Dv(\mathfrak{B}, \mathfrak{C})$ is com-lete. But if $\mathfrak{F}_{\mathfrak{A}}$ does not contain a point of $\mathfrak{C}$, $\mathfrak{c} = Dv(\mathfrak{B}, \mathfrak{C})$. Thus on this hypothesis, $\mathfrak{c}$ is complete. Now $c = Dv(A, \mathfrak{C})$ is omplete in any case. Thus $\mathfrak{C} = \mathfrak{c} + c$, which contradicts 585, 4.

2. *If* $\mathfrak{P}_\iota$, $\mathfrak{P}_e$, *the interior and exterior of a simple polygon* $\mathfrak{P}$ *each ontain a point of a connex* $\mathfrak{C}$, *then* $\mathfrak{P}$ *contains a point of* $\mathfrak{C}$.

3. *Let* $\mathfrak{K}$ *be complete and not connected. There exists a simple polygon* $\mathfrak{P}$ *such that no point of* $\mathfrak{K}$ *lies on* $\mathfrak{P}$, *while a part of* $\mathfrak{K}$ *lies in* $\mathfrak{P}_\iota$ *and another part in* $\mathfrak{P}_e$.

For let $\mathfrak{K}_1$, $\mathfrak{K}_2$ be two non-connected parts of $\mathfrak{K}$ whose distance rom each other is $\rho > 0$. Let Δ be a quadrate division of the lane of norm δ, so small that no cell contains a point of $\mathfrak{K}_1$ and $\mathfrak{K}_2$. Let Δ_1 denote the cells of Δ containing points of $\mathfrak{K}_1$. It is omplete, and the complementary set $\Delta_2 = \mathfrak{C} - \Delta_1$ is formed of one r more precincts. No point of $\mathfrak{K}_1$ lies in Δ_2 or on its frontier.

Let P_1, P_2 be points in $\mathfrak{K}_1$, $\mathfrak{K}_2$ respectively. Let D be that recinct containing P_2. Then $\mathfrak{F}_D$ embraces a simple polygon $\mathfrak{P}$ which separates P_1 and P_2.

4. *Let* $\mathfrak{K}_1$, $\mathfrak{K}_2$ *be two detached connexes. There exists a simple polygon* $\mathfrak{P}$ *which separates them. One of them is in* $\mathfrak{P}_\iota$, *the other in* $\mathfrak{P}_e$, *and no point of either connex lies on* $\mathfrak{P}$.

For the previous theorem shows that there is a simple polygon $\mathfrak{P}$ which separates a point P_1 in $\mathfrak{K}_1$ from a point P_2 in $\mathfrak{K}_2$ and no oint of $\mathfrak{K}_1$ or $\mathfrak{K}_2$ lies on $\mathfrak{P}$. Call this fact F.

Let now P_1 lie in $\mathfrak{P}_\iota$. Then every point of $\mathfrak{K}_1$ lies in $\mathfrak{P}_\iota$. For therwise $\mathfrak{P}_\iota$ and $\mathfrak{P}_e$ each contain a point of the connex $\mathfrak{K}_1$. Then 2 shows that a point of $\mathfrak{K}_1$ lies on $\mathfrak{P}$, which contradicts F.

5. *Let* $\mathfrak{B}$ *be a precinct determined by the connex* $\mathfrak{C}$. *Then* $= \text{Front}\ \mathfrak{B}$ *is a connex.*

* Since the initial sets are all limited, their complements may be taken with reference to a sufficiently large square $\mathfrak{Q}$; and when dealing with frontier points, points n the edge of $\mathfrak{Q}$ may be neglected.

For suppose $\mathfrak{b}$ is not a connex. Then by 3, there exists a simple polygon $\mathfrak{P}$ which contains a part of $\mathfrak{b}$ in $\mathfrak{P}_\iota$ and another in $\mathfrak{P}_e$, while no point of $\mathfrak{b}$ lies on $\mathfrak{P}$. Hence a point β' of $\mathfrak{b}$ lies in $\mathfrak{P}_\iota$, and another point β'' in $\mathfrak{P}_e$. As $\mathfrak{B}$ is a precinct, let us join β', β'' by a path v in $\mathfrak{B}$. Thus $\mathfrak{P}$ contains at least one point of v, that is, a point of $\mathfrak{B}$ lies on $\mathfrak{P}$. As $\mathfrak{b}$ and $\mathfrak{P}$ have no point in common, and as one point of $\mathfrak{P}$ lies in $\mathfrak{B}$, all the points of $\mathfrak{P}$ lie in $\mathfrak{B}$. Hence

$$Dv(\mathfrak{P}, \mathfrak{C}) = 0. \tag{1}$$

As $\mathfrak{b}$ is a part of $\mathfrak{C}$ and hence some of the points of $\mathfrak{C}$ are in $\mathfrak{P}_e$ and some in $\mathfrak{P}_\iota$, it follows from 2 that a part of $\mathfrak{P}$ lies in $\mathfrak{C}$. This contradicts 1).

6. Let $\mathfrak{K}_1$, $\mathfrak{K}_2$ be two connexes without double point. By 3 there exists a simple polygon $\mathfrak{P}$ which separates them and has one connex inside, the other outside $\mathfrak{P}$.

Now $\mathfrak{K} = \mathfrak{K}_1 + \mathfrak{K}_2$ is complete and defines one or more precincts. *One of these precincts contains* $\mathfrak{P}$.

For say $\mathfrak{P}$ lay in two of these precincts as $\mathfrak{A}$ and $\mathfrak{B}$. Then the precinct $\mathfrak{A}$ and its complement (in which $\mathfrak{B}$ lies) each contain a point of the connex $\mathfrak{P}$. Thus $\mathfrak{F}_{\mathfrak{A}}$ contains a point of $\mathfrak{P}$. But $\mathfrak{F}_{\mathfrak{A}}$ is a part of $\mathfrak{K}$, and no point of $\mathfrak{K}$ lies on $\mathfrak{P}$.

That precinct in Comp $\mathfrak{K}$ which contains $\mathfrak{P}$ we call the *intermediate precinct* determined by $\mathfrak{K}_1$, $\mathfrak{K}_2$, or more shortly the precinct between $\mathfrak{K}_1$, $\mathfrak{K}_2$ and denote it by Inter $(\mathfrak{K}_1, \mathfrak{K}_2)$.

7. *Let* $\mathfrak{K}_1$, $\mathfrak{K}_2$ *be two detached connexes, and let* $\mathfrak{k} = \text{Inter}\ (\mathfrak{K}_1, \mathfrak{K}_2)$. *Then* $\mathfrak{K}_1$, $\mathfrak{K}_2$ *can be joined by a path lying in* $\mathfrak{k}$, *except its end points which lie on the frontiers of* $\mathfrak{K}_1$, $\mathfrak{K}_2$ *respectively.*

For by hypothesis $\rho = \text{Dist}(\mathfrak{K}_1, \mathfrak{K}_2) > 0$. Let P_1 be a point of $\mathfrak{F}_{\mathfrak{K}_1}$ such that some domain $\mathfrak{d}$ of P_1 contains only points of $\mathfrak{K}_1$ and of $\mathfrak{k}$. Let Q_1 be a point of $\mathfrak{k}$ in $\mathfrak{d}$. Join P_1, Q_1 by a right line, let it cut $\mathfrak{F}_{\mathfrak{K}_1}$ first at the point P'. In a similar way we may reason on $\mathfrak{K}_2$, obtaining the points P'', Q_2. Then $P'Q_1Q_2P''$ is the path in question. If we denote it by v, we may let v^* denote this path after removing its two end points.

8. *Let* $\mathfrak{K}_1$, $\mathfrak{K}_2$ *be two detached connexes. A path* v *joining* $\mathfrak{K}_1$, $\mathfrak{K}_2$ *and lying in* $\mathfrak{k} = \text{Inter}(\mathfrak{K}_1, \mathfrak{K}_2)$, *end points excepted, determines one and only one precinct in* $\mathfrak{k}$.

For from an arbitrary point P in $\mathfrak{k}$, let us draw all possible
aths to v. Those paths ending on the same side (left or right)
v certainly lie in one and the same precinct $\mathfrak{k}_r$ or $\mathfrak{k}_l$ in $\mathfrak{k}$. Then
nce one end point of v is inside, the other end point outside $\mathfrak{P}$,
iere must be a part of $\mathfrak{P}$ which is not met by v and which joins
ie right and left sides of v. We take this as an evident property
finite broken lines and polygons without double points.

Thus $\mathfrak{k}_l$ and $\mathfrak{k}_r$ are not detached; they are parts of one precinct.

9. *Two paths* v_1, v_2 *without common point, lying in* $\mathfrak{k}$ *and joining*
$_1$, $\mathfrak{K}_2$, *split* $\mathfrak{k}$ *into two precincts.*

Let $\mathfrak{i} = \mathfrak{k} - v_1$; this we have just seen is a precinct. From any
oint of it let us draw paths to v_2. Those paths ending on the
ame side of v_2 determine precincts $\mathfrak{i}_l$, $\mathfrak{i}_r$ which may be identical.
uppose they are. Then the two sides of v_2 can be joined by a
ath lying in $\mathfrak{k}$, which does not touch v_2 (end points excepted),
as no point in common with v_1, and together with a segment of
forms a simple polygon $\mathfrak{P}$ which has one end point of v_1 in $\mathfrak{P}_i$,
ie other end point in $\mathfrak{P}_e$. Thus by 2, $\mathfrak{P}$ contains a point of the
onnex v_1. This is contrary to hypothesis.

Similar reasoning shows that

10. *The n paths* $v_1 \cdots v_n$ *pairwise without common point, lying in*
and joining the connexes $\mathfrak{K}_1$, $\mathfrak{K}_2$ *split* $\mathfrak{k}$ *into n precincts.*

Let us finally note that the reasoning of 595, 4, being independ-
nt of an analytic representation of a Jordan curve, enables us to
se the geometric definition of 591, 5, and we have therefore the
ieorem

11. *Let* $\mathfrak{A}$ *be a precinct whose frontier* $\mathfrak{F}$ *is a Jordan curve. Then*
ere exists a path in $\mathfrak{A}$ *joining an arbitrary point of* $\mathfrak{A}$ *with any arc*
$\mathfrak{F}$.

Having established these preliminary theorems, we may now
ke up the body of the proof.

600. 1. *Let* $\mathfrak{A}$ *be a precinct determined by a closed Jordan curve*
Then $\mathfrak{F} =$ Front $\mathfrak{A}$ *is identical with* J.

If J determines but one precinct $\mathfrak{A}$ which is pantactic in $\mathfrak{E}$, we
ave obviously $\mathfrak{F} = J$.

Suppose then that $\mathfrak{A}$ is a precinct, not pantactic in $\mathfrak{E}$. Let $\mathfrak{B}$ be a precinct $\neq \mathfrak{A}$ determined by $\mathfrak{F}$. Let $\mathfrak{b} = \text{Front } \mathfrak{B}$. Then $\mathfrak{b} \leq \mathfrak{F} \leq J$. *Suppose now* $\mathfrak{b} < J$. As J is a connex by 591, 1, $\mathfrak{F}$ is a connex by 599, 5. Similarly since $\mathfrak{F}$ is a connex, $\mathfrak{b}$ is a connex. Since $\mathfrak{b} < J$, let $b \sim \mathfrak{b}$ on the circle Γ whose image is J. We divide b into three arcs b_1, b_2, b_3 to which $\sim \mathfrak{b}_1, \mathfrak{b}_2, \mathfrak{b}_3$ in $\mathfrak{b}$.

Let
$$\beta = \text{Inter}\,(\mathfrak{b}_1, \mathfrak{b}_3).$$

Then by 599, 11, we can join $\mathfrak{b}_1, \mathfrak{b}_3$ by a path v_1 in $\mathfrak{A}$, and by a path v_2 in $\mathfrak{B}$. By 599, 9, these paths split β into two precincts β_1, β_2. We can join v_1, v_2 by a path u_1 lying in β_1, and by a path u_2 lying in β_2.

Now the precinct $\mathfrak{B}$ and its complement each contain a point of the connex u_1. Hence by 599, 1, $\mathfrak{b}$ contains a point of u_1. Similarly $\mathfrak{b}$ contains a point of u_2. Thus u_1, u_2 cut $\mathfrak{b}$, and as they do not cut $\mathfrak{b}_1, \mathfrak{b}_3$ by hypothesis, they cut $\mathfrak{b}_2$. Thus *at least one point of* β_1 *and one point of* β_2 *lie in* $\mathfrak{b}_2$.

Let $\mathfrak{p}$ be a point of β_1 lying in $\mathfrak{b}_2$, let $p \sim \mathfrak{p}$ on the circle. Let b' be an arc of b_2 containing p. Let $\mathfrak{b}' \sim b'$. As the connex $\mathfrak{b}'$ has no point in common with Front β_1, $\mathfrak{b}'$ must lie entirely in β_1 by 599, 1. This is independent of the choice of $\mathfrak{b}'$, hence the connex $\mathfrak{b}_2$, except its end points, lies in β_1. Thus β_2 can contain no point of $\mathfrak{b}_2$, which contradicts the result in italics above.

Thus the supposition that $\mathfrak{b} < J$ is impossible. Hence $\mathfrak{b} = J$, and therefore $\mathfrak{F} = J$.

As a corollary we have:

2. *A Jordan curve is apantactic in* $\mathfrak{E}$.

3. *A closed Jordan curve J cannot determine more than two precincts.*

For suppose there were more than two precincts

$$\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3 \cdots \tag{1}$$

Let us divide the circle Γ into four arcs whose images call J_1, J_2, J_3, J_4.

Then by 1, the frontier of each of the precincts 1) is J. Thus by 599, 9, there is a path in each of the precincts $\mathfrak{A}_1, \mathfrak{A}_2 \cdots$ joining J_1 and J_3. These paths split

$$\mathfrak{k} = \text{Inter}\,(J_1, J_3)$$

to precincts $\mathfrak{k}_1, \mathfrak{k}_2 \cdots$

Now as in 1, we show on the one hand that each $\mathfrak{k}_\iota$ must contain point of J_2 or J_4, and on the other hand neither J_2 nor J_4 can e in more than one $\mathfrak{k}_\iota$.

4. *A closed Jordan curve J must determine at least two precincts.*

Suppose that J determines but a single precinct $\mathfrak{A}$. From a oint a of $\mathfrak{A}$ we may draw two non-intersecting paths u_1, u_2 to oints b_1, b_2 of J.

Since the point a may be regarded as a connex, a and J are two etached connexes. Hence by 599, 6, the paths u_1, u_2 split $\mathfrak{A}$ into wo precincts $\mathfrak{A}_1, \mathfrak{A}_2$. Let $j = (u_1, u_2, J)$. The points b_1, b_2 vide J into two arcs J_1, J_2, and

$$j_1 = (u_1, u_2, J_1) \quad , \quad j_2 = (u_1, u_2, J_2) \ .$$

o closed Jordan curves. Regarding a and J_1 as two detached nnexes, we see j_1 determines two precincts, α_1, α_2. By 599, 1, a th which joins a point a_1 of α_1 with a point a_2 of α_2 must cut j_1 d hence j. It cannot thus lie altogether in $\mathfrak{A}_1$ or in $\mathfrak{A}_2$. Thus th a_1, a_2 do not lie in $\mathfrak{A}_1$, nor both in $\mathfrak{A}_2$. Let us therefore y for example that $\mathfrak{A}_1$ lies in α_1, and $\mathfrak{A}_2$ in α_2. Hence by 2, is pantactic in α_1, and $\mathfrak{A}_2$ in α_2. By 1, each point of j_1 is comon to the frontiers of α_1 and of α_2, and hence of $\mathfrak{A}_1$ and of $\mathfrak{A}_2$, these are pantactic.

Let P be a point of J_2. It lies either in α_1 or α_2. Suppose it es in α_1. Then it lies neither in α_2 nor on Front α_2, and hence ither in $\mathfrak{A}_2$ nor on Front $\mathfrak{A}_2$. But every point of j_2 and also ery point of j_1 lies on Front $\mathfrak{A}_2$. We are thus brought to a ntradiction. Hence the supposition that J determines but a ngle precinct is untenable.

Dimensional Invariance

601. 1. In 247 we have seen that the points of a unit interval and of a unit square S may be put in one to one correspondence. his fact, due to Cantor, caused great astonishment in the mathe- atical world, as it seemed to contradict our intuitional views

egarding the number of dimensions necessary to define a figure. 'hus it was thought that a curve required *one* variable to define t, a surface *two*, and a solid *three*.

The correspondence set up by Cantor is not continuous. On he other hand the curves invented by Peano, Hilbert, and others cf. 573) establish a continuous correspondence between I and S, ut this correspondence is not one to one. Various mathematiians have attempted to prove that a continuous one to one correpondence between spaces of m and n dimensions cannot exist. Ve give a very simple proof due to *Lebesgue*.*

It rests on the following theorem:

2. *Let* $\mathfrak{A}$ *be a point set in* $\mathfrak{R}_m$. *Let* $\mathfrak{Q} \leq \mathfrak{A}$ *be a standard cube*

$$0 \leq x_\iota - a_\iota \leq 2\sigma \quad , \quad \iota = 1, 2 \cdots m.$$

Let $\mathfrak{C}_1, \mathfrak{C}_2 \cdots$ *be a finite number of complete sets so small that each ies in a standard cube of edge* σ. *If each point of* $\mathfrak{A}$ *lies in one of he* $\mathfrak{C}$'s, *there is a point of* $\mathfrak{A}$ *which lies in at least* $m+1$ *of them.*

Suppose first that each $\mathfrak{C}_\iota$ is the union of a finite number of tandard cubes. Let $\mathfrak{E}_1$ denote those $\mathfrak{C}$'s containing a point of he face $\mathfrak{f}_1$ of $\mathfrak{Q}$ lying in the plane $x_1 = a_1$. The frontier $\mathfrak{F}_1$ of $\mathfrak{E}_1$ s formed of a part of the faces of the $\mathfrak{C}$'s. Let F_1 denote that art of $\mathfrak{F}_1$ which is parallel to $\mathfrak{f}_1$. Let $\mathfrak{Q}_1 = Dv(\mathfrak{Q}, F_1)$. Any oint of it lies in at least two $\mathfrak{C}$'s.

Let $\mathfrak{E}_2$ denote those of the $\mathfrak{C}$'s not lying altogether in $\mathfrak{E}_1$ and ontaining a point of the face $\mathfrak{f}_2$ of $\mathfrak{Q}$ determined by $x_2 = a_2$. Let F_2 denote that part of Front $\mathfrak{E}_2$ which is parallel to $\mathfrak{f}_2$. Let $\mathfrak{Q}_2 = Dv(\mathfrak{Q}_1, F_2)$. Any point of it lies in at least three of the $\mathfrak{C}$'s.

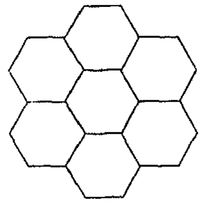
In this way we may continue, arriving finally at $\mathfrak{Q}_m$, any point f which lies in at least $m+1$ of the $\mathfrak{C}$'s.

Let us consider now the *general case*. We effect a cubical livision of space of norm $d < \sigma$. Let C_ι denote those cells of D vhich contain a point of $\mathfrak{C}_\iota$. Then by the foregoing, there is a oint of $\mathfrak{A}$ which lies in at least $m+1$ of the C's. As this is true, owever small d is taken, and as the $\mathfrak{C}$'s are complete, there is at east one point of $\mathfrak{A}$ which lies in $m+1$ of the $\mathfrak{C}$'s.

* *Math Annalen*, vol. 70 (1911), p. 166.

. We now note that *the space $\mathfrak{R}_m$ may be divided into congruent s so that no point is in more than $m+1$ cells.*

'or $m=1$ it is obvious. For $m=2$ we may a hexagonal pattern. We may also use uadrate division of norm δ of the plane. ese squares may be grouped in horizontal ds. Let every other band be slid a distance to the right. Then no point lies in more n 3 squares. For $m=3$ we may use a ical division of space, etc.



n each case no point of space is in more than $m+1$ cells.
,et us call such a division a *reticulation* of $\mathfrak{R}_m$.

. *Let $\mathfrak{A}$ be a point set in $\mathfrak{R}_m$ having an inner point a. There is continuous unifold image $\mathfrak{B}$ of $\mathfrak{A}$ in $\mathfrak{R}_n$, $n \neq m$, such that $b \sim a$ is inner point of $\mathfrak{B}$.*

'or let $n > m$. Let us effect a reticulation R of $\mathfrak{R}_m$ of norm ρ. > 0 is taken sufficiently small $\Delta = D_{2\delta}(a)$ lies in $\mathfrak{A}$. Let $= D_\delta(a)$; if ρ is taken sufficiently small, the cells

$$C_1,\ C_2 \cdots C_s \qquad (1$$

R which contain points of E, lie in Δ. Let the image of E be nd that of the cells 1) be

$$\mathfrak{C}_1,\ \mathfrak{C}_2 \cdots \mathfrak{C}_s. \qquad (2$$

'hese are complete. Each point of $\mathfrak{C}$ lies in one of the sets 2). ice by 2, they contain a point β which lies in $n+1$ of them. en $\alpha \sim \beta$ lies in $n+1$ of the cells 1). But these, being part of reticulation R, are such that no point lies in more than $m+1$ hem. Hence the contradiction.

02. 1. *Schönfliess' Theorem.* Let

$$u = f(x, y) \quad , \quad v = g(x, y) \qquad (1$$

one-valued and continuous in a unit square A whose center is origin. These equations define a transformation T. If T is ular, we have seen in I, 742, that the domain $D_\rho(P)$ of a point $= (x, y)$ within A goes over into a set E such that if $Q \sim P$ n $D_\sigma(Q)$ lies in E, if $\sigma > 0$ is sufficiently small.

These conditions on f, g which make T regular are sufficient, but they are much more than necessary as the following theorem due to *Schönfliess* * shows.

2. *Let* $A = B + c$ *be a unit square in the* x, y *plane, whose center is the origin and whose frontier is* c.

Let

$$u = f(x, y) \quad , \quad v = g(x, y)$$

be one-valued continuous functions in A. *As* (x, y) *ranges over* A, *let* (u, v) *range over* $\mathfrak{A} = \mathfrak{B} + \mathfrak{c}$ *where* $\mathfrak{c} \sim c$. *Let the correspondence between* A *and* $\mathfrak{A}$ *be uniform. Then* $\mathfrak{c}$ *is a closed Jordan curve and the interior* $\mathfrak{c}_\iota$ *of* $\mathfrak{c}$ *is identical with* $\mathfrak{B}$.

That $\mathfrak{c}$ is a closed Jordan curve follows from 576 seq., or 598 seq. Obviously if one point of $\mathfrak{B}$ lies in $\mathfrak{c}_\iota$, all do. For if β_ι, β_e are points of $\mathfrak{B}$, one within $\mathfrak{c}$ and the other without, let $b_\iota \sim \beta_\iota$, $b_e \sim \beta_e$. Then b_ι, b_e lying in B can be joined by a path in B which has no point in common with c. The image of this path is a continuous curve which has no point in common with $\mathfrak{c}$, which contradicts 578, 2.

Let

$$\rho = \phi(\theta)$$

be the equation of c in polar coördinates.

If $0 \leq \mu \leq 1$, the equation

$$\rho = \mu\phi(\theta)$$

defines a square, call it c_μ, concentric with c and whose sides are in the ratio $\mu : 1$ with those of c. The equations of $\mathfrak{c}_\mu \sim c_\mu$ are

$$u = f\{\mu\phi(\theta)\cos\theta \quad , \quad \mu\phi(\theta)\sin\theta\} = F(\mu, \theta),$$

$$v = g\{\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\} = G(\mu, \theta).$$

These $\mathfrak{c}_\mu$ curves have now the following property:

If a point (p, q) *is exterior (interior) to* $\mathfrak{c}_{\mu_0}$, *it is exterior (interior) to* $\mathfrak{c}_\mu$, *for all* μ *such that*

$$|\mu - \mu_0| \leq \text{some } \epsilon > 0.$$

For let ρ_μ be the distance of (p, q) from a point (u, v) on $\mathfrak{c}_\mu$. Then

$$\rho_\mu = \sqrt{(u_\mu - p)^2 + (v_\mu - q)^2}$$

***Goettingen Nachrichten*, 1899. The demonstration here given is due to *Osgood*, *Goett. Nachr.*, 1900.

is a continuous function of θ, μ which does not vanish for $\mu = \mu_0$, when $0 \leq \theta \leq 2\pi$. But being continuous, it is uniformly continuous. It therefore does not vanish in the rectangle

$$-\epsilon + \mu_0 \leq \mu \leq \mu_0 + \epsilon \quad , \quad 0 \leq \theta \leq 2\pi.$$

We can now show that *if* $\mathfrak{B} \leq \mathfrak{c}_\iota$, *it is identical with* $\mathfrak{c}_\iota$. To this end we need only to show that any point β of $\mathfrak{c}_\iota$ lies on some $\mathfrak{c}_\mu$. In fact, as $\mu \doteq 0$, $\mathfrak{c}_\mu$ contracts to a point. Thus β is an outer point of some $\mathfrak{c}_\mu$, and an inner point of others. Let μ_0 be the maximum of the values of μ such that β is exterior to all $\mathfrak{c}_\mu$, if $\mu < \mu_0$. Then β lies on $\mathfrak{c}_{\mu_0}$. For if not, β is exterior to $\mathfrak{c}_{\mu_0+\epsilon}$, by what we have just shown, and μ_0 is not the maximum of μ.

Let us suppose that $\mathfrak{B}$ *lay without* $\mathfrak{c}$. We show this leads to a contradiction. For let us invert with respect to a circle $\mathfrak{k}$, lying in $\mathfrak{c}_\iota$. Then $\mathfrak{c}$ goes over into a curve $\mathfrak{f}$, and $\mathfrak{A}$ goes over into $\mathfrak{D} = \mathfrak{E} + \mathfrak{f}$. Then $\mathfrak{E}$ lies inside $\mathfrak{f}$. Let ξ, η be coördinates of a point of $\mathfrak{D}$. Obviously they are continuous functions of x, y in A, and

$$A \sim \mathfrak{D} \quad , \quad c \sim \mathfrak{f}, \qquad \text{uniformly.}$$

By what we have just proved, $\mathfrak{E}$ must fill all the interior of $\mathfrak{f}$. This is impossible unless $\mathfrak{A}$ is unlimited.

3. We may obviously extend the theorem 2 to the case

$$u_1 = f_1(x_1 \cdots x_m) \cdots u_m = f_m(x_1 \cdots x_m)$$

and A is a cube in m-way space $\mathfrak{R}_m$, *provided* we assume that $\mathfrak{c}$, the image of the boundary of A, divides space into two precincts whose frontier is $\mathfrak{c}$.

Area of Curved Surfaces

603. 1. *The Inner Definition.* It is natural to define the area of a curved surface in a manner analogous to that employed to define the length of a plane curve, viz. by inscribing and circumscribing the surface with a system of polyhedra, the area of whose faces converges to 0. It is natural to expect that the limits of the area of these two systems will be identical, and this common limit would then forthwith serve as the definition of the area of the surface. The consideration of the inner and the outer sys-

tems of polyhedra afford thus two types of definitions, which may be styled the inner and the outer definitions. Let us look first at the inner definition.

Let the equations of the surface S under consideration be

$$x=\phi(u, v) \quad , \quad y=\psi(u, v) \quad , \quad z=\chi(u, v), \tag{1}$$

the parameters ranging over a complete metric set $\mathfrak{A}$, and x, y, z being one-valued and continuous in $\mathfrak{A}$.

Let us effect a rectangular division D of norm d of the u, v plane. The rectangles fall into triangles t_κ on drawing the diagonals. Such a division of the plane we call *quasi rectangular*. Let

$$P_0=(u_0, v_0) \quad , \quad P_1=(u_0+\delta, v) \quad , \quad P_2=(u_0, v_0+\eta)$$

be the vertices of t_κ. To these points in the u, v plane correspond three points $\mathfrak{P}_\iota=(x_\iota, y_\iota, z_\iota)$, $\iota=1, 2, 3$, of S which form the vertices of one of the triangular faces τ_κ of the inscribed polyhedron Π_D corresponding to the division D. Here, as in the following sections, we consider only triangles lying in $\mathfrak{A}$. We may do this since $\mathfrak{A}$ is metric.

Let X_κ, Y_κ, Z_κ be the projections of τ_κ on the coördinate planes. Then, as is shown in analytic geometry,

$$\tau_\kappa^2=X_\kappa^2+Y_\kappa^2+Z_\kappa^2$$

where

$$2X_\kappa=\begin{vmatrix} y_0 & z_0 & 1 \\ y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \end{vmatrix}=\begin{vmatrix} y_1-y_0 \, , & z_1-z_0 \\ y_2-y_0 \, , & z_2-z_0 \end{vmatrix}=\begin{vmatrix} \Delta' y \, , & \Delta' z \\ \Delta'' y \, , & \Delta'' z \end{vmatrix}$$

and similar expressions for Y_κ, Z_κ.

Thus the area of Π_D is

$$S_D=\Sigma\sqrt{X_\kappa^2+Y_\kappa^2+Z_\kappa^2},$$

the summation extending over all the triangles t_κ lying in the set $\mathfrak{A}$.

Let x, y, z have continuous first derivatives in $\mathfrak{A}$. Then

$$\Delta' x=x_1-x_0=\frac{\partial x}{\partial u}\delta+\alpha'\delta \,; \quad \Delta'' x=x_2-x_0=\frac{\partial x}{\partial v}\eta+\alpha''\eta,$$

with similar expressions for the other increments. Let

$$A = \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} \quad , \quad B = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} \quad , \quad C = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \qquad (2$$

Then

$$X_\kappa = (A_\kappa + \alpha_\kappa)t_\kappa \quad , \quad Y_\kappa = (B_\kappa + \beta_\kappa)t_\kappa \quad , \quad Z_\kappa = (C_\kappa + \gamma_\kappa)t_\kappa$$

where $\alpha_\kappa\ \beta_\kappa\ \gamma_\kappa$ are uniformly evanescent with d in $\mathfrak{A}$. Thus if A, B, C do not simultaneously vanish at any point of $\mathfrak{A}$, we have as area of the surface S

$$\lim_{d=0} S_D = \int_{\mathfrak{A}} \sqrt{A^2 + B^2 + C^2}\, du\, dv. \qquad (3$$

2. An objection which at once arises to this definition lies in the fact that we have taken the faces of our inscribed polyhedra in a very restricted manner. We cannot help asking, Would we get the same area for S if we had chosen a different system of polyhedra?

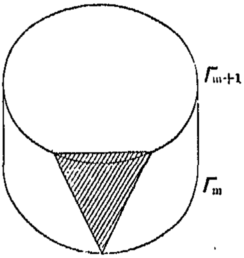
To lessen the force of this objection we observe that by replacing the parameters u, v by two new parameters u', v' we may replace the above quasi rectangular divisions which correspond to the family of right lines u = constant, v = constant by the infinitely richer system of divisions corresponding to the family of curves u' = constant, v' = constant. In fact, by subjecting u', v' to certain very general conditions, we may transform the integral 3) to the new variables u', v' without altering its value.

But even this does not exhaust all possible ways of dividing $\mathfrak{A}$ into a system of triangles with evanescent sides. Let us therefore take at pleasure a system of points in the u, v plane having no limiting points, and join them in such a way as to cover the plane without overlapping with a set of triangles t_κ. If each triangle lies in a square of side d, we may call this a triangular division of norm d. We may now inquire if S_D still converges to the limit 3), as $d \doteq 0$, for this more general system of divisions. It was generally believed that such was the case, and standard treatises even contained demonstrations to this effect. These demonstrations are wrong; for Schwarz * has shown that by

* *Werke*, vol. 2, p. 309.

properly choosing the triangular divisions D, it is possible to make S_D converge to a value large at pleasure, for an extensive class of simple surfaces.

604. 1. *Schwarz's Example.* Let C be a right circular cylinder of radius 1 and height 1. A set of planes parallel to the base at a distance $\frac{1}{n}$ apart cuts out a system of circles $\Gamma_1, \Gamma_2 \cdots$ Let us divide each of these circles into m equal arcs, in such a way that the end points of the arcs on $\Gamma_1, \Gamma_3, \Gamma_5 \cdots$ lie on the same vertical generators, while the end points of $\Gamma_2, \Gamma_4, \Gamma_6 \cdots$ lie on generators halfway between those of the first set. We now inscribe a polyhedron so that the base of one of the triangular facets lies on one circle while the vertex lies on the next circle above or below, as in the figure.



The area t of one of these facets is

$$t = \tfrac{1}{2} bh \quad , \quad b = 2 \sin \frac{\pi}{m} \quad , \quad h = \sqrt{\frac{1}{n^2} + \left(1 - \cos \frac{\pi}{m}\right)^2}.$$

Thus

$$t = \sin \frac{\pi}{m} \sqrt{\frac{1}{n^2} + 4 \sin^4 \frac{\pi}{2m}}.$$

There are $2m$ such triangles in each layer, and there are n layers. Hence the area of the polyhedron corresponding to this triangular division D is

$$S_D = \Sigma t_\kappa = 2\, mn \sin \frac{\pi}{m} \sqrt{\frac{1}{n^2} + 4 \sin^4 \frac{\pi}{2m}}.$$

Since the integers m, n are independent of each other, let us consider various relations which may be placed on them.

Case 1°. Let $n = \lambda m$. Then

$$S_D = 2\, m^2 \lambda \sin \frac{\pi}{m} \sqrt{\frac{1}{\lambda^2 m^2} + 4 \sin^4 \frac{\pi}{2m}}$$

$$= 2\, m^2 \lambda \cdot \frac{\pi}{m} \cdot \frac{\sin \frac{\pi}{m}}{\frac{\pi}{m}} \sqrt{\frac{1}{\lambda^2 m^2} + 4 \frac{\pi^4}{2^4 m^4} \left[\frac{\sin \frac{\pi}{2m}}{\frac{\pi}{2m}}\right]^4}$$

$$\doteq 2\pi \quad , \quad \text{as } m \doteq \infty.$$

Case 2°. Let $n = \lambda m^2$. Then

$$S_D = 2\lambda m^3 \cdot \frac{\pi}{m}\left(\frac{\sin\frac{\pi}{m}}{\frac{\pi}{m}}\right)\sqrt{\frac{1}{\lambda^2 m^4} + 4\frac{\pi^4}{2^4 m^4}\left(\frac{\sin\frac{\pi}{2m}}{\frac{\pi}{2m}}\right)^4}$$

$$\doteq 2\pi\sqrt{1 + \frac{\pi^4}{4}\lambda^2} \quad , \quad \text{as } m \doteq \infty.$$

Case 3°. Let $n = \lambda m^3$. Then

$$S_D = 2\pi\left(\frac{\sin\frac{\pi}{m}}{\frac{\pi}{m}}\right)\sqrt{1 + \frac{\pi^4}{2^2} m^2\lambda^2\left(\frac{\sin\frac{\pi}{2m}}{\frac{\pi}{2m}}\right)^4}$$

$$\doteq +\infty \quad , \quad \text{as } m \doteq \infty.$$

2. Thus only in the first case does S_D converge to 2π, which the area of the cylinder C as universally understood. In the ° and 3° cases the ratio $h/b \doteq 0$. As equations of C we may ke

$$x = \cos u \quad , \quad y = \sin u \quad , \quad z = v.$$

Then to a triangular facet of the inscribed polyhedron will corspond a triangle in the u, v plane. In cases 2° and 3° this trigle has an angle which converges to π as $m \doteq \infty$. This is not in case 1°. Triangular divisions of this latter type are of great nportance. Let us call then a triangular division of the u, v lane such that no angle of any of its triangles is greater than $-\epsilon$, where $\epsilon > 0$ is small at pleasure but fixed, *positive triangur divisions*. We employ this term since the sine of one of the ngles is > some fixed positive number.

605. *The Outer Definition.* Having seen one of the serious diffiılties which arise from the inner definition, let us consider briefly ıe outer definition. We begin with the simplest case in which ıe equation of the surface S is

$$z = f(x, y), \tag{1}$$

being one-valued and having continuous first derivatives. Let s effect a metric division Δ of the x, y plane of norm δ, and on

each cell d_κ as base, we erect a right cylinder C, which cuts out an element of surface δ_κ from S. Let $\mathfrak{P}_\kappa$ be an arbitrary point of δ_κ and $\mathfrak{T}_\kappa$ the tangent plane at this point. The cylinder C cuts out of $\mathfrak{T}_\kappa$ an element ΔS_κ. Let ν_κ be the angle that the normal to $\mathfrak{T}_\kappa$ makes with the z-axis. Then

$$\cos \nu_\kappa = \frac{1}{\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)_\kappa^2 + \left(\frac{\partial z}{\partial y}\right)_\kappa^2}}$$

and

$$\Delta S_\kappa = \frac{d_\kappa}{\cos \nu_\kappa}.$$

The area of S is now defined to be

$$\lim_{\delta=0} \Sigma \Delta S_\kappa \tag{2}$$

when this limit exists. The derivatives being continuous, we have at once that this limit is

$$\int_{\mathfrak{A}} dxdy \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \tag{3}$$

which agrees with the result obtained by the inner definition in 603, 3).

The advantages of this form of definition are obvious. In the first place, the nature of the divisions Δ is quite arbitrary; however they are chosen, one and the same limit exists. Secondly, the most general type of division is as easy to treat as the most narrow, viz. when the cells d_κ are squares.

Let us look at its disadvantages. In the first place, the elements ΔS_κ do not form a circumscribing polyhedron of S. On the contrary, they are little patches attached to S at the points $\mathfrak{P}_\kappa$, and having in general no contact with one another. Secondly, let us suppose that S has tangent planes parallel to the z-axis. The derivatives which enter the integral 603, 3) are no longer continuous, and the reasoning employed to establish the existence of the limit 2) breaks down. Thirdly, we have the case that z is not one-valued, or that the tangent planes to S do not turn continuously, or do not even exist at certain points.

To get rid of these disadvantages various other forms of outer efinitions have been proposed. One of these is given by *Goursat* ı his *Cours d'Analyse.* Instead of projecting an arbitrary lement of surface on a fixed plane, the xy plane, it is projected on ne of the tangent planes belonging to that element. Hereby the nore general type of surfaces defined by 603, 1) instead of those efined by 1) above is considered. The restriction is, however, nade that the normals to the tangent planes cut the elements of urface but once, also the first derivatives of the coördinates are ssumed to be continuous in $\mathfrak{A}$. Under these conditions we get ıe same value for the area as that given in 603, 3).

When the first derivatives of x, y, z are not continuous or do ot exist, this definition breaks down. To obviate this difficulty *e la Vallée-Poussin* has proposed a third form of definition in his *'ours d'Analyse*, vol. 2, p. 30 seq. Instead of projecting the lement of surface on a tangent plane, let us project it on a plane or which the projection is a maximum. In case that S has a con- inuously turning tangent plane nowhere parallel to the z-axis, de ı Vallée-Poussin shows that this definition leads to the same alue of the area of S as before. He does not consider other cases ı detail.

Before leaving this section let us note that Jordan in his *Cours* nploys the form of outer definition first noted, using the paramet- c form of the equations of S. In the preface to this treatise the ıthor avows that the notion of area is still somewhat obscure, and ıat he has not been able "à définir d'une manière satisfaisante aire d'une surface gauche que dans le cas où la surface a un plan ıngent variant suivant une loi continue."

606. 1. *Regular Surfaces.* Let us return to the inner definition onsidered in 603. We have seen in 604 that not every system of riangular divisions can be employed. Let us see, however, if we annot employ divisions much more general than the quasi rec- ıngular. We suppose the given surface is defined by

$$x = \phi(u, v) \quad , \quad y = \psi(u, v) \quad , \quad z = \chi(u, v) \tag{1}$$

ıe functions ϕ, ψ, χ being one-valued, totally differentiable func- ons of the parameters u, v which latter range over the complete

metric set $\mathfrak{A}$. Surfaces characterized by these conditions we shall call *regular*. Let

$$P_0=(u_0, v_0) \quad , \quad P_1=(u_0+\delta', v_0+\eta') \quad , \quad P_2=(u_0+\delta'', v_0+\eta'')$$

be the vertices of one of the triangles t_κ, of a triangular division D of norm d of $\mathfrak{A}$. As before let $\mathfrak{P}_0, \mathfrak{P}_1, \mathfrak{P}_2$ be the corresponding points on the surface S. Then

$$\Delta' x = x_1 - x_0 = \frac{\partial x}{\partial u}\delta' + \frac{\partial x}{\partial v}\eta' + \alpha_x'\delta' + \beta_x'\eta',$$

$$\Delta'' x = x_2 - x_0 = \frac{\partial x}{\partial u}\delta'' + \frac{\partial x}{\partial v}\eta'' + \alpha_x''\delta + \beta_x''\eta'',$$

and similar expressions hold for the other increments. Also

$$2\, X_\kappa = \begin{vmatrix} \dfrac{\partial y}{\partial u}\delta' + \dfrac{\partial y}{\partial v}\eta' \quad , & \dfrac{\partial z}{\partial u}\delta' + \dfrac{\partial z}{\partial v}\eta' \\ \dfrac{\partial y}{\partial u}\delta'' + \dfrac{\partial y}{\partial v}\eta'' \quad , & \dfrac{\partial z}{\partial u}\delta'' + \dfrac{\partial z}{\partial v}\eta'' \end{vmatrix} + 2\, X_\kappa',$$

where X_κ' denotes the sum of several determinants, involving the infinitesimals

$$\alpha_y' \quad , \quad \alpha_y'' \quad , \quad \beta_z' \quad , \quad \beta_z''.$$

Similar expressions hold for Y_κ, Z_κ. We get thus

$$X_\kappa = A_\kappa t_\kappa + X_\kappa' \quad , \quad Y_\kappa = B_\kappa t_\kappa + Y_\kappa' \quad , \quad Z_\kappa = C_\kappa t_\kappa + Z_\kappa'$$

where A, B, C are the determinants 2) in 603. Then the area of the inscribed polyhedron corresponding to this division D is

$$S_D = \Sigma t_\kappa \sqrt{\left(A_\kappa + \frac{X_\kappa'}{t_\kappa}\right)^2 + \left(B_\kappa + \frac{Y_\kappa'}{t_\kappa}\right)^2 + \left(C_\kappa + \frac{Z_\kappa'}{t_\kappa}\right)^2}.$$

Let us suppose that

$$A^2 + B^2 + C^2 \geq q \quad , \quad q > 0 \tag{2}$$

as u, v ranges over $\mathfrak{A}$. Also let us assume that

$$\frac{X_\kappa'}{t_\kappa} \quad , \quad \frac{Y_\kappa'}{t_\kappa} \quad , \quad \frac{Z_\kappa'}{t_\kappa} \tag{3}$$

main numerically $< \epsilon$ for any division D of norm $d < d_0$, ϵ small pleasure, except in the vicinity of a discrete set of points, that , let 3) be *in general uniformly evanescent* in $\mathfrak{A}$, as $d \doteq 0$. Then

$$S_D = \Sigma t_\kappa \sqrt{A_\kappa^2 + B_\kappa^2 + C_\kappa^2} + \Sigma \epsilon_\kappa t_\kappa,$$

here *in general*

$$|\epsilon_\kappa| < \frac{\epsilon}{\text{Cont}\, \mathfrak{A}}.$$

If now A, B, C are limited and R-integrable in $\mathfrak{A}$, we have at ice

$$\lim_{d=0} S_D = \int_{\mathfrak{A}} du dv \sqrt{A^2 + B^2 + C^2}$$

in 603.

2. We ask now under what conditions are the expressions 3) general uniformly evanescent in $\mathfrak{A}$? The answer is pretty eviant from the example given by Schwarz. In fact the equation the tangent plane $\mathfrak{T}$ at $\mathfrak{P}_0$ is

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

On the other hand the equation of the plane $T = (\mathfrak{P}_0, \mathfrak{P}_1, \mathfrak{P}_2)$

$$\begin{vmatrix} x & y & z & 1 \\ x_0 & y_0 & z_0 & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \end{vmatrix} = 0,$$

$$xX_\kappa + yY_\kappa + zZ_\kappa + U_\kappa = 0,$$

finally

$$x\left(A_\kappa + \frac{X_\kappa'}{t_\kappa}\right) + y\left(B_\kappa + \frac{Y_\kappa'}{t_\kappa}\right) + z\left(C_\kappa + \frac{Z_\kappa'}{t_\kappa}\right) + \frac{U_\kappa}{t_\kappa} = 0.$$

Thus for 3) to converge in general uniformly to zero, it is necsary and sufficient that the secant planes T converge in general iformly to tangent planes. Let us call divisions such that the ces of the corresponding inscribed polyhedra converge in general iformly to tangent planes *uniform triangular* divisions. For ch divisions the expressions 3) are in general uniformly evanesnt, as $d \doteq 0$. We have therefore the following theorem :

3. *Let $\mathfrak{A}$ be a limited complete metric set. Let the coördinates y, z be one-valued totally differentiable functions of the parame-*

ters u, v in $\mathfrak{A}$, *such that* $A^2+B^2+C^2$ *is greater than some positive constant, and is limited and R-integrable in* $\mathfrak{A}$. *Then*

$$S=\lim_{d=0} S_D=\int_{\mathfrak{A}}\sqrt{A^2+B^2+C^2}\,du\,dv,$$

D denoting the class of uniform triangular divisions of norms d.

This limit we shall call the *area of S.* From this definition we have at once a number of its properties. We mention only the following:

4. *Let* $\mathfrak{A}_1, \cdots \mathfrak{A}_m$ *be unmixed metric sets whose union is* $\mathfrak{A}$. *Let* $S_1, \cdots S_m$ *be the pieces of S corresponding to them. Then each* S_κ *has an area and their sum is S.*

5. *Let* $\mathfrak{A}_\lambda$ *be a metric part of* $\mathfrak{A}$, *depending on a parameter* $\lambda \doteq 0$, *such that* $\widehat{\mathfrak{A}}_\lambda \doteq \widehat{\mathfrak{A}}$. *Then*

$$\lim_{\lambda=0} S_\lambda = S.$$

6. *The area of S remains unaltered when S is subjected to a displacement or a transformation of the parameters as in* I, 744 *seq.*

607. 1. *Irregular Surfaces.* We consider now surfaces which do not have tangent planes at every point, that is, surfaces for which one or more of the first derivatives of the coördinates x, y, z do not exist, and which may be styled *irregular surfaces.* We prove now the theorem:

Let the coördinates x, y, z be one-valued functions of u, v having limited total difference quotients in the metric set $\mathfrak{A}$. *Let D be a positive triangular division of norm* $d \le d_0$. *Then*

$$\text{Max } S_D$$

is finite and evanescent with $\mathfrak{A}$.

For let the difference quotients remain $\le \mu$. We have

$$S_D \le \Sigma|X_\kappa| + \Sigma|Y_\kappa| + \Sigma|Z_\kappa|.$$

But

$$|X_\kappa| = \tfrac{1}{2}|\Delta'y\Delta''z - \Delta'z\Delta''y| \le \tfrac{1}{2}\{|\Delta'y|\cdot|\Delta''z| + |\Delta'z|\cdot|\Delta''y|\}$$

$$\le \mu^2\, \overline{P_0P_1}\cdot\overline{P_0P_2} = 2\,\mu^2 t_\kappa|\operatorname{cosec}\theta_\kappa|$$

where θ_κ is the angle made by the sides $\overline{P_0P_1}$, $\overline{P_0P_2}$. As D is a positive division, one of the angles of t_κ is such that cosec θ_κ is numerically less than some positive number M. Thus

$$|X_\kappa| < 2\,\mu^2 M t_\kappa,$$

where μ, M are independent of κ and d. Similar relations hold for $|Y_\kappa|$, $|Z_\kappa|$. Hence

$$S_D < \Sigma\, 6\,\mu^2 M \cdot t_\kappa = 6\,\mu^2 M(\overline{\mathfrak{A}} + \eta)$$

where $\eta > 0$ is small at pleasure, for d_0 sufficiently small.

2. *Let* $\mathfrak{A}$ *and* x, y, z *be as in* 606, 3, *except at certain points forming a discrete set* $\mathfrak{a}$, *the first partial derivatives do not exist. Let their total difference quotients be limited in* $\mathfrak{A}$. *Then*

$$\lim_{d=0} S_D = \int \sqrt{A^2 + B^2 + C^2}\, du dv,$$

where D denotes a positive triangular division of norm d.

Let us first show that the limit on the left exists. We may choose a metric part $\mathfrak{B}$ of $\mathfrak{A}$ such that $\mathfrak{C} = \mathfrak{A} - \mathfrak{B}$ is complete and exterior to $\mathfrak{a}$ and such that $\widehat{\mathfrak{B}}$ is as small as we please. Let $S_{\mathfrak{C}}$ denote the area of the surface corresponding to $\mathfrak{C}$. The triangles t_κ fall into two groups: G_1 containing points of $\mathfrak{B}$; G_2 containing only points of $\mathfrak{C}$. Then

$$S_D = \Sigma \sqrt{X_\kappa^2 + Y_\kappa^2 + Z_\kappa^2} = \sum_{G_1} + \sum_{G_2}.$$

But $\widehat{\mathfrak{B}}$ may be chosen so small that the first sum is $< \epsilon/4$ for any $d < d_0$. Moreover by taking d_0 still smaller if necessary, we have

$$|\sum_{G_2} - S_{\mathfrak{C}}| < \epsilon/4.$$

Hence

$$|S_D - S_{\mathfrak{C}}| < \epsilon/2 \quad , \quad d < d_0. \tag{1}$$

Similarly for any other division D' of norm d',

$$|S_{D'} - S_{\mathfrak{C}}| < \epsilon/2 \quad , \quad d' < d_0$$

decreasing d_0 still farther if necessary. Thus

$$|S_{D'} - S_D| < \epsilon \quad , \quad d, d' < d_0.$$

Hence $\lim S_D$ exists, call it S. Since S exists we may take d_0 so small that

$$|S - S_D| < \epsilon/2 \quad , \quad d < d_0.$$

This with 1) gives

$$|S - S_{\mathfrak{C}}| < \epsilon,$$

that is,

$$S = \lim S_{\mathfrak{C}} = \lim \int_{\mathfrak{C}} \sqrt{A^2 + B^2 + C^2}\, dudv$$

$$= \int_{\mathfrak{A}} \sqrt{A^2 + B^2 + C^2}\, dudv$$

by I, 724.

608. 1. The preceding theorem takes care of a large class of irregular surfaces whose total difference quotients are limited. In case they are not limited we may treat certain cases as follows:

Let us effect a quadrate division of the u, v plane of norm d, and take the triangles t_κ so that for any triangular division D associated with d, no square contains more than n triangles, and no triangle lies in more than ν squares; n, ν being arbitrarily large constants independent of d. Such a division we call a *quasi quadrate division of norm d*. If we replace the quadrate by a rectangular division, we get a quasi rectangular division.

We shall also need to introduce a new classification of functions according to their variation in $\mathfrak{A}$, or along lines parallel to the u, v axes. Let D be a quadrate division of the u, v plane of norm $d < d_0$. Let

$$\omega_\kappa = \text{Osc} f(u, v) \quad , \quad \text{in the cell } d_\kappa.$$

Then

$$\text{Max}\ \Sigma \omega_\kappa d$$

is the variation of f in $\mathfrak{A}$. If this is not only finite, but evanescent with $\bar{\mathfrak{A}}$, we say f has *limited fluctuation* in $\mathfrak{A}$. Obviously this may be extended to any limited point set in m-way space.

Let us now restrict ourselves to the plane. Let $\mathfrak{a}$ denote the points of $\mathfrak{A}$ on a line parallel to the u-axis. Let us effect a division D' of norm d'. Let $\omega'_\kappa = \text{Osc} f(u, v)$ in one of the intervals of D'. Then

$$\eta_{\mathfrak{a}} = \text{Max}\ \Sigma \omega'_\kappa$$

is the variation of f in $\mathfrak{a}$.

Let us now consider all the sets $\mathfrak{a}$ lying on lines parallel to the axis, and let

$$\bar{\mathfrak{a}} \leq \sigma \quad , \quad \sigma \doteq 0.$$

If now there exists a constant G independent of $\mathfrak{a}$ such that

$$\eta_{\mathfrak{a}} < \sigma G,$$

at is, if $\eta_{\mathfrak{a}}$ is uniformly evanescent with σ, we say that $f(u, v)$ s *limited fluctuation* in $\mathfrak{A}$ *with respect to* u.

With the aid of these notions we may state the theorems:

2. *Let the coördinates x, y, z be one-valued limited functions in e limited complete set $\mathfrak{A}$. Let x, y have limited total difference otients, while z has limited variation in $\mathfrak{A}$. Let D denote a quasi adratic division of norm $d \leq d_0$. Then*

$$\operatorname*{Max}_{D} S_D$$

finite.

For, as before,

$$2 \,|\, X_\kappa \,| \leq |\, \Delta'_y \,| \cdot |\, \Delta''_z \,| + |\, \Delta''_y \,| \cdot |\, \Delta'_z \,|.$$

But μ denoting a sufficiently large constant,

$$|\, \Delta'_y \,|, \qquad |\, \Delta''_y \,| \qquad \text{are} < \mu d.$$

Let $\omega_\iota = \text{Osc}\, z$ in the square s_ι. If the triangle t_κ lies in the uares $s_{\iota_1}, \cdots s_{\iota_k}$,

$$|\, \Delta'_z \,|, \; |\, \Delta''_z \,| \leq \omega_{\iota_1} + \cdots + \omega_{\iota_k}.$$

Thus, n denoting a sufficiently large constant,

$$\sum_\kappa |\, X_\kappa \,| < \mu \sum_\kappa d(\omega_{\iota_1} + \cdots \omega_{\iota_k})$$

$$< n\mu \sum_\iota \omega_\iota d,$$

e summation extending over those squares containing a triangle D. But z having limited variation,

$$\Sigma \omega_\iota d < \text{some } M.$$

Hence $\qquad \Sigma \,|\, X_\kappa \,| \quad , \quad \Sigma \,|\, Y_\kappa \,| \qquad \text{are} < n\mu M.$

Finally, as in 607,

$$\Sigma \,|\, Z \,| < \text{some } M'.$$

The theorem is thus established.

3. *The coördinates x, y, z, being as in 2, except that z has limited fluctuation in $\mathfrak{A}$, and D denoting a quasi quadrate division of norm $d \leq d_0$,*

$$\operatorname*{Max}_D S_D$$

is finite and evanescent with $\overline{\mathfrak{A}}$.

The reasoning is the same as in 2 except that now M, M' are evanescent with $\overline{\mathfrak{A}}$.

4. *Let the coördinates x, y, z have limited total difference quotients in $\mathfrak{A}$, while the variation of z along any line parallel to the u or v axis is $< M$. Let $\mathfrak{A}$ lie in a square of side $s \doteq 0$. Then*

$$\operatorname*{Max}_D S_D < sG,$$

where G is some constant independent of s, and D is a quasi rectangular division of norm $d \leq d_0$.

For here

$$2\Sigma \,|\, X_\kappa \,| \leq \Sigma \,|\, \Delta' y \,| \cdot |\, \Delta'' z \,| + \Sigma \,|\, \Delta'' y \,| \cdot |\, \Delta' z \,|$$

$$< M' \Sigma \omega_u d_v + M' \Sigma \omega_v d_u,$$

where M' denotes a sufficiently large constant; d_u, d_v denote the length of the sides of one of the triangles t_κ parallel respectively to the u, v axes, and ω_u, ω_v the oscillation of z along these sides. Since the variation is $< M$ in both directions,

$$\Sigma \omega_u d_v = \sum_v d_v \Sigma \omega_u < M \Sigma d_v < Ms.$$

Similarly

$$\Sigma \omega_v d_u < M_s.$$

The rest of the proof follows as before.

5. *The symbols having the same meaning as before, except that z has limited fluctuation with respect to u, v,*

$$\operatorname*{Max}_D S_D < s^2 G.$$

The demonstration is similar to the foregoing. Following the line of proof used in establishing 607, 2 and employing the theorems just given, we readily prove the following theorems:

6. *Let $\mathfrak{A}$ be a metric set containing the discrete set $\mathfrak{a}$. Let $\mathfrak{b}$ be a metric part of $\mathfrak{A}$, containing $\mathfrak{a}$ such that $\mathfrak{B} = \mathfrak{A} - \mathfrak{b}$ is exterior to $\mathfrak{a}$, and $\widehat{\mathfrak{b}} \doteq 0$. Let the coördinates x, y, z be one-valued totally differentiable functions in $\mathfrak{B}$, such that $A^2 + B^2 + C^2$ never sinks below a positive constant in any $\mathfrak{B}$, is properly R-integrable in any $\mathfrak{B}$, and improperly integrable in $\mathfrak{A}$. Let x, y have limited total difference quotients, and z limited fluctuation in $\mathfrak{b}$. Then*

$$\lim_{d=0} S_D = \int_{\mathfrak{A}} \sqrt{A^2 + B^2 + C^2}\,dudv$$

where A, B, C are the determinants in 603, 2), *and D is any quasi quadrate division of norm d.*

7. *Let the symbols have the same meaning as in* 6, *except that*

1° $\mathfrak{a}$ *reduces to a finite set.*

2° *z has limited variation along any line parallel to the u, v axes.*

3° *D denotes a uniform quasi rectangular division. Then*

$$\lim_{d=0} S_D = \int_{\mathfrak{A}} \sqrt{A^2 + B^2 + C^2}\,dudv.$$

8. *The symbols having the same meaning as in* 6, *except that*

1° *z has limited fluctuation with respect to u, v in $\mathfrak{b}$.*

2° *D denotes a uniform quasi rectangular division. Then*

$$\lim_{d=0} S_D = \int_{\mathfrak{A}} \sqrt{A^2 + B^2 + C^2}\,dudv.$$

9. *If we call the limits in theorems* 6, 7, 8, *area, the theorems* 606, 3, 4, 5 *still hold.*

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(Numbers refer to pages)

SYMBOLS EMPLOYED IN VOLUME II

(Numbers refer to pages)

The following symbols are defined in Volume I and are repeated here for the convenience of the reader.

$\mathrm{Dist}(a, x)$ is the distance between a and x

$D_\delta(a)$, called the *domain* of the point a of norm δ is the set of points x, such that $\mathrm{Dist}\,(a, x) < \delta$

$V_\delta(a)$, called the *vicinity* of the point a of norm δ, refers to some set $\mathfrak{A}$, and is the set of points in $D_\delta(a)$ which lie in $\mathfrak{A}$

$D_\delta^*(a)$, $V_\delta^*(a)$ are the same as the above sets, omitting a. They are called *deleted* domains, *deleted* vicinities

$a_n \doteq \alpha$ means a_n converges to α

$f(x) \doteq \alpha$, means $f(x)$ converges to α

A line of symbols as:

$$\epsilon < 0,\ m,\ |\alpha - a_n| < \epsilon,\ n > m$$

is of constant occurrence, and is to be read: for each $\epsilon > 0$, there exists an index m, such that $|\alpha - a_n| < \epsilon$, for every $n > m$

Similarly a line of symbols as:

$$\epsilon > 0,\ \delta > 0,\ |f(x) - \alpha| < \epsilon,\ x \text{ in } V_\delta^*(a)$$

is to be read: for each $\epsilon > 0$, there exists a $\delta > 0$, such that

$$|f(x) - \alpha| < \epsilon,$$

for every x in $V_\delta^*(a)$